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Small Deviations for Gaussian Markov Processes Under the Sup-Norm[1]

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Let $\{X(t); 0 \leqslant t \leqslant 1\}$ be a real-valued continuous Gaussian Markov process with mean zero and covariance $\sigma(s, t) = EX(s)\, X(t) \neq 0$ for $0 < s,\ t < 1$. It is known that we can write $\sigma(s, t) = G(\min(s, t))\, H(\max(s, t))$ with $G > 0$, $H > 0$ and G/H nondecreasing on the interval $(0, 1)$. We show that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log P(\sup_{0 < t \leqslant 1} |X(t)| < \varepsilon) = -(\pi^2/8) \int_0^1 (G'H - H'G)\, dt$$

In the critical case, i.e. this integral is infinite, we provide the correct rate (up to a constant) for $\log P(\sup_{0 < t \leqslant 1} |X(t)| < \varepsilon)$ as $\varepsilon \to 0$ under regularity conditions.

KEY WORDS: Small ball problem; Gaussian Markov processes; Brownian motion; weighted norms.

1. INTRODUCTION

Let $\{X(t); 0 \leqslant t \leqslant 1\}$ be a real-valued mean-zero Gaussian process, and let "$\|\cdot\|$" be a semi-norm on the space of real functions on $[0, 1]$. The so called "small ball estimates" or "small deviation estimates" refer to the asymptotic behavior of

$$\log P(\|X\| < \varepsilon), \qquad \text{for} \quad \varepsilon \to 0 \tag{1.1}$$

This type of problem is quite delicate, and the asymptotic decay rate in (1.1), up to a constant, depends heavily on the process $X(t)$ and the

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semi-norm being used. The papers [see Berthet and Shi;[1] Dunker *et al.*;[4] Kuelbs *et al.*;[8] Li;[13] Li and Shao;[16] Li and Linde[15]], together with their combined references, cover much of the recent progress in this area. In most of these papers, the main results determine the asymptotic behavior in (1.1) up to some constant factor in front of the rate. Even for Brownian motion and the Brownian bridge under various norms, these constants are known in only a few cases.

Although small ball estimates have generated considerable interest recently, relatively little is known, and there are very few general results available. Actually, as it was established in a paper of Kuelbs and Li[7] (see also Li and Linde[14] for further remarks), the rough behavior in (1.1) is determined by the metric entropy of the unit ball of the reproducing kernel Hilbert space associated with X, and *vice versa*. Thus one should *not* expect to see the complete asymptotics, including constants, for small balls of an arbitrary Gaussian process X. However, in this paper, the Gaussian Markov processes are studied, and in this situation we are able to determine even some constants.

Let us introduce some basic facts about real-valued continuous Gaussian Markov processes $X(t)$ on the interval $[0, 1]$ with mean zero. It is known [cf. Lévy;[12] Feller;[5] and Borisov[2]] that the covariance function $\sigma(s, t) = EX(s)\, X(t) < \infty$, $0 \leqslant s, t \leqslant 1$, satisfies the relation

$$\sigma(s, t)\, \sigma(t, u) = \sigma(t, t)\, \sigma(s, u), \qquad 0 \leqslant s < t < u \leqslant 1$$

and this relation actually implies the Markov property of $X(t)$. Hence it is easy to obtain and *characterize* the Gaussian Markov process $X(t)$ with $\sigma(s, t) \neq 0$, $0 < s \leqslant t < 1$, by

$$\sigma(s, t) = G(\min(s, t))\, H(\max(s, t)) \tag{1.2}$$

with $G > 0$, $H > 0$ and G/H nondecreasing on the interval $(0, 1)$. Moreover, the functions G and H are unique up to a constant multiple. Now we can state the first result of this paper.

Theorem 1. Let the Gaussian Markov process $X(t)$ be defined as earlier. Assume H and G are absolutely continuous and G/H is strictly increasing on the interval $[0, 1]$. If

$$\sup_{0 < t \leqslant 1} H(t) < \infty, \qquad \text{or} \tag{1.3}$$

$$H(t) \text{ is nonincreasing in a neighborhood of } 0 \tag{1.4}$$

then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log P(\sup_{0 \leqslant t \leqslant 1} |X(t)| < \varepsilon) = -\frac{\pi^2}{8} \int_0^1 (G'H - H'G)\, dt \tag{1.5}$$

First we observe that there is nothing special about the interval $[0, 1]$ and it can be replaced by any interval $[a, b]$ as long as $\sigma(s, t) \neq 0$ for s, t in (a, b) and the analogous regularity condition (1.3) or (1.4) holds.

Next we apply Theorem 1 to some well known Gaussian Markov processes. Let $\{W(t); 0 \leqslant t \leqslant 1\}$ be the standard Brownian motion and $\{B(t); 0 \leqslant t \leqslant 1\}$ be a standard Brownian bridge, which can be realized as $\{W(t) - tW(1); 0 \leqslant t \leqslant 1\}$.

Example 1. Consider $X_1(t) = t^{-\alpha} W(t)$ and $X_2(t) = t^{-\alpha} B(t)$ on the interval $[0, 1]$ for $\alpha < 1/2$. Then we have for $0 \leqslant s \leqslant t \leqslant 1$

$$\begin{aligned} \sigma_1(s, t) &= Cov(X_1(s)\, X_1(t)) = Cov(s^{-\alpha} W(s), t^{-\alpha} W(t)) \\ &= s^{1-\alpha} t^{-\alpha} = G_1(s)\, H_1(t) \end{aligned}$$

and

$$\begin{aligned} \sigma_2(s, t) &= Cov(X_2(s)\, X_2(t)) = Cov(s^{-\alpha} B(s), t^{-\alpha} B(t)) \\ &= s^{1-\alpha} t^{-\alpha} (1-t) = G_2(s)\, H_2(t) \end{aligned}$$

which imply

$$\int_0^1 (G_1' H_1 - H_1' G_1)\, dt = \int_0^1 (G_2' H_2 - H_2' G_2)\, dt = \frac{1}{(1-2\alpha)}$$

and the corresponding results in (1.5).

Example 2. Stationary Gaussian Markov processes or the Ornstein–Uhlenbeck process. In this case,

$$\sigma(s, t) = \sigma^2 e^{-\lambda |t-s|} = \sigma^2 e^{\lambda \min(s, t)} e^{-\lambda \max(s, t)} = G_3(s)\, H_2(t)$$

for $0 \leqslant s \leqslant t \leqslant 1$ and $\int_0^1 (G_3' H_3 - H_3' G_3)\, dt = 2\sigma^2 \lambda$.

Example 3. Nondegenerate Gaussian processes with independent increments. In this case, we have $\sigma(s, t) = G(\min(s, t))$ and thus the constant is $(G(1) - G(0))\, \pi^2/8$.

Next we mention that Theorem 1 is close related to the following result about small ball estimates for Brownian motion under a weighted sup-norm (More details are given in the next section).

Theorem A. Let $W(t)$, $t \geqslant 0$, be the standard Brownian motion. If $f\colon (0, 1] \mapsto (0, \infty)$ satisfies either of the conditions:

$$\inf_{0 < t \leqslant 1} f(t) > 0 \tag{H1}$$

$$f \text{ is nondecreasing in a neighborhood of } 0 \tag{H2}$$

Then,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log P\left(\sup_{0 < t \leqslant 1} \frac{|W(t)|}{f(t)} < \varepsilon\right) = -\frac{\pi^2}{8} \int_0^1 \frac{dt}{f^2(t)} \tag{1.6}$$

Theorem A was proved by Mogulskii[17] under essentially condition (H1) and by Berthet and Shi[1] under condition (H2). It is clear that the rate ε^2 is not correct in both Theorems 1 and A if the integrals in them are infinite. When the integral in Theorem A is infinite, partial estimates are given[1] and it was called critical case. Our next theorem deals with this critical case for both Theorems 1 and A as they are closely related. Here we formulate our next result in terms of a weighted sup-norm for Brownian motion. This is purely for the easy statement of the result and simpler notations in the proof. The precise statement for general Gaussian Markov processes in the critical case can be handled similarly as in the proof of Theorem 1.

Before we state our next result that provides tight estimates in the critical cases under regularity conditions, we need some notation. We use $a(x) \approx b(x)$ as $x \to 0$ if

$$0 < \liminf_{x \to 0} a(x)/b(x) \leqslant \limsup_{x \to 0} a(x)/b(x) < \infty.$$

For any positive nondecreasing function f on $(0, \delta)$ with $\delta > 0$ fixed, we define

$$N(\varepsilon) = \max\{n : t_n > 0\} \tag{1.7}$$

with $t_0 = \delta$ and

$$t_i = t_{i-1} - \varepsilon^2 f^2(t_{i-1}) \qquad \text{for} \quad i \geqslant 1 \tag{1.8}$$

Note that the sequence t_i is decreasing and $f(t_i) \leqslant f(t_{i-1})$ for $1 \leqslant i \leqslant N(\varepsilon)$.

Theorem 2. Assume that $f(x)$ is a positive nondecreasing function on $(0, \delta)$ with $\delta > 0$ fixed and $x/f^2(x)$ is also nondecreasing. Let $N(\varepsilon) < \infty$ be defined as in (1.7). If there exist some constants $1 < c_1 \leqslant c_2 < \infty$ such that for all $\varepsilon > 0$ small

$$c_1 N(\varepsilon) \leqslant N(\varepsilon/2) \leqslant c_2 N(\varepsilon) \tag{1.9}$$

then, for some constants $0 < C_1 \leqslant C_2 < \infty$ and every $\varepsilon > 0$,

$$-C_2 N(\varepsilon) \leqslant \log P\left(\sup_{0 < t \leqslant \delta} \frac{|W(t)|}{f(t)} < \varepsilon\right) \leqslant -C_1 N(\varepsilon)$$

Note that the assumptions on $f(x)$, namely $f(x)$ is a positive nondecreasing function and $x/f^2(x)$ is nondecreasing, are natural. This can be seen from the fact that the function $f(x)$ has to pass the Kolmogorov's integral test [cf. Itô and McKean,[6] p. 33]:

$$\int_0^\delta \frac{f(x)}{x^{3/2}} \exp\left(-\frac{f^2(x)}{2x}\right) dx < \infty \tag{1.10}$$

which requires the same assumptions. Otherwise, we will have $P(\sup_{0 < t \leqslant \delta} |W(t)|\ f(t) < \varepsilon) = 0$ for $\varepsilon > 0$ small.

Next, just to give a flavor of what this theorem covers, we have for example as $\varepsilon \to 0$

$$\log P\left(\sup_{0 < t \leqslant 1/e} \frac{|W(t)|}{\sqrt{t \log(1/t)}} < \varepsilon\right) \approx -\frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon} \tag{1.11}$$

Other examples and the proof of the theorem will be given in Section 2. In Section 3, we provide some final remarks and point out some applications. In particular, we also provide some results on weighted fractional Brownian motion.

2. PROOF OF THEOREMS AND MORE EXAMPLES

One of the key facts that relates Theorems 1 and A along with its proof is the following representation for Gaussian Markov processes

$$X(t) = h(t)\ W(g(t)) \tag{2.1}$$

with $g(t) > 0$ nondecreasing on the interval $(0, 1)$ and $h(t) > 0$ on the interval $(0, 1)$. It is easy to see the connection between (1.2) and (2.1),

$$h(t) = H(t) \qquad \text{and} \qquad g(t) = G(t)/H(t) \tag{2.2}$$

For our small deviation problem, we thus have by our representation, (2.1)

$$P(\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\varepsilon)=P(\sup_{0\leqslant t\leqslant 1}h(t)\,|W(g(t))|\leqslant\varepsilon)$$
$$=P(\sup_{g(0)\leqslant t\leqslant g(1)}h(g^{-1}(t))\,|W(t)|\leqslant\varepsilon)\qquad(2.3)$$

Note that there is no problem for the finiteness of $\sup_{g(0)\leqslant t\leqslant g(1)}h(g^{-1}(t))\,|W(t)|$ if

$$\sup_{g(0)\leqslant t\leqslant g(1)}h(g^{-1}(t))=\sup_{0\leqslant t\leqslant 1}H(t)<\infty$$

which is the assumption (1.3) in Theorem 1. However, when condition (1.3) fails, we must have $g(0)=0$ and $h(0)=\infty$, and the situation is somewhat delicate. For simplicity and easy notation to compare here and in the next section, let

$$1/f(t)=h(g^{-1}(t))=H((G/H)^{-1}(t))\qquad(2.4)$$

Then we are essentially (if $g(1)=1$) examining the small ball estimates for W under the weighted norm. Our assumption (1.4) in Theorem 1 and the assumption (H2) in Theorem A are same in this case. Thus our Theorem 1 follows from Theorem A under the assumption (1.4), and in this case ($g(0)=0$) we have for $g(1)=1$

$$\int_0^1\frac{dt}{f^2(t)}=\int_0^1 h^2(g^{-1}(t))\,dt=\int_0^1 h^2(t)\,g'(t)\,dt=\int_0^1 (G'H-H'G)\,dt$$

by using the relation (2.2).

Now more generally, for $g(0)\neq 0$ we would obtain the constant (without the $\pi^2/8$ in it)

$$\int_{g(0)}^{g(1)}\frac{dt}{f^2(t)}=\int_{g(0)}^{g(1)} h^2(g^{-1}(t))\,dt=\int_0^1 h^2(t)\,g'(t)\,dt=\int_0^1 (G'H-H'G)\,dt$$

Thus we need a weighted result with $g(0)\neq 0$ under the assumption, (1.3) in Theorem 1. This can be done by two different ways. The first is along the line given by Berthet and Shi.[1] We can use the following more general estimate given in Li[13] for $0\leqslant a<b$

$$\log P(\sup_{a\leqslant t\leqslant b}|W(t)|<\varepsilon)\sim-\frac{\pi^2(b-a)}{8\varepsilon^2}\qquad\text{as}\quad\varepsilon\to 0$$

rather than the one with $a=0$ used in the proof given by Berthet and Shi.[1] We also need to take care of the starting position $W(g(0))$ compared with $W(0)=0$, but this can be done easily since it makes no contribution to the asymptotics at the logarithmic level. The second is along the line given by Mogulskii[17] with necessary modifications. Thus we will not give a detailed proof here but just point out that the length of the interval considered in Theorem 1 is not important as long as the covariance on the interior of the interval does not vanish, which is important in the arguments.

Next we turn to the proof of Theorem 2. Let us first consider the upper bound. By independent increments and scaling property of Brownian motion, it follows that

$$\begin{aligned}
P\left(\sup_{0<t\leqslant\delta}\frac{|W(t)|}{f(t)}<\varepsilon\right) &\leqslant P(|W(t_i)|<\varepsilon f(t_i),\ 0\leqslant i\leqslant N(\varepsilon))\\
&\leqslant P(|W(t_{i-1})-W(t_i)|<2\varepsilon f(t_{i-1}),\ 1\leqslant i\leqslant N(\varepsilon))\\
&=\prod_{i=1}^{N(\varepsilon)} P(|W(t_{i-1})-W(t_i)|<2\varepsilon f(t_{i-1}))\\
&=\prod_{i=1}^{N(\varepsilon)} P(|W(1)|<2\varepsilon f(t_{i-1})/\sqrt{t_{i-1}-t_i})\\
&=P(|W(1)|<2)^{N(\varepsilon)}
\end{aligned}$$

and thus the upper bound follows. Note that if $N(\varepsilon)=\infty$ for some $\varepsilon>0$, then these arguments show that

$$P\left(\sup_{0<t\leqslant\delta}\frac{|W(t)|}{f(t)}<\varepsilon\right)=0$$

To prove the lower bound, we use the following general lower bound on supremum of, Gaussian processes under entropy conditions. It was established by Talagrand[18] and this formulation is given by Ledoux,[10] [p. 257].

Lemma 1. Let $(X_t)_{t\in T}$ be a centered Gaussian process. For every $\varepsilon>0$, let $N(T,d;\varepsilon)$ denote the minimal number of balls of radius ε, under the metric $d(s,t)=(E|X_s-X_t|^2)^{1/2}$, that are necessary to cover T. Assume that there is a nonnegative function ψ on R_+ such that

$$N(T,d;\varepsilon)\leqslant\psi(\varepsilon),\qquad \varepsilon>0$$

and such that for some constants $1 < c_1 \leqslant c_2 < \infty$ and all $\varepsilon > 0$

$$c_1 \psi(\varepsilon) \leqslant \psi(\varepsilon/2) \leqslant c_2 \psi(\varepsilon)$$

Then, for some constant $K > 0$ and every $\varepsilon > 0$,

$$P(\sup_{s,\, t \in T} |X_s - X_t| \leqslant \varepsilon) \geqslant \exp(-K\psi(\varepsilon))$$

Now back to the proof of the lower bound and note that for any s, $t \in T = (0, \delta)$, $s \leqslant t$,

$$d_f^2(s, t) = E\left(\frac{W(s)}{f(s)} - \frac{W(t)}{f(t)}\right)^2 = \frac{s}{f^2(s)} + \frac{t}{f^2(t)} - \frac{2s}{f(s)\, f(t)}$$

Thus using the fact that $f(x)$ and $x/f^2(x)$ are both nondecreasing, we have for $0 < s < t < \delta$

$$d_f^2(s, t) \leqslant \frac{2t}{f^2(t)} - \frac{2s}{f(s)\, f(t)} \leqslant \frac{2(t-s)}{f^2(t)}$$

In particular, we have $d_f(t_i, t_{i-1}) \leqslant \sqrt{2}\, \varepsilon$ for $1 \leqslant i \leqslant N(\varepsilon)$ and $d_f(0, t_N) \leqslant \varepsilon$ by using the construction (1.7) and (1.8). Hence using t_i, $1 \leqslant i \leqslant N(\varepsilon)$, as centers, we get

$$N(T, d_f; \sqrt{2}\, \varepsilon) \leqslant N(\varepsilon), \qquad \varepsilon > 0$$

Now using the condition (1.9) in Theorem 2, it follows that

$$N(T, d_f; \varepsilon) \leqslant N(\varepsilon/\sqrt{2}) \leqslant c_2 N(\varepsilon), \qquad \varepsilon > 0$$

which implies the lower bound by the lemma. Thus we finish the proof by noting that with $X_0 = 0$, $P(\sup_{t \in T} |X_t| \leqslant \varepsilon) \geqslant P(\sup_{s,\, t \in T} |X_s - X_t| \leqslant \varepsilon)$.

Next let us compute some examples and keep in mind that a power type function satisfies the condition (1.9).

Example 4. Let $f(t) = t^\alpha$ on $(0, 1)$ with $\alpha < 1/2$. Then it is not hard to see that as $\varepsilon \to 0$, $N(\varepsilon) \approx 1/\varepsilon^2$. In fact, we have

$$\int_0^\delta \frac{1}{f^2(t)}\, dt \geqslant \sum_{i=1}^{N(\varepsilon)} \int_{t_i}^{t_{i-1}} \frac{1}{f^2(t)}\, dt \geqslant \sum_{i=1}^{N(\varepsilon)} \frac{t_{i-1} - t_1}{f^2(t_{i-1})} = \varepsilon^2 N(\varepsilon)$$

and hence if $\int_0^\delta 1/f^2(t)\, dt < \infty$, then $N(\varepsilon)$ is of order no more than $1/\varepsilon^2$. In the opposite direction, we have $t_{i-1} - t_i = \varepsilon^2 f^2(t_{i-1}) \leqslant \varepsilon^2 f^2(t_0)$ for $1 \leqslant i \leqslant N$ which implies that $N(\varepsilon)$ is of order at least $1/\varepsilon^2$. On the other hand, if $N(\varepsilon)$

is of order ε^{-2}, then by Theorem 2, $\log P(\sup_{0 \leqslant t \leqslant \delta} |W(t)|/f(t) < \varepsilon)$ is of order ε^{-2} as well. Thus according to Theorem A, $\int_0^1 dt/f^2(t)$ must be finite.

Example 5. Let $f(t) = \sqrt{t \log 1/t}$ on $(0, e^{-1})$. Then we have as $\varepsilon \to 0$

$$N(\varepsilon) \approx \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}$$

To see the upper bound, note that $\log(1-x) \leqslant -x$ for $x < 1$ and thus for $1 \leqslant i \leqslant N$ and $\varepsilon > 0$ small,

$$\log \frac{1}{t_i} = \log \frac{1}{t_{i-1}} - \log \left(1 - \varepsilon^2 \log \frac{1}{t_{i-1}}\right) > (1+\varepsilon^2) \log \frac{1}{t_{i-1}}$$

Iterating this inequality, we have

$$\log \frac{1}{t_{N-1}} > (1+\varepsilon^2)^{N-1} \tag{2.5}$$

On the other hand, by the definition of N, we have

$$\log \frac{1}{t_{N-1}} < \frac{1}{\varepsilon^2} \tag{2.6}$$

Combine (2.5) and (2.6) together, we have

$$\frac{1}{\varepsilon^2} > (1+\varepsilon^2)^{N-1}$$

which provides the upper bound for $\varepsilon > 0$ small.

To obtain the lower bound, let

$$m = m(\varepsilon) = \min \left\{ i : \log \frac{1}{t_m} > \frac{1}{2\varepsilon^2} \right\}$$

Then $\log(1/t_{m-1}) \leqslant 1/(2\varepsilon^2)$ and thus for $\varepsilon > 0$ small,

$$\begin{aligned} \log \frac{1}{t_m} &= \log \frac{1}{t_{m-1}} - \log \left(1 - \varepsilon^2 \log \frac{1}{t_{m-1}}\right) \\ &\leqslant \frac{1}{2\varepsilon^2} - \log \left(1 - \varepsilon^2 \cdot \frac{1}{2\varepsilon^2}\right) \\ &< \frac{1}{2\varepsilon^2} + 1 < \frac{1}{\varepsilon^2} \end{aligned}$$

which implies that $m \leqslant N(\varepsilon)$. Now using the fact that $\log(1-x) \geqslant -2x$ for $0 \leqslant x \leqslant 1/2$, we have for $1 \leqslant i \leqslant m$

$$\log \frac{1}{t_i} = \log \frac{1}{t_{i-1}} - \log \left(1 - \varepsilon^2 \log \frac{1}{t_{i-1}} \right) < (1 + 2\varepsilon^2) \log \frac{1}{t_{i-1}}$$

Iterating this inequality, we have

$$\frac{1}{2\varepsilon^2} < \log \frac{1}{t_m} < (1 + 2\varepsilon^2)^m$$

which provides the lower bound for $\varepsilon > 0$ small.

Example 6. Let $f(t) = \sqrt{t \log\log 1/t}$. Then we have as $\varepsilon \to 0$

$$\log N(\varepsilon) \approx \frac{1}{\varepsilon^2}$$

We omit the proofs since they are similar to the proof of Example 6.

Example 7. Let $f(t) = \sqrt{t}$ on (0, 1). Then we have for any $0 < \varepsilon < 1$, $t_i = (1 - \varepsilon^2)\, t_{i-1}$. Thus we have $N(\varepsilon) = \infty$ and

$$P\left(\sup_{0 < t \leqslant 1} \frac{|W(t)|}{\sqrt{t}} < \varepsilon \right) = 0$$

which can also be seen from Kolmogorov's integral test.

3. SOME REMARKS

First note that for the upper bound of the Theorem to hold, we do not need the regularity condition (1.9) on the number $N(\varepsilon)$. In fact, the regularity condition (1.9) appears as an assumption in the lemma, and even there it is not always needed. A slightly different lower bound can be obtained based on the upper bound of the entropy number $N(T, d; \varepsilon)$ when the bound is of exponential type. This can be seen from the proof given by Talagrand,[18] which is based on Sidak's lemma and a chaining argument.

For the so called critical case covered by our Theorem 2, the following upper bound is given by Berthet and Shi[1]

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon^2}{F(\varepsilon)} \log P\left(\sup_{0 \leqslant t \leqslant \delta} \frac{|W(t)|}{f(t)} < \varepsilon \right) \leqslant -\frac{\pi^2}{8}$$

where

$$F(\varepsilon) = \int_{\varepsilon}^{\delta} dt/f^2(t)$$

But this is not sharp in the light of (1.11) for the function $f(t) = \sqrt{t \log(1/t)}$ on $[0, 1/e]$. On the other hand, for the particular example $f(t) = \sqrt{t \log(1/t)}$, there is a way to get the correct lower bound as outlined by Berthet and Shi,[1] based on scaling and Kolmogorov's integral test given in (1.10). The trouble is that we do not know how it could be done in general. But it seems that the upper bound estimate and the structure of the partition point have something to do with Kolmogorov's integral test, as can also be seen from Example 7.

Finally, as given by Berthet and Shi,[1] the applications of Theorem A to Chung's functional iterated logarithm law, [cf. Kuelbs *et al.*[9]], and to the study of the local increments of W can all be extended to the critical case. It is also seems possible to provide some interesting critical weights for the empirical processes similar to the work of Csáki.[3]

Now for the remaining of this paper, we give some estimates along the line of Theorem 2 for the α-fractional, Brownian motion $Y(t)$, $t \geqslant 0$, with $Y(0) = 0$ and $0 < \alpha < 1$. Note that $\{Y(t) : t \geqslant 0\}$ has covariance function

$$E(Y(s)\; Y(t)) = \tfrac{1}{2}(s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha})$$

For $s, t \geqslant 0$, In particular, $\{Y(t) : t \geqslant 0\}$ has stationary increments with

$$EY^2(t) = \sigma^2(t) = t^{2\alpha}, \qquad t \geqslant 0$$

and is a standard Brownian when $\alpha = 1/2$.

We first define a sequence analogous to the Brownian motion case. For any positive nondecreasing function f on $(0, \delta)$ with $\delta > 0$ fixed, we can define

$$N(\varepsilon) = \max\{n : t_n > 0\} \tag{3.1}$$

with $t_0 = \delta$ and

$$t_i = t_{i-1} - (\varepsilon f(t_{i-1}))^{1/\alpha} \qquad \text{for} \quad i \geqslant 1 \tag{3.2}$$

Note that the sequence t_i is decreasing and $f(t_i) \leqslant f(t_{i-1})$ for $1 \leqslant i \leqslant N(\varepsilon)$.

Proposition 1. Assume that $f(x)$ is a positive nondecreasing function on $(0, \delta)$ with $\delta > 0$ fixed. Let $N(\varepsilon) < \infty$ be defined as in (3.1) with $0 < \alpha \leqslant 1/2$. Then, for some constants $0 < C_1 < \infty$ and $\varepsilon > 0$ small,

$$\log P\left(\sup_{0 < t \leqslant \delta} \frac{|Y(t)|}{f(t)} < \varepsilon\right) \leqslant -C_1 N(\varepsilon)$$

Proof. The proof depends on Slepian's lemma which can be found, for example, see Ledoux and Talagrand.[11] It is easy to see that

$$\begin{aligned} P\left(\sup_{0 < t \leqslant \delta} \frac{|Y(t)|}{f(t)} < \varepsilon\right) &\leqslant P(|Y(t_i)| < \varepsilon f(t_i),\ 0 \leqslant i \leqslant N(\varepsilon)) \\ &\leqslant P(|Y(t_{i-1}) - Y(t_i)| < 2\varepsilon f(t_{i-1}),\ 1 \leqslant i \leqslant N(\varepsilon)) \\ &\leqslant P(Y(t_{i-1}) - Y(t_i) < 2\varepsilon f(t_{i-1}),\ 1 \leqslant i \leqslant N(\varepsilon)) \end{aligned}$$

Now let

$$\xi_i = Y(t_{i-1}) - Y(t_i), \qquad 1 \leqslant i \leqslant N(\varepsilon)$$

Then

$$E(\xi_i^2) = (t_{i-1} - t_i)^{2\alpha}, \qquad 1 \leqslant i \leqslant N(\varepsilon)$$

and since the function $y = x^{2\alpha}$, $x \geqslant 0$ is concave for $0 \leqslant \alpha \leqslant 1/2$ it follows fairly easily that $E(\xi_i \xi_j) \leqslant 0$ for $1 \leqslant i \leqslant N(\varepsilon)$. Therefore, by Slepian's lemma, with ξ denoting the standard normal random variable,

$$\begin{aligned} P(\xi_i < 2\varepsilon f(t_{i-1}),\ 1 \leqslant i \leqslant N(\varepsilon)) &\leqslant \prod_{i=1}^{N(\varepsilon)} P(\xi_i < 2\varepsilon f(t_{i-1})) \\ &= \prod_{i=1}^{N(\varepsilon)} P(\xi < 2\varepsilon f(t_{i-1})/(t_{i-1} - t_i)^{\alpha}) \\ &= P(\xi < 2)^{N(\varepsilon)} \end{aligned}$$

and thus the upper bound follows. Note that if $N(\varepsilon) = \infty$ for some $\varepsilon > 0$, then these arguments show that for $0 \leqslant \alpha \leqslant 1/2$

$$P\left(\sup_{0 < t \leqslant \delta} \frac{|Y(t)|}{f(t)} < \varepsilon\right) = 0$$

To obtain a lower bound, we again try to use the general lower bound on supremum of Gaussian processes under entropy conditions. Note that in this case

$$d_f^2(s, t) = E\left(\frac{Y(s)}{f(s)} - \frac{Y(t)}{f(t)}\right)^2 = \frac{s^{2\alpha}}{f^2(s)} + \frac{t^{2\alpha}}{f^2(t)} - \frac{s^{2\alpha} + t^{2\alpha} - (t-s)^{2\alpha}}{f(s)\, f(t)}$$

for $0 < s < t < \delta$.

Proposition 2. Under assumptions similar to those in the Brownian motion case for the lower bound and the additional assumption that t_{i-1}/t_i is uniformly bounded for $i \leqslant N(\varepsilon) - 1$, we have

$$\log P\left(\sup_{0 < t \leqslant \delta} \frac{|Y(t)|}{f(t)} < \varepsilon\right) \geqslant -C_2 N(\varepsilon)$$

Note that the condition in Proposition 2 is easy to check for $f(t) = t^\beta$ with $\beta < \alpha$. This case is known and is mentioned by Kuelbs *et al.*[8] in connection with Hölder norms. However, the exact formula is not given there, and the argument is somewhat different.

Proposition 3. Define a new sequence s_i such that, $s_0 = \delta$

$$s_i + (\varepsilon f(s_i))^{1/\alpha} = s_{i-1} \qquad \text{for} \quad i \geqslant 1$$

Let

$$M(\varepsilon) = \max\{n : s_n > 0\}$$

Then under similar assumptions to the Brownian motion case (on $M(\varepsilon)$), we have

$$\log P\left(\sup_{0 < t \leqslant \delta} \frac{|Y(t)|}{f(t)} < \varepsilon\right) \geqslant -C_2 M(\varepsilon)$$

Since the proofs of Propositions 2 and 3 are similar to that of Theorem 2, we omit the details here.

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REFERENCES

1. Berthet, P., and Shi, Z. (1997). Small ball estimates for Brownian motion under a weighted sup-norm. Preprint.
2. Borisov, I. S. (1982). On a criterion for Gaussian random processes to be Markovian. *Theor. Prob. Appl.* **27**, 863–865.
3. Csáki, E. (1994). Some limit theorems for empirical processes. In Vilaplana, J. P., and Puri, M. L. (eds.), *Recent Advances in Statistics and Probability*, Proc. 4th IMSIBAC, VSP, Utrecht, pp. 247–254.
4. Dunker, T., Linde, W., Kühn, T., and Lifshits, M. A. (1998). Metric, entropy of integration operator and small ball probabilities for the Brownian sheet. *C. R. Acad. Sci. Paris* **326**, 347–352.
5. Feller, W. (1967). *Introduction to Probability Theory and Its Application*, Vol. II, Wiley, New York.
6. Itô, K., and McKean, H. P. (1965). *Diffusion Processes and their Sample Paths*, Springer, Berlin.
7. Kuelbs, J., and Li, W. V. (1993). Metric entropy and the small ball problem for Gaussian measures. *J. Funct. Anal.* **116**, 133–157.
8. Kuelbs, J., Li, W. V., and Shao, Q.-M. (1995). Small ball probabilities for Gaussian processes with stationary increments under Hölder norms. *J. Theor. Prob.* **8**, 361–386.
9. Kuelbs, J., Li, W. V., and Talagrand, M. (1994). Lim inf results for Gaussian samples and Chung's functional LIL. *Ann. Probab.* **22**, 1879–1903.
10. Ledoux, M. (1996). Isoperimetry and Gaussian analysis. Lectures on Probability Theory and Statistics, Springer-Verlag, *Lecture Notes in Math.* **1648**, 165–294.
11. Ledoux, M., and Talagrand, M. (1991). *Probability on Banach Spaces*. Springer, Berlin.
12. Lévy, P. (1956). A special problem of Brownian motion, and a general theory of Gaussian random functions, University of California Press, Berkeley, *Proc. 3rd Berkeley Symp. Math. Statist. Probab. Th.* **2** 133–175.
13. Li, W. V. (1998). A lim inf result for the Brownian motion. *Asymptotic Methods in Probability and Statistics*, pp. 281–292, North-Holland.
14. Li, W. V., and Linde, W. (1998). A note on approximation, metric entropy and small ball estimates for Gaussian measures. To appear in *Ann. Probab.*
15. Li, W. V., and Linde, W. (1998b). Existence of small ball constants for fractional Brownian motions. *C. R. Acad. Sci. Paris* **326**, 1329–1334.
16. Li, W. V., and Shao, Q.-M. (1998). Small ball estimates for Gaussian processes under the Sobolev norm. *J. Th. Prob.* **12**, 699–720.
17. Mogulskii, A. A. (1974). Small deviations in space of trajectories. *Th. Prob. Appl.* **19**, 726–736.
18. Talagrand, M. (1993). New Gaussian estimates for enlarged balls. *Geometric Funct. Anal.* **3**, 502–526.