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A lim inf result for the Brownian motion

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Dedicated to Miklós Csörgő on the occasion of his 65th birthday

Abstract: Let $B(t)$ be a standard Brownian motion in R^1. We prove that

$$\varliminf_{T\to\infty}\left(\frac{\log(T/a(T))+2\log\log T}{a(T)}\right)^{1/2}\sup_{T-a(T)\le t\le T}|B(t)|=\frac{\pi}{2}\quad a.s.$$

under suitable conditions on $a(T)$.

1. Introduction

Let $\{B(t), t\geq 0\}$ be a standard Brownian motion in R^1. There are various types of laws of iterated logarithm that are known for $B(t)$ for different types of norms. For sup-norms, we know (see Chung [2])

$$\varlimsup_{T\to\infty}\frac{1}{(T\log\log T)^{1/2}}\sup_{0\le t\le T}|B(t)|=\sqrt{2}\quad a.s. \tag{1.1}$$

and (see Chung [1], Jain and Pruitt [7])

$$\varliminf_{T\to\infty}\left(\frac{\log\log T}{T}\right)^{1/2}\sup_{0\le t\le T}|B(t)|=\frac{\pi}{\sqrt{8}}\quad a.s. \tag{1.2}$$

For L_2-norms, we know (see Strassen [10])

$$\varlimsup_{T\to\infty}\frac{1}{T^2\log\log T}\int_0^T B^2(t)dt=\frac{8}{\pi^2}\quad a.s. \tag{1.3}$$

and (see Donsker and Varadhan [5])

$$\varliminf_{T\to\infty}\frac{\log\log T}{T^2}\int_0^T B^2(t)dt=\frac{1}{8}\quad a.s. \tag{1.4}$$

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They have the same nature that $B(t)$ is being considered only on $[0,T]$. What happens if $B(t)$ is on $[T-a(T),T]$ for $a(T)\geq 0$? The case regarding (1.1) on $[T-a(T),T]$ can be obtained easily by using (1.1) and the Levy's laws of iterated logarithm

$$\overline{\lim_{T\to\infty}} \frac{1}{(T\log\log T)^{1/2}} |B(T)| = \sqrt{2} \quad a.s. \tag{1.5}$$

The case regarding (1.3) and (1.4) on $[T-a(T),T]$ are investigated in Li [8] and [9]. In this paper we investigate the case regarding (1.2) on $[T-a(T),T]$. We have the following results.

Theorem. *Let $a(T)$ satisfy the conditions*

(*i*) $0 < a(T) \leq T$, *$a(T)$ is a non-decreasing function of T, for $0 < T < \infty$;*

(*ii*) *$a(T)/T$ is non-increasing as $T\to\infty$; or*

(*ii*)$'$ $\lim_{T\to\infty} a(T)/T = \rho$, $0<\rho\leq 1$.

If $\lim_{T\to\infty}\log(T/a(T))\cdot(\log\log T)^{-1}=\infty$, *or if* $\lim_{T\to\infty}\log(T/a(T))\cdot(\log\log T)^{-1}<\infty$ *and* $\overline{\lim}_{T\to\infty}a(\gamma T)/a(T)<\infty$ *for some* $\gamma>1$, *then*

$$\underline{\lim}_{T\to\infty} \phi(T) \sup_{T-a(T)\leq t\leq T} |B(t)| = \frac{\pi}{2} \quad a.s. \tag{1.6}$$

where

$$\phi(T) = \left(\frac{\log\,(T/a(T)) + 2\log\log T}{a(T)}\right)^{1/2}.$$

After circulating the preprint of this paper, Prof. A. Főldes kindly pointed out a close related result in [3]. Namely

$$\underline{\lim}_{T\to\infty} \phi(T) \inf_{0<t\leq T-a(T)} \sup_{0\leq s\leq a(T)} |B(t+s)| = \frac{\pi}{2} \quad a.s. \tag{1.7}$$

The difference is that we only look at the last interval of length $a(T)$. Thus (1.7) provides a lower bound for our result. Since the lower bound is the relative easy part to prove, we also include a proof for completeness.

Next we give here the following examples to illustrate what our theorem tells us.

Example 1. For $x\geq 0$, let $a(T)=(1+x)^{-1}T$, then (1.6) tells us by the change of variable that

$$\underline{\lim}_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{xT\leq t\leq(x+1)T} |B(t)| = \frac{\pi}{\sqrt{8}} \quad a.s. \tag{1.8}$$

If $x=0$, (1.8) becomes (1.2). It is somewhat strange that (1.8) is true no matter what $x\geq 0$ is. One might expect (1.8) has something to do with the zeros of $B(t)$. In fact, for almost all $\omega\in\Omega$, there exist $T_k(\omega)$ such that

$$B(xT_k(\omega))=0 \quad k=1,2,\ldots \quad , \quad \lim_{k\to\infty} T_k(\omega)=\infty.$$

Hence we can see in a very rough sense (we use $\simeq$), for $\psi(T) = (T^{-1}\log\log T)^{1/2}$,

$$\begin{aligned}\varliminf_{T\to\infty} \psi(T) \sup_{xT\le t\le (x+1)T} |B(t)| &\simeq \varliminf_{k\to\infty} \psi(T_k(\omega)) \sup_{xT_k(\omega)\le t\le (x+1)T_k(\omega)} |B(t)| \\ &\simeq \varliminf_{k\to\infty} \psi(T_k(\omega)) \sup_{0\le t\le T_k(\omega)} |B(t)| \simeq \frac{\pi}{\sqrt{8}}.\end{aligned}$$

The problem, however, is to make this precise.

Example 2. Let $a(T) = T - T^\alpha$ where $0 < \alpha < 1$. Then (1.6) says that

$$\varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{T^\alpha\le t\le T} |B(t)| = \frac{\pi}{\sqrt{8}} \quad a.s. \tag{1.9}$$

Note that we can give a trivial proof of (1.9) by (1.1) and (1.2) as below

$$\begin{aligned}&\varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{T^\alpha\le t\le T} |B(t)| \\ \le\ &\varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{0\le t\le T} |B(t)| = \frac{\pi}{\sqrt{8}} \quad a.s. \\ \le\ &\varlimsup_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{0\le t\le T^\alpha} |B(t)| + \varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{T^\alpha\le t\le T} |B(t)|\end{aligned}$$

and

$$\varlimsup_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{0\le t\le T^\alpha} |B(t)| = 0 \quad a.s.$$

But our next example shows that the above proof is not always going to work when the interval $[T - a(T), T]$ becomes shorter.

Example 3. Let $a(T) = T - T(\log\log\log T)^{-1}$ and $a(T) = T/(\log T)^\alpha$ separately where $\alpha > 0$. Then (1.6) gives that

$$\varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{T(\log\log\log T)^{-1}\le t\le T} |B(t)| = \frac{\pi}{\sqrt{8}} \quad a.s.$$

and

$$\varliminf_{T\to\infty} \left(\frac{(\log T)^\alpha\cdot\log\log T}{T}\right)^{1/2} \sup_{T-T(\log T)^{-\alpha}\le t\le T} |B(t)| = \frac{\pi}{2\sqrt{2+\alpha}} \quad a.s.$$

Example 4. Let $a_1(T) = c$, $a_2(T) = c\log T$, $a_3(T) = cT^\alpha$ where $0 < \alpha < 1$ and $c > 0$ is a constant. Then (1.6) says that

$$\varliminf_{T\to\infty} (\log T)^{1/2} \sup_{T-c\le t\le T} |B(t)| = \frac{\pi}{2}\cdot\sqrt{c} \quad a.s. \quad ;$$

$$\varliminf_{T\to\infty} \sup_{T-c\log T\le t\le T} |B(t)| = \frac{\pi}{2}\cdot\sqrt{c} \quad a.s. \quad ;$$

$$\varlimsup_{T\to\infty}\left(\frac{\log T}{T^\alpha}\right)^{1/2}\sup_{T-cT^\alpha\le t\le T}|B(t)|=\frac{\pi}{2}\cdot\sqrt{\frac{c}{1-\alpha}}\quad a.s.$$

Hence we see from (1.6) that $a(T) = c\log T$ is the critical function, i.e. under our conditions (i) and (ii),

$$\varlimsup_{T\to\infty}\sup_{T-a(T)\le t\le T}|B(t)|=\begin{cases}0 & a.s. \text{ if } \lim_{T\to\infty} a(T)/\log T=0\\ \pi\sqrt{c}/2 & a.s. \text{ if } \lim_{T\to\infty} a(T)/\log T=c\\ \infty & a.s. \text{ if } \lim_{T\to\infty} a(T)/\log T=\infty\end{cases}.$$

We give the proof of our theorem and some lemmas in next section. Our Lemma 1 and Lemma 2 are the useful probability estimates for Brownian motion, which have independent interest. Now we need some notation for next section. Let ε stand for a small positive number given arbitrarily, and C denote various positive constants independent of k and n , whose values might change from line to line. $f(\varepsilon)\sim g(\varepsilon)$ as $\varepsilon\to 0$ means $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon)=1$.

2. Proof of Theorem

The key estimates for the proof of our theorem is the following lemma of small ball type.

Lemma 1. *Let $a=a_\varepsilon$, $b=b_\varepsilon$ and $b>a>0$. Then as $(b-a)\varepsilon^{-2}\to\infty$, $a\varepsilon^{-2}\to\infty$ and $\varepsilon\to 0$,*

$$P\left(\sup_{a\le t\le b}|B(t)|\le\varepsilon\right)\sim K\cdot\frac{\varepsilon}{\sqrt{a}}\cdot\exp\left(-\frac{\pi^2}{8}\cdot\frac{b-a}{\varepsilon^2}\right)$$

where K is a positive constant independent of ε.

Proof. Define $\tau=\tau(x)=\inf\{t\ge 0: |B_x(t)|=\varepsilon\}$ where $B_x(t)$ is standard Brownian motion starting from x, $|x|\le b$. Since $\exp\{\lambda B_x(t)-\lambda^2 t/2\}$ is a martingale, we have by optional stopping theorem, $E\exp\{\lambda B_x(\tau)-\lambda^2\tau/2\}=\exp\{-\lambda x\}$. Replace λ by $-\lambda$ and solve the two equations, we have

$$E\exp\{-\lambda^2\tau/2\}=(e^{\lambda x}+e^{-\lambda x})/(e^{\lambda\varepsilon}+e^{-\lambda\varepsilon}).$$

Thus we have by making a partial fraction expansion and inversion of this Laplace transform, the density function of the random variable τ,

$$f_\tau(t)=\frac{\pi}{2\varepsilon^2}\sum_{k=0}^{\infty}(-1)^k(2k+1)\cos\left((2k+1)\frac{\pi x}{2\varepsilon}\right)\exp\{-(2k+1)^2\cdot\frac{\pi^2 t}{8\varepsilon^2}\}.$$

By integration on t we obtain

$$\begin{aligned}P\left(\sup_{0\le t\le c}|B_x(t)|\le\varepsilon\right) &= P(\tau(x)>c)\\ &= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left((2k+1)\frac{\pi x}{2\varepsilon}\right)\exp\{-(2k+1)^2\cdot\frac{\pi^2 c}{8\varepsilon^2}\}.\end{aligned}$$

In particular, for ε small, the dominating term is, at $k = 0$,

$$\frac{4}{\pi} \cos\left(\frac{\pi x}{2\varepsilon}\right) \exp\{-\frac{\pi^2 c}{8\varepsilon^2}\}.$$

Now by conditioning,

$$\begin{aligned} P\left(\sup_{a\le t\le b} |B(t)| \le \varepsilon\right) &= \int_{-\varepsilon}^{\varepsilon} P\left(\sup_{a\le t\le b} |B(t)| \le \varepsilon\,, B(a) = x\right) dP(B(a) < x) \\ &= \int_{-\varepsilon}^{\varepsilon} P\left(\sup_{0\le t\le b-a} |B_x(t)| \le \varepsilon\right) \cdot \frac{1}{\sqrt{2\pi a}} e^{-x^2/2a} dx. \end{aligned}$$

Hence as $(b-a)\varepsilon^{-2} \to \infty$, $a\varepsilon^{-2} \to \infty$ and $\varepsilon \to 0$,

$$\begin{aligned} & P\left(\sup_{a\le t\le b} |B(t)| \le \varepsilon\right) \\ \sim\ & \frac{4}{\pi} \exp\{-\frac{\pi^2(b-a)}{8\varepsilon^2}\} \int_{-\varepsilon}^{\varepsilon} \cos\left(\frac{\pi x}{2\varepsilon}\right) \cdot \frac{1}{\sqrt{2\pi a}} e^{-x^2/2a} dx \\ =\ & \frac{4}{\pi} \exp\{-\frac{\pi^2(b-a)}{8\varepsilon^2}\} \int_{-1}^{1} \cos(\pi y/2) \cdot \frac{\varepsilon}{\sqrt{2\pi a}} e^{-\varepsilon^2 y^2/2a} dy \\ \sim\ & K \cdot \frac{\varepsilon}{\sqrt{a}} \cdot \exp\left(-\frac{\pi^2}{8} \cdot \frac{b-a}{\varepsilon^2}\right) \end{aligned}$$

where $K = 16/\pi^2\sqrt{2\pi}$. This finishes the proof.

Our next lemma provides estimates for the correlations.

Lemma 2 For any $b' > a' \ge b > a \ge 0$ and $s' > 0$, $s > 0$, we have

$$\begin{aligned} & P\left(\sup_{a\le t\le b} |B(t)| \le s\ ,\ \sup_{a'\le t\le b'} |B(t)| \le s'\right) \\ \le\ & P\left(\sup_{a\le t\le b} |B(t)| \le s\right) \cdot P\left(\sup_{a'-b\le t\le b'-b} |B(t)| \le s'\right) \\ \le\ & \left(\frac{a'}{a'-b}\right)^{1/2} \cdot P\left(\sup_{a\le t\le b} |B(t)| \le s\right) \cdot P\left(\sup_{a'\le t\le b'} |B(t)| \le s'\right). \end{aligned}$$

Proof. First, let us prove the first inequality. By the fact that for Brownian motion the past and the future are conditionally independent given the present (see Theorem 9.2.4 in Chung [2]), we have

$$\begin{aligned} & P\left(\sup_{a\le t\le b} |B(t)| \le s\ ,\ \sup_{a'\le t\le b'} |B(t)| \le s'\right) \\ =\ & \int_{-s}^{s} P\left(\sup_{a\le t\le b} |B(t)| \le s\ ,\ \sup_{a'\le t\le b'} |B(t)| \le s' \mid B(b) = x\right) dP(B(b) < x) \\ =\ & \int_{-s}^{s} P\left(\sup_{a\le t\le b} |B(t)| \le s \mid B(b) = x\right) \cdot P\left(\sup_{a'\le t\le b'} |B(t)| \le s' \mid B(b) = x\right) dP(B(b) < x). \end{aligned}$$

Now note the following inequality for any $b > a \geq 0$, $\varepsilon > 0$ and $x \in R$,

$$P\left(\sup_{a\leq t\leq b}|B(t)+x|\leq\varepsilon\right)\leq P\left(\sup_{a\leq t\leq b}|B(t)|\leq\varepsilon\right).$$

It is a particular case of Theorem 2.1 of Hoffmann-Jørgensen, Shepp and Dudley [6] which is a well known fact about the measure of the translated ball for the centered Gaussian measures. Hence by using the fact that the Brownian motion has independent and stationary increments, we have

$$\begin{aligned}
& P\left(\sup_{a'\leq t\leq b'}|B(t)|\leq s' \mid B(b)=x\right) \\
= \; & P\left(\sup_{a'\leq t\leq b'}|B(t)-B(b)+x|\leq s' \mid B(b)=x\right) \\
= \; & P\left(\sup_{a'\leq t\leq b'}|B(t)-B(b)+x|\leq s'\right) \\
= \; & P\left(\sup_{a'-b\leq t\leq b'-b}|B(t)+x|\leq s'\right) \\
\leq \; & P\left(\sup_{a'-b\leq t\leq b'-b}|B(t)|\leq s'\right).
\end{aligned}$$

We thus obtain

$$\begin{aligned}
& P\left(\sup_{a\leq t\leq b}|B(t)|\leq s\,,\;\sup_{a'\leq t\leq b'}|B(t)|\leq s'\right) \\
\leq \; & \int_{-s}^{s} P\left(\sup_{a\leq t\leq b}|B(t)|\leq s \mid B(b)=x\right)\cdot P\left(\sup_{a'-b\leq t\leq b'-b}|B(t)|\leq s'\right)dP(B(b)<x) \\
= \; & P\left(\sup_{a\leq t\leq b}|B(t)|\leq s\right)\cdot P\left(\sup_{a'-b\leq t\leq b'-b}|B(t)|\leq s'\right).
\end{aligned}$$

To prove the second inequality, we have by the basic properties of the Brownian motion

$$\begin{aligned}
& P\left(\sup_{a'-b\leq t\leq b'-b}|B(t)|\leq s'\right) \\
= \; & \int_{-\infty}^{\infty} P\left(\sup_{a'-b\leq t\leq b'-b}|B(t)|\leq s' \mid B(a'-b)=x\right)dP(B(a'-b)<x) \\
= \; & \int_{-\infty}^{\infty} P\left(\sup_{a'-b\leq t\leq b'-b}|B(t)-B(a'-b)+x|\leq s' \mid B(a'-b)=x\right)dP(B(a'-b)<x) \\
= \; & \int_{-\infty}^{\infty} P\left(\sup_{a'-b\leq t\leq b'-b}|B(t)-B(a'-b)+x|\leq s'\right)dP(B(a'-b)<x) \\
= \; & \int_{-\infty}^{\infty} P\left(\sup_{0\leq t\leq b'-a'}|B(t)+x|\leq s'\right)dP(B(a'-b)<x) \\
= \; & \int_{-\infty}^{\infty} P\left(\sup_{0\leq t\leq b'-a'}|B(t)+x|\leq s'\right)\cdot\frac{1}{\sqrt{2\pi(a'-b)}}\exp(-\frac{1}{2}\frac{x^2}{a'-b})dx
\end{aligned}$$

$$\leq \left(\frac{a'}{a'-b}\right)^{1/2} \int_{-\infty}^{\infty} P\left(\sup_{0\leq t\leq b'-a'} |B(t)+x| \leq s'\right) \cdot \frac{1}{\sqrt{2\pi a'}} \exp(-\frac{1}{2}\frac{x^2}{a'})dx$$
$$= \left(\frac{a'}{a'-b}\right)^{1/2} \int_{-\infty}^{\infty} P\left(\sup_{0\leq t\leq b'-a'} |B(t)+x| \leq s'\right) dP(B(a') < x)$$
$$= \left(\frac{a'}{a'-b}\right)^{1/2} P\left(\sup_{a'\leq t\leq b'} |B(t)| \leq s'\right)$$

where the last equality follows from the first part of this proof backward. This finishes the proof.

The following is a well known version of the Borel-Cantelli lemma.

Lemma 3 If A_k are events such that $\sum_{k\geq 1} P(A_k) = \infty$ and

$$\varliminf_{n\to\infty} \frac{\sum_{k=1}^n \sum_{l=1}^n P(A_k A_l)}{\sum_{k=1}^n \sum_{l=1}^n P(A_k)P(A_l)} \leq 1,$$

then $P(A_k \ i.o.) = 1$.

Now we turn to the proofs of our theorem. They are similar to that in Li [9] for L_2-norm given Lemma 1 and Lemma 2. So we only provide main steps of the proof and crucial differences for completeness.

Let us note that under our conditions (i) and $(ii)'$, our theorem becomes

$$\varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{T-a(T)\leq t\leq T} |B(t)| = \frac{\pi}{\sqrt{8}}\rho \quad a.s. \quad , \quad 0 < \rho \leq 1.$$

This can be easily derived as follows if our theorem holds under our conditions (i) and (ii) . For $0 < \rho < 1$ and $\varepsilon > 0$ small, we have $0 < T-(\rho+\varepsilon)T \leq T-a(T) \leq T-(\rho-\varepsilon)T < T$ if T is large and thus

$$\varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{T-(\rho+\varepsilon)T\leq t\leq T} |B(t)| = \frac{\pi}{\sqrt{8}}(\rho+\varepsilon) \quad a.s.$$
$$\geq \varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{T-a(T)\leq t\leq T} |B(t)|$$
$$\geq \varliminf_{T\to\infty} \left(\frac{\log\log T}{T}\right)^{1/2} \sup_{T-(\rho-\varepsilon)T\leq t\leq T} |B(t)| = \frac{\pi}{\sqrt{8}}(\rho-\varepsilon) \quad a.s.$$

For $\rho = 1$, the above argument also works by using (1.2) as the upper bound. Hence, for the rest of this section, we assume conditions (i) and (ii) hold and $\lim_{T\to\infty} a(T)/T = \rho < 1$. Now we formulate the following three statements which together imply our theorem.

(I) $\varliminf_{T\to\infty} \phi(T) \sup_{T-a(T)\leq t\leq T} |B(t)| \geq \frac{\pi}{2} \quad a.s.$

(II) If $\lim_{T\to\infty} a(T)/T = 0$, then

$$\varliminf_{T\to\infty} \left(\frac{\log(T/a(T))}{a(T)}\right)^{1/2} \sup_{T-a(T)\leq t\leq T} |B(t)| \leq \frac{\pi}{2} \quad a.s.$$

(III) If $\lim_{T\to\infty} \log(T/a(T)) \cdot (\log\log T)^{-1} < \infty$ and $\overline{\lim}_{T\to\infty} a(\gamma T)/a(T) < \infty$ for some $\gamma > 1$, then

$$\underline{\lim}_{T\to\infty} \phi(T) \sup_{T-a(T)\le t\le T} |B(t)| \le \frac{\pi}{2} \quad a.s.$$

Let us first show (I). Define

$$T_1 = 1 \quad , \quad T_{k+1} - \varepsilon_1 a(T_{k+1}) = T_k \tag{2.1}$$

where $\varepsilon_1 = 1-(1-\varepsilon^2)^{1/2}$ and $0<\varepsilon<1$. Note that $T-\varepsilon_1 a(T)$ is a strictly increasing and continuous function by our conditions (i) and (ii). Hence T_k in (2.1) is well defined and $T_{k+1} > T_k$, $\lim_{k\to\infty} T_k = \infty$. Since

$$\phi(T) \ge \left(\left(\log\left(T_k/a(T_k)\right) + 2\log\log T_k\right)/a(T_{k+1})\right)^{1/2}$$

and $T - a(T) \le T_{k+1} - a(T_{k+1})$ for $T_{k+1} > T \ge T_k$, it is sufficient to show

$$\underline{\lim}_{k\to\infty} \left(\frac{\log\left(T_k/a(T_k)\right) + 2\log\log T_k}{a(T_{k+1})}\right)^{1/2} \sup_{T_{k+1}-a(T_{k+1})\le t\le T_k} |B(t)| \ge \frac{\pi}{2}(1-\varepsilon) \quad a.s. \tag{2.2}$$

Note that for k large, $T_k - a(T_k) \ge T_k(1-\rho)/2$. Thus by Lemma 1, we have for k large

$$\begin{aligned}
& P\left(\left(\frac{\log\left(T_k/a(T_k)\right) + 2\log\log T_k}{a(T_{k+1})}\right)^{1/2} \sup_{T_{k+1}-a(T_{k+1})\le t\le T_k} |B(t)| \le \frac{\pi}{2}(1-\varepsilon)\right) \\
= \;& P\left(\sup_{T_{k+1}-a(T_{k+1})\le t\le T_k} |B(t)| \le \frac{(1-\varepsilon)\pi}{2} \cdot \left(\frac{a(T_{k+1})}{\log(T_k/a(T_k)) + 2\log\log T_k}\right)^{1/2}\right) \\
\le \;& C \cdot \left(\frac{a(T_{k+1})}{T_{k+1} - a(T_{k+1})}\right)^{1/2} \cdot \exp\left(-\frac{\log(T_k/a(T_k)) + 2\log\log T_k}{2(1-\varepsilon)^2} \cdot \frac{(T_k - T_{k+1} + a(T_{k+1}))}{a(T_{k+1})}\right) \\
= \;& C \cdot \left(\frac{a(T_{k+1})}{T_{k+1} - a(T_{k+1})}\right)^{1/2} \cdot \exp\left(-\frac{\log(T_k/a(T_k)) + 2\log\log T_k}{2(1-\varepsilon)^2} \cdot (1-\varepsilon_1)\right) \\
\le \;& C \cdot \left(\frac{a(T_{k+1})}{T_{k+1} - a(T_{k+1})}\right)^{1/2} \cdot \left(\frac{a(T_k)}{T_k}\right)^{(1+\varepsilon)/2} \cdot \left(\frac{1}{\log T_k}\right)^{1+\varepsilon} \\
\le \;& C \cdot \left(\frac{a(T_k)}{T_k - a(T_k)}\right)^{1/2} \cdot \left(\frac{a(T_k)}{T_k}\right)^{1/2} \cdot \left(\frac{1}{\log T_k}\right)^{1+\varepsilon} \\
\le \;& C \cdot \frac{T_k - T_{k-1}}{(T_k - a(T_k))^{1/2} T_k^{1/2} \cdot (\log T_k)^{1+\varepsilon}} \\
\le \;& C \cdot \frac{T_k - T_{k-1}}{T_k\,(\log T_k)^{1+\varepsilon}} \le C \cdot \int_{T_{k-1}}^{T_k} \frac{dx}{x\,(\log x)^{1+\varepsilon}}.
\end{aligned}$$

Hence by the Borel-Cantelli lemma, we obtain (2.2) which shows (I).

Now turn to the proof of (II). Let T_k be the unique solution of the equation

$$x/a(x) = k^\beta \quad \text{where} \quad \beta = 2(1+\varepsilon)/(2+\varepsilon) > 1. \tag{2.3}$$

Then $T_{k+1} > T_k$ and $\lim_{k\to\infty} T_k = \infty$. Define the events

$$A_k = \left\{ \sup_{T_k - a(T_k) \le t \le T_k} |B(t)| \le \frac{(1+\varepsilon)\pi}{2} \cdot \left(\frac{a(T_k)}{\log(T_k/a(T_k))} \right)^{1/2} \right\}.$$

We then show $P(A_k\ i.o.) = 1$ by Lemma 3 which in turn gives us (II). Note that $T_k/a(T_k) = k^\beta$. Hence by Lemma 1 and the choice of β in (2.3), we have for k large

$$P(A_k) \ge C \cdot \left(\frac{a(T_k)}{(T_k - a(T_k)) \log(T_k/a(T_k))} \right)^{1/2} \cdot \exp\left(-\frac{\log(T_k/a(T_k))}{2(1+\varepsilon)^2} \right) \ge C \cdot (k \log k)^{-1}$$

which shows $\sum_{k\ge1} P(A_k) = \infty$.

For given $\delta > 0$ small, define k_0 large such that for $l > k > k_0$, we have

$$\sum_{k<l<(\delta^{-1}+1)k} \left(l^\beta - 1 - k^\beta\right)^{-1/2} \cdot l^{-1+\beta/2} \le 2 \int_{k+1}^{(\delta^{-1}+1)k} \left(x^\beta - 1 - k^\beta\right)^{-1/2} \cdot x^{-1+\beta/2} dx \le C. \tag{2.4}$$

Note that for $l > k$ and $\beta > 1$,

$$T_l - a(T_l) \ge T_{k+1} - a(T_{k+1}) = (k+1)^\beta a(T_{k+1}) - a(T_{k+1}) \ge k^\beta a(T_k) = T_k.$$

Hence for given k , $k_0 < k \le n$, we can split the set $\{l\ :\ k_0 < k < l \le n\}$ into two parts,

$$L_1 = \{l\ :\ k_0 < k < l\ ,\ T_l - a(T_l) \ge T_k > \delta(T_l - a(T_l))\} \quad ;$$
$$L_2 = \{l\ :\ k_0 < k < l\ ,\ \delta(T_l - a(T_l)) \ge T_k\}.$$

If $l \in L_2$, then by Lemma 2,

$$P(A_k A_l) \le \left(\frac{T_l - a(T_l)}{T_l - a(T_l) - T_k} \right)^{1/2} P(A_k)P(A_l) \le (1-\delta)^{-1/2} P(A_k)P(A_l).$$

Note that by the proof of Lemma 2, we also have for $k_0 < k < l$

$$P(A_k A_l) \le P(A_k) \cdot P\left(\sup_{T_l - a(T_l) - T_k \le t \le T_l - T_k} |B(t)| \le \frac{(1+\varepsilon)\pi}{2} \left(\frac{a(T_l)}{\log(T_l/a(T_l))} \right)^{1/2} \right). \tag{2.5}$$

If $l \in L_1$, then

$$\delta^{-1} k^\beta a(T_k) = \delta^{-1} T_k \ge T_l - a(T_l) = (l^\beta - 1) a(T_l) \ge (l^\beta - 1) a(T_k)$$

which gives $k < l < (\delta^{-1}+1)k$. Now for $l \in L_1$, we have by Lemma 1,

$$\begin{aligned} & P\left(\sup_{T_l - a(T_l) - T_k \le t \le T_l - T_k} |B(t)| \le \frac{(1+\varepsilon)\pi}{2} \left(\frac{a(T_l)}{\log(T_l/a(T_l))} \right)^{1/2} \right) \qquad (2.6) \\ \le\ & C \cdot \left(\frac{a(T_l)}{T_l - a(T_l) - T_k} \right)^{1/2} \cdot \exp\left(-\frac{\log(T_l/a(T_l))}{2(1+\varepsilon)^2} \right) \\ \le\ & C \cdot (T_l/a(T_l) - 1 - T_k/a(T_k))^{-1/2} \cdot (T_l/a(T_l))^{-1/(2+2\varepsilon)} \\ =\ & C \cdot \left(l^\beta - 1 - k^\beta\right)^{-1/2} \cdot l^{-1+\beta/2}. \end{aligned}$$

Hence we have by combining (2.4), (2.5) and (2.6)

$$\begin{aligned}
&\sum_{k_0<k\le n}\ \sum_{l\in L_1} P(A_kA_l)\\
\le\ &\sum_{k_0<k\le n}\Big(C\cdot P(A_k)\sum_{k<l<(\delta^{-1}+1)k}\Big(l^\beta-1-k^\beta\Big)^{-1/2}\cdot l^{-1+\beta/2}\Big)\\
\le\ &C\sum_{k_0<k\le n} P(A_k).
\end{aligned}$$

Now by Lemma 3, (II) follows from $\lim_{n\to\infty}\sum_{k=1}^n P(A_k)=\infty$ and the estimates

$$\begin{aligned}
&\sum_{k=1}^n\sum_{l=1}^n P(A_kA_l)\\
=\ &\sum_{k=1}^n P(A_k)+2\sum_{1\le k<l\le n} P(A_kA_l)\\
=\ &\sum_{k=1}^n P(A_k)+2\sum_{k=1}^{k_0}\sum_{l=k+1}^{n} P(A_kA_l)+2\sum_{k=k_0+1}^{n}\sum_{l\in L_1} P(A_kA_l)+2\sum_{k=k_0+1}^{n}\sum_{l\in L_2} P(A_kA_l)\\
\le\ &(1+2k_0+2C)\sum_{k=1}^n P(A_k)+(1-\delta)^{-1/2}\sum_{k=1}^n\sum_{l=1}^n P(A_k)P(A_l).
\end{aligned}$$

Now turn to the proof of (III). Define $T_1=1$, $T_{k+1}-a(T_{k+1})=T_k$. Then for k large,

$$\lim_{k\to\infty} T_k=\infty \quad\text{and}\quad 1>T_k/T_{k+1}>(1-\rho)/2>0. \tag{2.7}$$

Thus

$$T_k\cdot(\log T_k)\cdot(\log\log T_k)^{1/2}\le C\cdot T_{k-1}\cdot(\log T_{k-1})\cdot(\log\log T_{k-1})^{1/2}. \tag{2.8}$$

Define the events

$$B_k=\left\{\sup_{T_k-a(T_k)\le t\le T_k}|B(t)|\le\frac{(1+\varepsilon)\pi}{2}\cdot\left(\frac{a(T_k)}{4(\log(T_k/a(T_k))+2\log\log T_k)}\right)^{1/2}\right\}.$$

We then show $P(B_k\ i.o.)=1$ by Lemma 2. By our assumptions for case (III) and the fact that if $\varepsilon>0$ is small enough, we have

$$\log(T/a(T))\le 2\varepsilon^{-1}\cdot\log\log T. \tag{2.9}$$

Hence by Lemma 1, (2.9) and (2.8), we have for k large

$$\begin{aligned}
P(B_k)\ \ge\ &C\cdot\left(\frac{a(T_k)}{(T_k-a(T_k))(\log(T_k/a(T_k))+2\log\log T_k)}\right)^{1/2}\\
&\cdot\exp\left(-\frac{\log(T_k/a(T_k))+2\log\log T_k}{2(1+\varepsilon)^2}\right)\\
\ge\ &C\cdot\frac{a(T_k)}{T_k\cdot\log T_k\cdot(\log\log T_k)^{1/2}}\\
\ge\ &C\cdot\int_{T_{k-1}}^{T_k}\frac{dx}{x\cdot\log x\cdot(\log\log x)^{1/2}}
\end{aligned}$$

which shows $\sum_{k\geq 1} P(B_k) = \infty$.

Since $a(T)$ is non-decreasing, we observe that $a(2(1-\rho)^{-1}T)/a(T) \leq C$ for T large by iterating $\overline{\lim}_{T\to\infty} a(\gamma T)/a(T) < \infty$ for some $\gamma > 1$ if necessary. Hence by (2.7), we can define k_0 large such that for $l > k_0$,

$$a(T_l) \leq C \cdot a(T_{l-1}). \tag{2.10}$$

Note that for $l > k+1$, $T_l - a(T_l) > T_{k+1} - a(T_{k+1}) = T_k$. Hence for given $\delta > 0$ small and $k_0 < k \leq n$, we can split the set $\{l \ : \ k_0 + 1 < k+1 < l \leq n\}$ into two parts,

$$\begin{aligned} L_1 &= \{l \ : \ k_0+1 < k+1 < l \ , \ T_l - a(T_l) > T_k > \delta(T_l - a(T_l))\} \quad ; \\ L_2 &= \{l \ : \ k_0+1 < k+1 < l \ , \ \delta(T_l - a(T_l)) \geq T_k\}. \end{aligned}$$

If $l \in L_2$, then by Lemma 2,

$$P(B_k B_l) \leq \left(\frac{T_l - a(T_l)}{T_l - a(T_l) - T_k}\right)^{1/2} P(B_k)P(B_l) \leq (1-\delta)^{-1/2} P(B_k)P(B_l). \tag{2.11}$$

Note that by Lemma 2, we also have for $k_0 < k < l$

$$\begin{aligned} & P(B_k B_l) \qquad\qquad (2.12) \\ \leq\ & P(B_k) \cdot P\left(\sup_{T_l - a(T_l) - T_k \leq t \leq T_l - T_k} |B(t)| \leq \frac{(1+\varepsilon)\pi}{2} \cdot \left(\frac{a(T_l)}{\log(T_l/a(T_l)) + 2\log\log T_l}\right)^{1/2}\right). \end{aligned}$$

If $l \in L_1$, then $T_k > \delta(T_l - a(T_l)) = \delta T_{l-1}$ which gives $T_k \leq T_{l-1} < \delta^{-1} T_k$. Now for $l \in L_1$, we have by Lemma 1, (2.9) and (2.10),

$$\begin{aligned} & P\left(\sup_{T_l - a(T_l) - T_k \leq t \leq T_l - T_k} |B(t)| \leq \frac{(1+\varepsilon)\pi}{2} \cdot \left(\frac{a(T_l)}{\log(T_l/a(T_l)) + 2\log\log T_l}\right)^{1/2}\right) \\ \leq\ & C \cdot \left(\frac{a(T_l)}{T_l - a(T_l) - T_k}\right)^{1/2} \cdot \exp\left(-\frac{\log(T_l/a(T_l)) + 2\log\log T_l}{2(1+\epsilon)^2}\right) \\ \leq\ & C \cdot \frac{a(T_l)}{(T_l - a(T_l) - T_k)^{1/2} \cdot T_l^{1/2}} \\ \leq\ & C \cdot \frac{a(T_{l-1})}{(T_l - a(T_l) - T_k)^{1/2} \cdot T_l^{1/2}} \\ \leq\ & C \cdot \int_{T_{l-2}}^{T_{l-1}} \frac{dx}{(x - T_k)^{1/2} \cdot x^{1/2}}. \end{aligned}$$

Hence we have by combining (2.12) and the above estimate,

$$\begin{aligned} & \sum_{k_0 < k \leq n} \sum_{l \in L_1} P(B_k B_l) \\ \leq\ & \sum_{k_0 < k \leq n} \left(C \cdot P(B_k) \sum_{T_k < T_{l-1} < (\delta^{-1}+1)T_k} \int_{T_{l-2}}^{T_{l-1}} \frac{dx}{(x - T_k)^{1/2} \cdot x^{1/2}} \right) \qquad (2.13) \\ \leq\ & \sum_{k_0 < k \leq n} \left(C \cdot P(B_k) \cdot \int_{T_k}^{(\delta^{-1}+1)T_k} \frac{dx}{(x - T_k)^{1/2} \cdot x^{1/2}} \right) \\ \leq\ & C \sum_{k_0 < k \leq n} P(B_k). \end{aligned}$$

Now similarly to what we did at the end of the proof of (II), $P(B_k\ i.o.) = 1$ follows from (2.11), (2.13) and $\lim_{n\to\infty}\sum_{k=1}^{n} P(B_k) = \infty$. Thus we complete the proof of (III) and hence finish the proof of our theorem.

REFERENCES

1. K. L. Chung, On the maximum partial sums of sequence of independent random variables. Amer. Math. Soc. **64** (1948) 205-233.
2. K. L. Chung, A course in probability theory. New York: Academic Press, 1974
3. E. Csáki and A. Főldes, On the narrowest tube of a Wiener process. Coll. Math. Soc. J. Bolyai **36**. Limit theorems in probability and statistics (ed. P. Révész), Veszprém (Hungary), 173-197, (1982).
4. M. Csörgő and P. Révész, Strong Approximations in Probability and Statistics. New York, Academic Press, 1981.
5. M.D. Donsker and S. R. S. Varadhan, On laws of the iterated logarithm for local times. Comm. Pure. Appl. Math. **30** (1977) 707-753.
6. J. Hoffmann-Jørgensen, L. A. Shepp and r. M. Dudley, On the lower tail of Gaussian seminorms. Ann. Probab. **7**, (1979) 319-342.
7. N. C. Jain and W. E. Pruitt, The other law of the iterated logarithm. Ann. Probab. **3** (1975) 1046-1049.
8. W. V. Li, Limit theorems for the square integral of Brownian motion and its increments. Stochastic Process. Appl. **41** (1992) 223-239.
9. W. V. Li, Lim inf results for the Wiener process and its increments under the L_2-norm. Probab. Theory Relat. Fields. **92** (1992) 69-90.
10. V. Strassen, An invariance principle for the law of the iterated logarithm. Z. Wahrs. verw. Gebiete **3** (1964) 211-226.