Small Value Probabilities: Techniques and Applications

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1 Scientific Justification

Probability is both a fundamental way of viewing the world, and a core mathematical discipline, alongside geometry, algebra and analysis. Without any doubt, probability theory has become one of the most fascinating, fast growing and main stream areas of mathematics. Two fundamental problems in probability theory and statistical analysis are typical behaviors such as expectations, laws of large numbers, central limit theorems and approximated sampling distribution, and rare events such as large deviations, significant level and power.

One of the most important types of rare events is that positive random variables take smaller values. In the past twenty years significant developments in the area have transformed our understanding of rare events and have the potential to significantly expand the applicability of new techniques in the general setting. The three main goals are to systematically study the existing techniques and applications which are spread over various topics, to further develop new methods of estimating small value problems for Gaussian and closely related random processes, and to reformulate and investigate problems in other areas of mathematics from the point of view of small value problems. We believes that a theory of small value problems and their wide range of applications should be developed.

1.1 Small Value (Deviation) Probability

Small value (deviation) probability studies the asymptotic rate of approaching zero for rare events that positive random variables take smaller values. More precisely, let Y_n

be a sequence of *non-negative* random variables and suppose that some or all of the probabilities

$$\mathbb{P}\{Y_n \le \varepsilon_n\}, \quad \mathbb{P}\{Y_n \le C\}, \quad \mathbb{P}\{Y_n \le (1-\delta)\mathbb{E}Y_n\}$$

tend to zero as $n \to \infty$, for $\varepsilon_n \to 0$, some constant $C > 0$ and $0 < \delta \le 1$. Obviously, they are all special cases of $\mathbb{P}(Y_n \le h_n) \to 0$ for some function $h_n \ge 0$, but examples and applications show the benefits of the separate formulations.

What is often an important and interesting problem is the determination of just how "rare" the events $\{Y_n \le \varepsilon_n\}$, $\{Y_n \le C\}$ and $\{Y_n \le (1-\delta)\mathbb{E}Y_n\}$ are. That is, one wants to know the rates at which the probabilities above are tending to 0. In general, a detailed answer to this question is seldom available in the infinite or high dimensional setting. However, if one only asks for rates up to a constant and/or at the logarithmic level such as $\log \mathbb{P}(Y_n \le h_n) \to 0$, then one has a much better chance of finding a solution and one is studying the *small value probabilities* of Y_n associated with the sequence h_n.

In the case that all of the Y_n are the same and $Y_n = \|X\|$ is the norm of a centered Gaussian random element X on a separable Banach space E, then we are in the setting of small ball with $\varepsilon_n = \varepsilon$ and we can see that the analysis of small ball probabilities are relative hard. Similarly, in various physical models, one can consider the associated Hamiltonian (energy function) H which is a nonnegative function. The asymptotic behavior of the partition function (normalizing constant) $\mathbb{E}e^{-\lambda H}$ for $\lambda > 0$ is of great interests and it is directly connected with the small value behavior $\mathbb{P}(H \le \epsilon)$ for $\epsilon > 0$ under appropriate scaling.

For a given process $\{X_t, t \in T\}$ with index set T and fixed $t_0 \in T$, the small ball probability under the sup-norm studies the behavior of $\mathbb{P}(\sup_{t\in T} |X_t - X_{t_0}| \le \varepsilon)$ and the lower tail probability studies the behavior of $\mathbb{P}(\sup_{t\in T}(X_t - X_{t_0}) \le \varepsilon)$, as $\varepsilon \to 0^+$. Both types play fundamental role in many areas of probability and they can also be viewed as the first exit time problems (one or two sided) if the process has scaling property.

1.2 Applications of Small Value Probabilities

In the literature, small value probabilities of various types are studied and applied to many problems of interest under different names such as small deviation/ball probabilities, lower tail behaviors, boundary crossing probabilities, asymptotic evaluation of Laplace transform for large time, etc. In addition to applications discussed in details later, some recent ones in related area of mathematics and statistics include: Quantization problems for Gaussian measure, e.g. Dereich et al (2003); Classical, average and probabilistic Kolmogorov widths, e.g. Creutzig (2002); Decaying turbulent transport, Vanden-Eijnden (2001); Passive scalar transport in Majdas model, Bronski (2003a,b); Singularity of Burgers equation, e.g. Sinai (1997) and Molchan (1999); Capacity estimates in Wiener space, e.g. Khoshnevisan and Shi (1998); Duality of entropy numbers, e.g. Milman and Szarek (2001); Wiener-Hopf equation, e.g. Li and Zhang (2010+); Hamiltonian and partition function, e.g. Chen and Li (2010+); Intersection local time of independent Brownian paths and random walks, e.g. Lawler and Limic (2010); Lower tail of the Martingale limit of variants of branching processes, e.g. Chu, Li and Ren (2010+); Random graphs and random sum of vectors, e.g. Alon and Spencer (1992, p148); Littlewood and Offord type problems, Tao and Vu (2006); Discrepancy theory, e.g. Chazelle (2000). Many other applications are also surveyed in Li and Shao (2001), including Chung's law of the iterated logarithm, lower limits for empirical processes rates of convergence of Strassen's FLIL, volume of Wiener sausage and fractional Brownian sausage.

In the following, we use three unexpected examples to show case the essential use of the small value probabilities in geometric functional analysis, in analysis and in PDEs, among many others.

First, probabilistic methods have been applied to geometric functional analysis successfully. For example, large deviation estimates are by now a standard tool in the asymptotic convex geometry. And precise links are established in Kuelbs and Li (1993) and completed in Li and Linde (1999), between small ball probability for Gaussian measure and the metric entropy of the unit ball of the associated reproducing kernel Hilbert space. This connection turns out to be one of the most powerful techniques and solves significant problems associated with compactness of linear operators. Very recently, novel applications of small deviation estimates to problems related to the diameters of random sections of high dimensional convex bodies are realized. They imply distinction between the lower and the upper inclusions in the celebrated Dvoretzky theorem, which says that any n-dimensional convex body has a section of dimension $c \log n$ that is approximately a Euclidean ball. See Latala and Oleszkiewicz (2005) and Klartag and Vershynin (2007). Results of these types have significant applications in dimensional reduction analysis for high dimensional dates.

Second, the connections between heat equation, principle eigenvalue and asymptotic of the exit time are well known for smooth bounded open domains and strong elliptic operator by using Feynman-Kac formula. The asymptotic of the process staying inside a domain for a long time can be viewed as as small value probability. For smooth *unbounded* domains or for *degenerated* operators, there is no general theory available due to non-self-adjointness. However, significant progress has been made in Lifshits and Shi (2002) and Li (2003) by using techniques from small value probability for Brownian motion in $\mathbb{R}^d$. This provides a way to studying certain long time behavior of a nonlinear evolution equation with solution satisfied by the associated non-exit probability. The quest for understanding other processes is likely to stimulate and to challenge probabilists for years to come.

Third, the criterion for the smoothness of the density leads to a probabilistic proof of Hormanders hypoellipticity theorem. The main ingredients are Malliavin calculus and some small value estimates for the determinant of the Malliavin matrix which is positive semi-definite. In many applications, such as the smoothness of the density of the solutions of stochastic partial differential equations (SPDE), one needs to show the determinant of the Malliavin matrix has negative moments of all orders, e.g. Nualart (2009). It turns out that the negative moments estimates is equivalent to the upper bound small value estimates. Due to the need to understand the finer properties of SPDEs such as smoothness of theirs density, small value probability is sure to play a much bigger role in the area.

2 Description of Lectures

The purpose of these lectures is to present the state of the art of various powerful techniques on estimating small value probabilities, including traditional ones such as blocking, chaining, series expansions, Laplace/Taubirean theorems, classical Gaussian inequalities, Feynman-Kac formula, and newly developed ones such as metric entropy, weaker correction and reverse Slepian type inequalities, determinantal approaches, etc. Major applications include strong limit theorems in probability and statistics, smoothness of density via Malliavin calculus, approximation quantities for stochastic processes, exit time and boundary crossing asymptotics, deviations for local times. The guiding philosophy of these lectures is that analysis of concrete processes is the most effective way to explain even the most general methods or abstract principles.

In the first three lectures we will provide an overview of small value probabilities, basic estimates and techniques associated with independent random variables, together with various applications. In Lecture 4 we will present blocking techniques for the maximum partial sums and stable processes, following early work of Chung and recent refined work for weighted sup-norms. In the next two lectures we will deal with Gaussian processes and related Gaussian measures in Banach space setting. Two highlights are the precise connections between small ball provability and metric entropy estimates of the associated generating compact operators, and the estimates for dependent sums via the weaker correlation inequality. Their far-reaching implications are explored. Lecture 7 and 8 are devoted to more general techniques for both upper and lower bounds, including chaining, locally non-determinant method, sup via l_2 method, Riesz representations for random fields, determinant method for smooth processes, Fourier analytic arguments. In lecture 9, we will develop techniques for the existence of precise constants, including orthogonal series expansions, the l_2 comparison theorems, and scaling/subsdditive method. In Lecture 10 we will present small value probabilities of lower tail type and the one-sided exit asymptotics for both Markov and Gaussian processes. One special feature is that, at the end of each lecture, several open and important problems related to techniques discussed will be offered. This will ensure that the participants, especially the new or recent entrants to the field, can thinking and working on interesting problems and branch/connect into the area.

1. Introduction, overview and applications. We first define the small value (deviation) probability in several setting, which basically studies the asymptotic rate of approaching zero for rare events that *positive* random variables take smaller values. Many applications discussed in the scientific justification section are given. Benefits and differences of various formulations of small value probabilities are examined in details, together with connections to related fields.

2. Basic estimates and equivalent transformations. We first formulate several equivalent results for small value probability, including negative moments, exponential moments, Laplace transform and Taubirean theorems. The basic techniques involved are various useful inequalities, motivated from large deviation estimates. Some refinement of known results are given, including to the classical Paley-Zegmund inequality. Applications to regularity and smoothness of probability laws via small value estimates of the determinant of Malliavin matrix are discussed in the setting of stochastic (partial) differential equations.

3. Techniques associated with independent variables. We start with probabilistic arguments for algebraic properties of small value probabilism, such as independent sums and products. These estimates are non-asymptotic and hence they can be applied are in the setting of conditional probability. Separate treatments are analyzed for exponential and power decay rates. A newly discover symmetrization inequality is proved by Fourier analytic method. Littlewood and Offord type problems are discussed. We end with

Komlos Conjecture on balancing vectors in discrepancy theory.

4. Blocking techniques for the sup-norm. We first present the vary useful blocking techniques for the maximum of the absolute value of partial sums in both upper and bound setting. The lower bound is more involved since the end position of each block has to be controlled also. The resulting estimates play a critical role in the Chung's type strong limit theorems for sample paths. Similar techniques are applied to weighted and/or controlled sup-norms for Brownian motion and stable processes. Applications to the two-sided exit time and Wichura type functional limit theorems are indicated.

5. Links between small ball probabilities and metric entropy. For a continues centered Gaussian processes, the generating linear operator is compact and so is the unit ball of the associated reproducing kernel Hilbert space. The fundamental links between small ball probability for Gaussian measure and the metric entropy are given and various far-reaching implications are explored. Several purely probabilistic results, obtained via the analytic connection without direct probabilistic proofs, are analyzed.

6. Small deviation (ball) estimates for sums of correlated Gaussian elements. We treat the sum of two *not* necessarily independent Gaussian random vectors in a separable Banach space. The main ingredients are Anderson's inequality and the weaker correlation inequality developed by the lecturer. Various applicants are provided to show the power of the method. As a direct consequence, under the sup-norm or L_p-norm, Brownian motion and Brownian bridge have exact the same small ball behavior at the log level, and so do Brownian sheets and various tied down Brownian sheets including Kiefer process.

7. More lower bound techniques. We first establish a commonly used general lower bound estimate for the supremum of non-differentiable Gaussian process via the chain argument, as well as improvements for smooth Gaussian processes. Then we present a connection between small ball probabilities, discovered recently, that can be used to estimate small ball probabilities under any norm via a relatively easier L_2-norm estimate.

8. More upper bound techniques. We present three techniques: locally non-determinant method, determinant method for smooth processes and Riesz representations for Gaussian random fields. The key ideas are illustrated by several important processes: fractional Brownian motion, L-process (infinitely differentiable), and Brownian and/or Slepian sheet.

9. Evaluation and existence of precise constants. Most of techniques discussed so far are for the asymptotic decay rate (up to a constant factor). Here we present a few known methods in which the exact constants can be obtained or shown to exist. In

the Hilbert space l_2, the full asymptotic formula is developed. And with the help of a comparison result, most small deviation probabilities under the L_2-norm can thus be treated, and in particular when the Karhunen-Loeve expansion for a given Gaussian process can be found in some reasonable form. This is the case for Brownian motion, fractional Brownian motion and Brownian sheets, etc. A scaling argument, similar to the well known subadditive method, is established for the sup-norm of the fractional Brownian motion

10. Lower tail probabilities and one-sided exit asymptotics. There are only a handful of known examples (specific Gaussian processes) for the one-sided exit asymptotics and it is intellectual challenging workout more examples in order to find a theory. We focus on extending results classical for Brownian motion to the fractional Brownian motions. The main motivations are not only the importance of these processes, but also the force to find proofs that relay upon general principles at a more fundamental level by moving away from crucial properties (such as the Markov property) of Brownian motion. Fractional Brownian motion might not be an object of central mathematical importance but abstract principles are.