C. R. Acad. Sci. Paris, t. 326, Série I, p. 1329-1334, 1998
Probabilités/*Probability Theory*

Existence of small ball constants for fractional Brownian motions

Wenbo V. LI [a], **Werner LINDE** [b]

[a] Department of Mathematics, University of Delaware, Newark, DE 19711, USA
E-mail: wli@math.udel.edu

[b] Institut für Stochastik, FSU, Ernst-Abbe-Platz 1-4, 07743 Jena, Germany
E-mail: lindew@minet.uni-jena.de

(Reçu le 10 avril 1998, accepté le 12 mai 1998)

Abstract. Let $\{B_\gamma(t),\ 0 \le t \le 1\}$ be a fractional Brownian motion of order $\gamma \in (0,2)$, and let $B(t) = B_1(t)$ be the standard Brownian motion. We show the existence of a $C_\gamma \in (0,\infty)$ such that:

$$\lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon \right) = \lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0\le t\le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right) = -C_\gamma.$$

where a_γ is an explicit constant and

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma+1)/2)} \int_0^t (t-s)^{(\gamma-1)/2} \mathrm{d}B(s).$$

© Académie des Sciences/Elsevier, Paris

L'existence de la limite pour l'asymptotique des petites boules du mouvement brownien fractionnaire

Résumé. *Soit* $\{B_\gamma(t),\ 0 \le t \le 1\}$ *le mouvement brownien fractionnaire d'ordre* $\gamma \in (0,2)$ *et soit* $B(t) = B_1(t)$ *le mouvement brownien ordinaire. Nous montrons l'existence d'une constante* $C_\gamma \in (0,\infty)$ *telle que :*

$$\lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon \right) = \lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0\le t\le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right) = -C_\gamma.$$

où a_γ *est une constante explicite et*

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma+1)/2)} \int_0^t (t-s)^{(\gamma-1)/2} \mathrm{d}B(s).$$

© Académie des Sciences/Elsevier, Paris

Note présentée par Paul DEHEUVELS.

Version française abrégée

Soit $\{B_\gamma(t) : 0 \le t \le 1\}$ le mouvement brownien fractionnaire d'ordre $\gamma \in (0, 2)$. Alors, $\{B_\gamma(t) : 0 \le t \le 1\}$ est le processus gaussien centré avec la fonction de covariance

$$\mathbb{E}\,(B_\gamma(t)B_\gamma(s)) = \frac{1}{2}(|s|^\gamma + |t|^\gamma - |s-t|^\gamma).$$

Dans [7] et [6] on a démontré que

$$-\infty < \liminf_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon\Big) \le \limsup_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon\Big) < 0.$$

Naturellement, on se demande si la limite ci-dessus existe. Notre résultat principal dit que c'est vrai. Plus précisément, nous montrons :

THÉORÈME. – *Soit* $\{B_\gamma(t) : 0 \le t \le 1\}$ *le mouvement brownien fractionnaire d'ordre* $\gamma \in (0, 2)$ *et soit* $B(t) = B_1(t)$ *le mouvement brownien ordinaire. On a :*

$$\lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon\Big) = \lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma\Big) = -C_\gamma,$$

où $0 < C_\gamma = -\inf_{\varepsilon>0} \varepsilon^{2/\gamma} \log \mathbb{P}\big(\sup_{0\le t\le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma\big) < \infty$,

$$a_\gamma = \Gamma((\gamma+1)/2)\Big(\gamma^{-1} + \int_{-\infty}^{0} ((1-s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2})^2 \mathrm{d}s\Big)^{-1/2}$$

et

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma+1)/2)} \int_0^t (t-s)^{(\gamma-1)/2} \mathrm{d}B(s).$$

Une simple conséquence du théorème précédent et des résultats dans [9] est :

$$\liminf_{T\to\infty} \Big(\frac{\log\log T}{T}\Big)^{\gamma/2} \sup_{0\le t\le T} |B_\gamma(t)| = \liminf_{T\to\infty} a_\gamma \Big(\frac{\log\log T}{T}\Big)^{\gamma/2} \sup_{0\le t\le T} |W_\gamma(t)| = C_\gamma^{\gamma/2}.$$

1. Introduction

Let $\{B_\gamma(t) : t \ge 0\}$ denote the γ-fractional Brownian motion with $B_\gamma(0) = 0$ and $0 < \gamma < 2$. Then $\{B_\gamma(t) : t \ge 0\}$ is a Gaussian process with mean zero and covariance function

$$\mathbb{E}\,(B_\gamma(t)B_\gamma(s)) = \frac{1}{2}(|s|^\gamma + |t|^\gamma - |s-t|^\gamma).$$

It was proved in [7] and [6] that

$$-\infty < \liminf_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon\Big) \le \limsup_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon\Big) < 0. \quad (1.1)$$

A natural and important question is the existence of the above limit. It was shown in [4] that this limit exists if the Gaussian correlation conjecture holds. The main purpose of this Note is to show that the limit in (1.1) exists and is related to a more attractable Gaussian process. During the preparation of this paper, Professor Q. Shao has informed us that he obtained a different proof of the existence of the constant for $B_\gamma(t)$ and his paper is in preparation.

THEOREM 1.1. – *Let* $\{B_\gamma(t),\ 0 \le t \le 1\}$ *be a fractional Brownian motion of order* $\gamma \in (0,2)$ *and let* $B(t) = B_1(t)$ *be the standard Brownian motion. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon\Big) = \lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma\Big) = -C_\gamma,$$

where $0 < C_\gamma = -\inf_{\varepsilon>0} \varepsilon^{2/\gamma} \log \mathbb{P}\big(\sup_{0\le t\le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma\big) < \infty$,

$$a_\gamma = \Gamma((\gamma+1)/2)\Big(\gamma^{-1} + \int_{-\infty}^{0} ((1-s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2})^2 \mathrm{d}s\Big)^{-1/2} \tag{1.2}$$

and

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma+1)/2)} \int_0^t (t-s)^{(\gamma-1)/2} \mathrm{d}B(s). \tag{1.3}$$

In the Brownian motion case, i.e. $\gamma = 1$, it is well known that $C_1 = \pi^2/8$ and $a_1 = 1$.

As a simple consequence of the above theorem and results in [9], we have

$$\liminf_{T\to\infty} \Big(\frac{\log\log T}{T}\Big)^{\gamma/2} \sup_{0\le t\le T} |B_\gamma(t)| = \liminf_{T\to\infty} a_\gamma \Big(\frac{\log\log T}{T}\Big)^{\gamma/2} \sup_{0\le t\le T} |W_\gamma(t)| = C_\gamma^{\gamma/2}.$$

One of the key step in our investigation of $B_\gamma(t)$ is the following useful representation when $\gamma \ne 1$ (*see* [5]),

$$B_\gamma(t) = a_\gamma(W_\gamma(t) + Z_\gamma(t)), \quad 0 \le t \le 1, \tag{1.4}$$

where a_γ is given in (1.2), $W_\gamma(t)$ is given in (1.3), and

$$Z_\gamma(t) = \frac{1}{\Gamma((\gamma+1)/2)} \int_{-\infty}^{0} \{(t-s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2}\} \mathrm{d}B(s).$$

Furthermore, $W_\gamma(t)$ is independent of $Z_\gamma(t)$. Observe that the centered Gaussian process $W_\beta(t)$ is defined for all $\beta > 0$ as a fractional Wiener integral. Thus $W_\beta(t)$ will be called *fractional Wiener process of order* β.

The remaining of the paper is organized as follows. In section 2, we estimate the small ball rate for $W_\beta(t)$ for any $\beta > 0$ and show the existence of the constant for $W_\beta(t)$. In section 3, we give the proof of Theorem 1.1.

2. Fractional Wiener processes

THEOREM 2.1. – *For any* $\beta > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon^{2/\beta} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |W_\beta(t)| \le \varepsilon\Big) = -k_\beta,$$

where $0 < k_\beta = -\inf_{\varepsilon>0} \varepsilon^{2/\beta} \log \mathbb{P}\big(\sup_{0\le t\le 1} |W_\beta(t)| \le \varepsilon\big) < \infty$.

Proof. – The lower estimate for all $\beta > 0$

$$\liminf_{\varepsilon\to 0} \varepsilon^{2/\beta} \log \mathbb{P}\left(\sup_{0\le t\le 1} |W_\beta(t)| \le \varepsilon \right) > -\infty$$

is given in [3]. But when $\beta = \gamma < 2$, the estimate follows easily from (1.1) and

$$\mathbb{P}\left(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon \right) \le \mathbb{P}\left(\sup_{0\le t\le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right), \tag{2.1}$$

which is a direct consequence of representation (1.4) and Anderson's inequality. Thus we only need to show the upper bound and the existence of the constant.

Let $\widehat{W}_\beta(t) = \Gamma((\beta+1)/2)W_\beta(t)$ for simplicity. Then for any $x > 0$ and $0 < \lambda < 1$, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le 1} \left|\widehat{W}_\beta(t)\right| \le x \right) &= \mathbb{P}\left(\sup_{0\le t\le 1} \left| \int_0^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right) \\ &= \mathbb{P}\left(\sup_{0\le t\le \lambda} \left|\widehat{W}_\beta(t)\right| \le x,\ \sup_{\lambda\le t\le 1} \left| Y_{\beta,\lambda}(t) + \int_\lambda^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right)\end{aligned} \tag{2.2}$$

where $Y_{\beta,\lambda}(t) = \int_0^\lambda (t-s)^{(\beta-1)/2}\mathrm{d}B(s)$. Note that the processes $Y_{\beta,\lambda}(t)$, $0 \le t \le 1$, and $\widehat{W}_\beta(t) = \int_0^t (t-s)^{(\beta-1)/2}\mathrm{d}B(s)$, $0 \le t \le \lambda$, are jointly independent of the process $\widehat{W}_\beta(t) - Y_{\beta,\lambda}(t) = \int_\lambda^t (t-s)^{(\beta-1)/2}\mathrm{d}B(s)$, $\lambda \le t \le 1$. Furthermore, Anderson's inequality implies that for any real number a,

$$\mathbb{P}\left(\sup_{\lambda\le t\le 1} \left| a + \int_\lambda^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right) \le \mathbb{P}\left(\sup_{\lambda\le t\le 1} \left| \int_\lambda^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right). \tag{2.3}$$

Thus by first conditioning on $\mathrm{d}B(s)$, $0 \le s \le \lambda$, then using (2.3), we obtain from (2.2) that

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le 1} \left|\widehat{W}_\beta(t)\right| \le x \right) &\le \mathbb{P}\left(\sup_{0\le t\le \lambda} \left|\widehat{W}_\beta(t)\right| \le x \right) \mathbb{P}\left(\sup_{\lambda\le t\le 1} \left| \int_\lambda^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right) \\ &= \mathbb{P}\left(\sup_{0\le t\le 1} \left|\widehat{W}_\beta(t)\right| \le x/\lambda^{\beta/2} \right) \mathbb{P}\left(\sup_{0\le t\le 1-\lambda} \left|\widehat{W}_\beta(t)\right| \le x \right)\end{aligned}$$

where the last equality follows from simple substitution and scaling. Taking $\lambda = 1/n$ and iterating the above procedure, we have for any $x > 0$ and any integer n

$$\mathbb{P}\left(\sup_{0\le t\le 1} \left|\widehat{W}_\beta(t)\right| \le x \right) \le \mathbb{P}\left(\sup_{0\le t\le 1} \left|\widehat{W}_\beta(t)\right| \le n^{\beta/2} x \right)^n$$

which finishes the proof by following exactly the same argument as the proof of Prop. 2.1 in [1].

3. Proof of Theorem 1.1

From (2.1) and Theorem 2.1, it is clear that

$$\limsup_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon \right) \le \lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0\le t\le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right) = -k_\gamma a_\gamma^{2/\gamma}.$$

So we only need to show the lower estimate. We need the following result which was established in [8] and reformulated in [2], page 257.

LEMMA 3.1. – *Let $(X_t)_{t\in T}$ be a centered Gaussian process. For every $\varepsilon > 0$, let $N(T, d; \varepsilon)$ denote the minimal number of balls of radius ε, under the (pseudo-)metric*

$$\mathrm{d}(s,t) = (E|X_s - X_t|^2)^{1/2},$$

that are necessary to cover T. Assume that there is a nonnegative function ψ on $\mathbb{R}_+$ such that $N(T, d; \varepsilon) \le \psi(\varepsilon)$ and such that $c_1\psi(\varepsilon) \le \psi(\varepsilon/2) \le c_2\psi(\varepsilon)$ for some constants $1 < c_1 \le c_2 < \infty$. Then, for some $K > 0$ and every $\varepsilon > 0$ we have $\mathbb{P}(\sup_{s,t\in T} |X_s - X_t| \le \varepsilon) \ge \exp(-K\psi(\varepsilon))$.

LEMMA 3.2. – *For any $0 < \gamma < 2$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0\le t\le 1} |Z_\gamma(t)| \le \varepsilon\right) = 0.$$

Proof. – Note that for any $s, t \in (0,1) = T$ and $s \le t$, with $X_\gamma(t) = \Gamma((\gamma+1)/2)Z_\gamma(t)$,

$$\begin{aligned}\mathrm{d}_\gamma^2(s,t) &= E(X_\gamma(t) - X_\gamma(s))^2 = E\left(\int_{-\infty}^0 ((t-u)^{(\gamma-1)/2} - (s-u)^{(\gamma-1)/2})\mathrm{d}B(u)\right)^2 \\ &= \int_{-\infty}^0 \left((t-u)^{(\gamma-1)/2} - (s-u)^{(\gamma-1)/2}\right)^2 \mathrm{d}u = \int_s^\infty \left((t-s+u)^{(\gamma-1)/2} - u^{(\gamma-1)/2}\right)^2 \mathrm{d}u.\end{aligned}$$

Since by the mean value theorem

$$\left|(t-s+u)^{(\gamma-1)/2} - u^{(\gamma-1)/2}\right| \le |t-s|u^{(\gamma-3)/2},$$

we have

$$\mathrm{d}_\gamma(s,t) \le (2-\gamma)^{-1/2}(t-s)\, s^{-(2-\gamma)/2}$$

for $0 < s \le t < 1$. When $s = 0$, it follows

$$\mathrm{d}_\gamma^2(0,t) = t^\gamma \int_0^\infty \left((1+u)^{(\gamma-1)/2} - u^{(\gamma-1)/2}\right)^2 \mathrm{d}u,$$

which implies $\mathrm{d}_\gamma(0,t) \le Ct^{\gamma/2}$ with $C > 0$ only depending on γ. For any $\varepsilon > 0$ small, we define numbers $0 < t_0 < t_1 < \cdots$ by: $t_0 = (\varepsilon/C)^{2/\gamma}$, so that $\mathrm{d}_\gamma(0, t_0) \le \varepsilon$, and for $i \ge 1$ by

$$(2-\gamma)^{-1/2}(t_i - t_{i-1})t_{i-1}^{-(2-\gamma)/2} = \varepsilon.$$

Let $N(\varepsilon) = \min\{n : t_n > 1\}$. Then for $1 \le i \le N(\varepsilon)$ we obtain:

$$t_i = t_{i-1}(1 + (2-\gamma)^{1/2}\varepsilon t_{i-1}^{-\gamma/2}) \ge t_{i-1}(1 + (2-\gamma)^{1/2}\varepsilon),$$

thus by iterating:

$$1 \ge t_{N(\varepsilon)-1} \ge t_0(1 + (2-\gamma)^{1/2}\varepsilon)^{N(\varepsilon)-1} = (\varepsilon/C)^{2/\gamma}(1 + (2-\gamma)^{1/2}\varepsilon)^{N(\varepsilon)-1},$$

which implies $N(\varepsilon) \le c\varepsilon^{-1}\log(1/\varepsilon)$ for some $c > 0$. Hence, using t_i, $0 \le i \le N(\varepsilon) - 1$, as centers, we finally get $N(T, d_\gamma; \varepsilon) \le N(\varepsilon) \le c\varepsilon^{-1}\log(1/\varepsilon)$ which finishes the proof by Lemma 3.1.

To obtain the lower bound of Theorem 1.1, we have for any $0 < \delta < 1$:

$$\mathbb{P}\Big(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon\Big) \ge \mathbb{P}\Big(\sup_{0\le t\le 1} |W_\gamma(t)| \le (1-\delta)\varepsilon/a_\gamma\Big)\mathbb{P}\Big(\sup_{0\le t\le 1} |Z_\gamma(t)| \le \delta\varepsilon/a_\gamma\Big)$$

since $W_\gamma(t)$ and $Z_\gamma(t)$ are independent of each other. Thus

$$\liminf_{\varepsilon\to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |B_\gamma(t)| \le \varepsilon\Big) \ge -k_\gamma(1-\delta)^{-2/\gamma} a_\gamma^{2/\gamma}.$$

So we obtain the desired lower bound with $C_\gamma = k_\gamma a_\gamma^{2/\gamma}$ by taking $\delta \to 0$.

Acknowledgements. The authors are very grateful to Th. Dunker (Jena) for helpful discussions and for pointing out a misuse of notation in the early draft.

([1]) Supported in part by NSF.

References

[1] Kuelbs J., Li W.V., Small ball estimates for Brownian motion and the Brownian sheet, J. Th. Probab. 6 (1993) 547–577.
[2] Ledoux M., Isoperimetry and Gaussian Analysis, Lectures on Probability Theory and Statistics, Lect. Notes in Math. 1648, Springer-Verlag, 1996, pp. 165–294.
[3] Li W.V., Linde W., Approximation, metric entropy and small ball estimates for Gaussian measures, Preprint, 1998.
[4] Li W.V., Shao Q.M., A note on the Gaussian correlation conjecture and the existence of small ball constant for fractional Brownian motions, Preprint, 1997.
[5] Mandelbrot B.B., Van Ness J. W., Fractional Brownian motions, fractional noises, and applications, SIAM Rev. 10 (1968) 422–437.
[6] Monrad D., Rootzén H., Small values of Gaussian processes and functional laws of the iterated logarithm, Probab. Th. Rel. Fields 101 (1995) 173–192.
[7] Shao Q.M., A note on small ball probability of Gaussian processes with stationary increments, J. Th. Probab. 6 (1993) 595–602.
[8] Talagrand M., New Gaussian estimates for enlarged balls, Geom. Funct. Anal. 3 (1993) 502–526.
[9] Talagrand M., Lower classes for fractional Brownian motion, J. Th. Probab. 9 (1996) 191–213.