27
An Extension of Ehrhard's Theorem

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1 Introduction

Let E denote a real separable Banach space and assume μ is a centered non-degenerate Gaussian measure on E. In [2] Ehrhard proved that if A, B are non-empty convex Borel subsets of E and

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-u^2/2} du,$$

then for $0 \leq \lambda \leq 1$ we have

$$\Phi^{-1} \circ \mu(\lambda A + (1-\lambda)B) \geq \lambda \Phi^{-1} \circ \mu(A) + (1-\lambda)\Phi^{-1} \circ \mu(B) \tag{1.1}$$

where $\lambda A + (1-\lambda)B = \{\lambda a + (1-\lambda)b \,:\, a \in A, b \in B\}$.

This is a delicate result, which implies the isoperimetric inequality for Gaussian measures when the set enlarged is convex, and has some other interesting consequences as well, see, for example [3], [5] and [6]. Ehrhard's proof first obtains this result when $\mu = \gamma_n$, the canonical Gaussian measure on $\mathbb{R}^n$, and then extends it to the infinite dimensional setting using the convexity of A and B and the log-concavity of Gaussian measures. To establish (1.1) when $\mu = \gamma_n$, Ehrhard first obtained (1.1) for $(\mathbb{R}^1, \gamma_1)$. Then, by using Gaussian symmetrizations, the result is lifted to $(\mathbb{R}^n, \gamma_n)$ by showing that if C is an open (closed)

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convex set of $\mathbb{R}^n$, and f is any Gaussian symmetrization of $\mathbb{R}^n$ (see definitions below), then $f(C)$ is open (closed) and convex. After the convexity of $f(C)$ is established, the proof considers A and B as n-dimensional convex sets in parallel hyperplanes of $\mathbb{R}^{n+1}$ and takes $C = \cup_{0\leq\lambda\leq 1}(\lambda A + (1-\lambda)B)$. Hence C is convex, and if f is an n-symmetrization of $\mathbb{R}^{n+1}$ in the direction u parallel to the hyperplanes containing A and B, then $f(C)$ is convex in $\mathbb{R}^{n+1}$. Once one understands the Gaussian symmetrizations defined below, (1.1) is now immediate.

It has been a question of interest for some time as to whether (1.1) holds when A and B are arbitrary Borel sets. In fact, even if one set, say A, is convex, but B is Borel, the result is open and of interest. What we establish here is that (1.1) holds if A and B are Borel subsets of E such that A is convex and B is the complement of a convex set. This is a long way from allowing B to be an arbitrary Borel set, but the proof requires something new and proceeds somewhat differently than when A and B are both convex. In particular, we first obtain the result for $(\mathbb{R}^1, \gamma_1)$, and then we lift the result, using set inclusions and Gaussian symmetrizations to $(\mathbb{R}^n, \gamma_n)$. This alternative route is required since the set $C = \cup_{0\leq\lambda\leq 1}(\lambda A + (1-\lambda)B)$ of $\mathbb{R}^{n+1}$ mentioned above need not be convex. We also need a modification of Ehrhard's proof to lift (1.1) to the infinite dimensional setting, since we have less convexity available. In fact, our proof shows that if (1.1) holds for $(\mathbb{R}^1, \gamma_1)$ when A and B are arbitrary Borel sets (or A is convex set and B is a Borel set), then the same results can be lifted to the infinite dimensional setting. Indeed, it can be seen that such a result extends more easily, since it is a stronger result on $(\mathbb{R}^1, \gamma_1)$.

The theorem we prove is the following.

THEOREM 1 Let μ be a centered Gaussian measure on a real separable Banach space E, and assume A, B are non-empty Borel subsets of E such that A and B^c are convex. Then for $0 \leq \lambda \leq 1$

$$\Phi^{-1} \circ \mu(\lambda A + (1-\lambda)B) \geq \lambda\Phi^{-1} \circ \mu(A) + (1-\lambda)\Phi^{-1} \circ \mu(B). \tag{1.2}$$

REMARK The set $\lambda A + (1-\lambda)B$ need not be a Borel subset of E, but since E is complete and separable it is easy to see it is an analytic subset of E. Hence $\lambda A + (1-\lambda)B$ is always completion measurable for any Borel probability, [1, p391]. Of course, a similar result holds when (E, μ) is replaced by $(\mathbb{R}^n, \gamma_n)$. Thus the left-hand term in (1.1) or (1.2) makes perfect sense when μ is completed, and we therefore ignore further discussion of measurability for $\lambda A + (1-\lambda)B$ in the remainder of the paper.

Section 2 contains the definitions for Gaussian symmetrizations and a proof of (1.2) for $(\mathbb{R}^1, \gamma_1)$. Section 3 lifts the result to $(\mathbb{R}^n, \gamma_n)$ and finishes with the proof of Theorem 1.

2 Gaussian symmetrizations and Theorem 1 for $(\mathbb{R}^1, \gamma_1)$

We write $\langle x, y\rangle$ to denote the usual inner product on $\mathbb{R}^n$, and for u of length one in $\mathbb{R}^n$, $\lambda \in \mathbb{R}$, use $H(u,\lambda)$ to denote the open half-space

$$H(u,\lambda) = \{x \in \mathbb{R}^n : \langle x, u\rangle < \lambda\}.$$

γ_n is the canonical Gaussian measure on $\mathbb{R}^n$ and for each affine subspace L of dimension k we write $\gamma_{k,L}$ to denote the Gaussian measure (canonical) whose mean is the orthogonal projection of the origin onto L. To simplify notation we will use γ_k when the affine subspace L is understood.

If T is a subspace of $\mathbb{R}^n$ of dimension $n-k(1 \le k \le n)$ and u is orthogonal to T, $\langle u, u\rangle = 1$, we denote the k-symmetrization about T in the orthogonal direction u by $S(T,u)$. That is, if A is an open (or closed) subset of $\mathbb{R}^n$ we define $S(T,u)(A) = S(A)$ to be the subset of $\mathbb{R}^n$ satisfying:

For every $x \in T$, $T^\perp$ the orthogonal complement of T, and $\gamma_k = \gamma_{k,x+T^\perp}$, the set $S(A)\cap(x+T^\perp)$ is given by

(i) ϕ if $\gamma_k(A\cap(x+T^\perp)) = 0$

(2.1) (ii) $x+T^\perp$ if $\gamma_k(A\cap(x+T^\perp)) = 1$

(iii) $H(u,a)\cap(x+T^\perp)$ ($\bar{H}(u,a)\cap(x+T^\perp)$ when A is closed) if $\gamma_k(A\cap(x+T^\perp)) \in (0,1)$ and $a = a(x,A)$ satisfies

$$\gamma_k(H(u,a)\cap(x+T^\perp)) = \gamma_k(A\cap(x+T^\perp)).$$

The definition of $S(T,u)$ in (2.1) differs from that in [2] since $H(u,\lambda) = \{x : \langle x,u\rangle < \lambda\}$ rather than $\{x : \langle x,u\rangle > \lambda\}$ as in [2]. However, this change does not effect the regularity of the mappings $S(T,u)$, and hence by Theorem 2.1 of [2] each k-symmetrization $S = S(T,u)$ in $\mathbb{R}^n$ is both open and closed, i.e. it maps open (closed) sets of $\mathbb{R}^n$ to open (closed) sets of $\mathbb{R}^n$. A similar statement holds for all other results we quote from [2], but we will not bother to include such statements. We will call our symmetrizations left symmetrizations and those of Ehrhard right symmetrizations several times later in the paper. However, unless mentioned explicitly, symmetrizations should always be considered left symmetrizations.

Next we will prove the following lemma which is Theorem 1 for $(\mathbb{R}^1, \gamma_1)$.

LEMMA 1 Let $A = (a,b)$, $-\infty \le a < b \le +\infty$ and $B = (-\infty, c)\cup(d,+\infty)$, $-\infty \le c < d \le +\infty$. Then for $0 \le \lambda \le 1$

$$\gamma_1(\lambda A + (1-\lambda)B) \ge \Phi((\lambda\theta_1 + (1-\lambda)\theta_2). \tag{2.2}$$

where

$$\Phi(b) - \Phi(a) = \gamma_1(A) = \Phi(\theta_1) \qquad \Phi(c) + (1-\Phi(d)) = \gamma_1(B) = \Phi(\theta_2). \tag{2.3}$$

Proof. Since $\lambda A + (1-\lambda)B = (-\infty, \lambda b + (1-\lambda)c)\cup(\lambda a + (1-\lambda)d, +\infty)$, we only need to show

$$\Phi(\lambda b + (1-\lambda)c) + (1 - \Phi(\lambda a + (1-\lambda)d)) \ge \Phi(\lambda\theta_1 + (1-\lambda)\theta_2). \tag{2.4}$$

Let θ_1, θ_2 and λ be fixed with $0 < \lambda < 1$, since (2.2) holds if $\lambda = 0$ or 1. Consider $b = b_a$ as a function of a and d_c as a function of c given in (2.3). Since $\Phi'(x)$ is strictly positive, we have from (2.3) that the functions $b = b_a$ and $d = d_c$ are uniquely defined for $-\infty \le a \le -\theta_1$, $-\infty \le c \le \theta_2$ and

$$(2.5) \qquad b = \begin{cases} \theta_1 & \text{if } a = -\infty \\ +\infty & \text{if } a = -\theta_1 \end{cases} \qquad d = \begin{cases} -\theta_2 & \text{if } c = -\infty \\ +\infty & \text{if } c = \theta_2 \end{cases}.$$

Furthermore, for $-\infty < a < -\theta_1$ and $-\infty < c < \theta_2$

$$(2.6) \qquad b' = \exp((b^2 - a^2)/2) \qquad d' = \exp((d^2 - c^2)/2).$$

Let

$$(2.7) \qquad f(a,c) = \Phi(\lambda b + (1-\lambda)c) + (1 - \Phi(\lambda a + (1-\lambda)d)).$$

It is easy to check that (2.4) holds (i.e. $f(a,c) \ge \Phi(\lambda\theta_1 + (1-\lambda)\theta_2)$) on the boundary given by (2.5). So we only need to show that (2.4) holds for the pairs (a,c) such that $-\infty < a < -\theta_1$, $-\infty < c < \theta_2$ and

$$(2.8) \qquad \frac{\partial f(a,c)}{\partial a} = 0 \quad \text{and} \quad \frac{\partial f(a,c)}{\partial c} = 0.$$

Now we have from (2.8)

$$(2.9) \qquad \lambda b' \cdot \exp(-(\lambda b + (1-\lambda)c)^2/2) - \lambda \cdot \exp(-(\lambda a + ((1-\lambda)d)^2/2) = 0$$

and

$$(2.10) \quad (1-\lambda) \cdot \exp(-(\lambda b + (1-\lambda)c)^2/2) - (1-\lambda)d' \cdot \exp(-(\lambda a + ((1-\lambda)d)^2/2) = 0.$$

Substitute (2.6) into (2.9) and (2.10), we have

$$(2.11) \qquad (b^2 - a^2) - (\lambda b + (1-\lambda)c)^2 + (\lambda a + ((1-\lambda)d)^2 = 0$$

and

$$(2.12) \qquad (c^2 - d^2) - (\lambda b + (1-\lambda)c)^2 + (\lambda a + ((1-\lambda)d)^2 = 0.$$

Thus we see that

$$(2.13) \qquad b^2 - a^2 = c^2 - d^2.$$

On the other hand, we can simplify (2.11) by factoring out $(1-\lambda)$ to obtain

$$(b^2 - a^2 - c^2 + d^2) + \lambda((b-c)^2 - (a-d)^2) = 0.$$

Using (2.13), we thus have $(b-c)^2 - (a-d)^2 = 0$ which implies

$$(2.14) \qquad a + b = c + d$$

since $(b-a) + (d-c) > 0$. Combining (2.13), (2.14) and the fact that $b - a > 0$, $c - d < 0$, we obtain $b = -a$ and $d = -c$. Hence we only need to prove that

$$(2.15) \qquad \Phi(\lambda x - (1-\lambda)y) + (1 - \Phi(-\lambda x + (1-\lambda)y)) \ge \Phi(\lambda\theta_1 + (1-\lambda)\theta_2).$$

where $0 < x < \infty$, $0 < y < \infty$ and

$$(2.16) \qquad \Phi(x) - \Phi(-x) = \Phi(\theta_1) \qquad \Phi(-y) + (1 - \Phi(y) = \Phi(\theta_2).$$

Now using the isoperimetric inequality for Gaussian measures on the set $(-\infty, -y) \cup (y, \infty)$ enlarged by $(-\lambda(x+y), \lambda(x+y))$, we have

$$\begin{aligned}\Phi(\lambda x - (1-\lambda)y) + (1 - \Phi(-\lambda x + (1-\lambda)y)) &= \Phi(-y + \lambda(x+y)) + (1 - \Phi(y - \lambda(x+y))) \\ &\ge \Phi(\theta_2 + \lambda(x+y)).\end{aligned}$$

Hence (2.15) will hold if $\theta_2 + \lambda(x+y) \ge \lambda\theta_1 + (1-\lambda)\theta_2$, i.e. $x + y \ge \theta_1 - \theta_2$. This follows easily from (2.16) by noting $x \ge \theta_1$ and $y \ge -\theta_2$. Hence we finished the proof.

3 Proof of Theorem 1.

The one-dimensional result obtained in Lemma 1 can now be extended to general symmetrizations on $\mathbb{R}^n$.

LEMMA 2 Let T be a subspace of dimension $n-1$ in $\mathbb{R}^n, u \in T^\perp, \langle u, u\rangle = 1$, and $S(T,u)$ the 1-symmetrization about T in the direction u. If A and B^c are open or closed convex subsets in $\mathbb{R}^n$, then for $0 \le \lambda \le 1$

$$S(T,u)(\lambda A + (1-\lambda)B) \supseteq \lambda S(T,u)(A) + (1-\lambda)S(T,u)(B). \tag{3.1}$$

Proof. Let $f = S(T,u)$ to simplify notation and assume A, B are both open. Since f is a 1-symmetrization about T in the direction u, we have for each open subset of E that

$$f(E) = \bigcup_{x\in T} f(E \cap (x + T^\perp)) = \bigcup_{x \in T} (H(u, a(x)) \cap (x + T^\perp)) \tag{3.2}$$

where $H(u,a(x)) \cap (x+T^\perp)$ is a line segment such that

$$\gamma_{1,x+T^\perp}(H(u,a(x)) \cap (x+T^\perp)) = \gamma_{1,x+T^\perp}(E \cap (x + T^\perp)).$$

Similar statements hold for A, B and $\lambda A + (1-\lambda)B$ and hence by (2.2) we have

$$f((\lambda A + (1-\lambda)B) \cap (x+T^\perp)) \supseteq \lambda f(A \cap (x+T^\perp)) + (1-\lambda) f(B \cap (x + T^\perp)).$$

Thus (3.2) now imples (3.1) for 1-symmetrizations. If A or B is closed a similar argument works so the lemma is proved.

LEMMA 3 Let A, B be closed convex subsets of $\mathbb{R}^2$ such that A, B^c are both convex with $0 < \gamma_2(A) < 1$, $0 < \gamma_2(B) < 1$, and assume $\lambda A + (1-\lambda)B$ closed for $0 \le \lambda \le 1$. Let $f = S(\{0\}, u)$ where $\langle u, u \rangle = 1$. Then for $0 \le \lambda \le 1$

$$f(\lambda A + (1-\lambda)B) \supseteq \lambda f(A) + (1-\lambda) f(B), \tag{3.3}$$

and hence

$$\Phi^{-1} \circ \gamma_2(\lambda A + (1-\lambda)B) \ge \lambda \Phi^{-1} \circ \gamma_2(A) + (1-\lambda)\Phi^{-1} \circ \gamma_2(B). \tag{3.4}$$

REMARK If A is compact and B is closed then $\lambda A + (1-\lambda)B$ is easily seen to be closed for $0 \le \lambda \le 1$.

Proof. Since $0 < \gamma_2(A) < 1$, $0 < \gamma_2(B) < 1$ we have $0 < \gamma_2(\lambda A + (1-\lambda)B)$. If $\gamma_2(\lambda A + (1-\lambda)B) = 1$, then $f(\lambda A + (1-\lambda)B) = \mathbb{R}^2$ and (3.3) and (3.4) are obvious, so assume $0 < \gamma_2(\lambda A + (1-\lambda)B) < 1$.

Applying Theorem 1.6 in [2] we have a sequence of 1-symmetrizations $\{f_j : j \ge 1\}$ defined on $\mathbb{R}^2$ such that for each $\epsilon \in (0, 1/2)$ there exists $R(\epsilon) > 0$ such that for all closed sets F in R^2 with $\gamma_2(F) \in [\epsilon, 1-\epsilon]$ and $R \ge R(\epsilon)$ we have

$$\lim_{j\to\infty} \tilde{f}_j(F) \cap \{x : \langle x, x\rangle \le R\} = f(F) \cap \{x : \langle x, x \rangle \le R\} \tag{3.5}$$

where $\tilde{f}_j = f_j \circ f_{j-1} \circ \ldots \circ f_1$ and the convergence is in the Hausdorff metric. Furthermore, the convergence in (3.5) is uniform in all closed F with $\epsilon \le \gamma_2(F) \le 1-\epsilon$,

Now Lemma 2 implies

$$f_1(\lambda A + (1-\lambda)B) \supseteq \lambda f_1(A) + (1-\lambda) f_1(B).$$

and we can iterate the above set inclusion if $f_1(A)$ is convex and the complement of $f_1(B)$ is convex. To see this, observe that A open (closed) and convex implies $f_1(A)$ is also open (closed) and convex by Theorem 3.1 in [2]. If we use left symmetrization for B, and right symmetrization for B^c, then $f_1(B^c)$ is convex and $(f_1(B))^c = f_1(B^c)$. Hence $f_1(B^c)$ is still convex, so the set incusion can be iterated implying

$$f_j(\lambda A + (1-\lambda)B) \supseteq \lambda f_j(A) + (1-\lambda) f_j(B),$$

for all $j \ge 1$. Hence for all $R > 0$

$$\begin{aligned} & f_j(\lambda A + (1-\lambda)B) \cap \{x : \langle x, x\rangle \le R\} \\ & \qquad \supseteq \lambda f_j(A) \cap \{x : \langle x, x\rangle \le R\} + (1-\lambda) f_j(B) \cap \{x : \langle x, x\rangle \le R\}, \end{aligned} \tag{3.6}$$

and (3.5) and (3.6) combine to prove

$$\begin{aligned} & f(\lambda A + (1-\lambda)B) \cap \{x : \langle x, x\rangle \le R\} \\ & \qquad \supseteq \lambda f(A) \cap \{x : \langle x, x\rangle \le R\} + (1-\lambda) f(B) \cap \{x : \langle x, x\rangle \le R\} \end{aligned}$$

for all R sufficiently large. Letting $R \uparrow \infty$, we have (3.3), and since $f = S(\{0\}, u)$ this implies (3.4) as well. Hence Lemma 3 is proven.

LEMMA 4 Let $S = S(T, u)$ be a 2-symmetrization in $\mathbb{R}^n$. Let A, B be closed convex subsets of $\mathbb{R}^n$ such that A, B^c both convex with $0 < \gamma_2(A) < 1$, $0 < \gamma_2(B) < 1$, and $\lambda A + (1-\lambda)B$ closed for $0 \le \lambda \le 1$. Then for $0 \le \lambda \le 1$

$$S(\lambda A + (1-\lambda)B) \supseteq \lambda S(A) + (1-\lambda) S(B). \tag{3.7}$$

Proof. If $\lambda = 0$ or $\lambda = 1$, (3.7) is obvious so assume $0 < \lambda < 1$. Also $\gamma_n(\lambda A + (1-\lambda)B) = 1$ implies $S(\lambda A + (1-\lambda)B) = \mathbb{R}^n$, so (3.7) again is trivial. Hence we can assume $0 < \gamma_n(\lambda A + (1-\lambda)B) < 1$. Now

$$\begin{aligned} S(\lambda A + (1-\lambda)B) &= \bigcup_{x\in T} (S(\lambda A + (1-\lambda)B) \cap (x + T^\perp)) \\ &= \bigcup_{x \in T} S((\lambda A + (1-\lambda)B) \cap (x + T^\perp)) \end{aligned} \tag{3.8}$$

where $x + T^\perp$ is a 2-dimensional hyperplane through x and $S(\lambda A + (1-\lambda)B) \cap (x + T^\perp)$ is a 2-dimensional half-space of $(x + T^\perp)$ such that

$$\gamma_{2,x+T^\perp}(S(\lambda A + (1-\lambda)B)) = \gamma_{2,x+T^\perp}(\lambda A + (1-\lambda)B).$$

If $\gamma_{2,x+T^\perp}(\lambda A + (1-\lambda)B) = 0$ (or 1), then

$$S(\lambda A + (1-\lambda)B) \cap (x + T^\perp) = \phi \quad (\text{or} \quad x + T^\perp),$$

and hence Lemma 3 and (3.8) imply

$$S(\lambda A + (1-\lambda)B) \supseteq \bigcup_{x\in T} (\lambda S(A \cap (x+T^{\perp})) + (1-\lambda)S(B \cap (x+T^{\perp})).$$

Thus

$$\begin{aligned} S(\lambda A + (1-\lambda)B) &\supseteq \bigcup_{x\in T} (\lambda(S(A) \cap (x+T^{\perp})) + (1-\lambda)(S(B) \cap (x+T^{\perp})) \\ &= \bigcup_{x\in T} (\lambda S(A) + (1-\lambda)S(B)) \cap (x+T^{\perp})) \\ &= \lambda S(A) + (1-\lambda)S(B). \end{aligned} \tag{3.9}$$

Thus (3.7) holds and the lemma is proven.

LEMMA 5 Each n symmetrization in $\mathbb{R}^n$ can be written as the composition of $(n-1)$ 2-symmetrizations of $\mathbb{R}^n$ which point in the same direction. That is, given u with $\langle u,u\rangle = 1$ there exist subspaces $T_1, \cdots, T_{n-1}$ of dimension $n-2$ such that

$$S(\{0\}, u) = S_{n-1} \circ S_{n-2} \cdots \cdot S_2 \circ S_1, \tag{3.10}$$

where $S_i = S(T_i, u)$ for $i = 1, \cdots, n-1$.

Proof. Let $\alpha_1, \cdots, \alpha_n$ be an orthonormal basis of $\mathbb{R}^n$ with $\alpha_1 = u$ and define $T_i =$ span $(\{\alpha_2, \cdots, \alpha_n\} - \{\alpha_{i+1}\})$ for $i = 1, 2, \cdots n-1$. Then $T_2 =$ span $\{\alpha_2, \alpha_4, \cdots, \alpha_n\} \supseteq (T_1 + \mathbb{R}\alpha_1)^{\perp} =$ span $(\{\alpha_2\})$, and by Lemma 2.2 of [2] we have

$$S_2 \circ S_1 = S(T_1 \cap T_2, u).$$

Thus if $k < n-1$ we assume

$$S_{k-1} \circ \cdots \circ S_1 = S(T_1 \cap T_2 \cdots \cap T_{k-1}, u).$$

Now $(T_1 \cap \cdots \cap T_{k-1} + \mathbb{R}\alpha_1)^{\perp} =$ span $\{\alpha_2, \cdots, \alpha_k\}$ so

$$T_k = \text{span } \{\{\alpha_2, \cdots, \alpha_n\} - \{\alpha_{k+1}\}\} \supseteq (T_1 \cap \cdots \cap T_{k-1} + \mathbb{R}\alpha_1)^{\perp},$$

and hence by Lemma 2.2 of [2] we have

$$S_k \circ (S_{k-1} \circ \cdots \circ S_1) = S(T_1 \cap \cdots \cap T_{k-1} \cap T_k, u).$$

Hence

$$\begin{aligned} S_{n-1} \circ \cdots \circ S_1 &= S(T_1 \cap \cdots \cap T_{n-1}, u) \\ &= S(\{0\}, u), \end{aligned}$$

and (3.10) holds.

PROPOSITION 1 Let γ_n be the canonical Gaussian measure on $\mathbb{R}^n$ and A, B be non-empty Borel subsets of $\mathbb{R}^n$ such that A and B^c are convex. Then for $0 \le \lambda \le 1$

$$\Phi^{-1} \circ \gamma_n(\lambda A + (1-\lambda)B) \ge \lambda \Phi^{-1} \circ \gamma_n(A) + (1-\lambda)\Phi^{-1} \circ \gamma_n(B). \tag{3.11}$$

Proof. We can assume $0 < \lambda < 1$, $0 < \gamma_n(\lambda A + (1-\lambda)B) < 1$, $0 < \gamma_n(A) < 1$ and $0 < \gamma_n(B) < 1$, otherwise (3.11) holds trivially.

If A and B are both closed, let $A_r = A \cap \{x : \langle x, x\rangle \leq r\}$. Then A_r is convex and compact, and $\lambda A_r + (1-\lambda)B$ is closed. Now take r sufficiently large so that $0 < \gamma_n(A_r) < 1$ and let $f = S(\{0\}, u)$ denote the n-symmetrization in the direction u. Furthermore, we can express f as in (3.10), so Lemma 4 implies

$$S_1(\lambda A_r + (1-\lambda)B) \supseteq \lambda S_1(A_r) + (1-\lambda)S_1(B). \tag{3.12}$$

Using left symmetrization for B and right symmetrization for B^c as in the proof of Lemma 3, we see (3.12) can be iterated to obtain

$$S_2 \circ S_1(\lambda A_r + (1-\lambda)B) \supseteq \lambda S_2 \circ S_1(A_r) + (1-\lambda)S_2 \circ S_1(B).$$

Continuing, we obtain

$$f(\lambda A_r + (1-\lambda)B) \supseteq \lambda f(A_r) + (1-\lambda)f(B), \tag{3.13}$$

and hence that

$$\Phi^{-1} \circ \gamma_n(\lambda A_r + (1-\lambda)B) \geq \lambda \Phi^{-1} \circ \gamma_n(A_r) + (1-\lambda)\Phi^{-1} \circ \gamma_n(B). \tag{3.14}$$

Letting $r \to \infty$, we have (3.11) if A and B are closed with A, B^c both convex.

Now assume A and B^c are convex Borel subsets with $0 < \gamma_n(A) < 1$ and $0 < \gamma_n(B) < 1$. Fix $\epsilon > 0$. Then by [1] and Theorem 12.3 in [7] we can obtain P_ϵ compact, convex, and Q_ϵ closed with Q_ϵ^c convex such that $P_\epsilon \subseteq A$, $Q_\epsilon \subseteq B$ and $\lim_{\epsilon\to 0} \gamma_n(P_\epsilon) = \gamma_n(A)$, $\lim_{\epsilon \to 0} \gamma_n(Q_\epsilon) = \gamma_n(B)$. Hence applying (3.14) we have

$$\begin{aligned}\Phi^{-1} \circ \gamma_n(\lambda A + (1-\lambda)B) &\geq \Phi^{-1} \circ \gamma_n(\lambda P_\epsilon + (1-\lambda)Q_\epsilon) \\ &\geq \lambda\Phi^{-1} \circ \gamma_n(P_\epsilon) + (1-\lambda)\Phi^{-1} \circ \gamma_n(Q_\epsilon).\end{aligned} \tag{3.15}$$

Letting $\epsilon \to 0$ in (3.15), we obtain (3.11) and proposition 1 is proven.

Proof of Theorem 1. The proof will follow the proof of Theorem 3.3 in [2], but has some modifications since the set $\lambda A + (1-\lambda)B$ need not be convex.

Using the notation of Lemma 2.1 in [4] we consider the series $\sum_{j\geq 1} \alpha_j(x) S\alpha_j$, $x \in E$, $\alpha_j \in E^*$, where $\{\alpha_j, j \geq 1\}$ are i.i.d. $N(0,1)$ with respect to μ and $\{S\alpha_j : j \geq 1\}$ is complete and orthonormal in the Hilbert space H_μ which generates μ. Let $\Pi_n(x) = \sum_{j=1}^n \alpha_j(x) S\alpha_j$, $Q_n(x) = x - \Pi_n(x)$ and set $\mathcal{F}_n = \sigma\{\alpha_j : 1 \leq j \leq n\}$ for $n \geq 1$. If $\sigma(\alpha_j : j \geq 1)$ is the minimal σ-field generated by $\cup_{n\geq 1}\mathcal{F}_n$, then it is well known that $\sigma(\alpha_j : j \geq 1)$ restricted to $\overline{H}_\mu$ is the Borel subsets of $\overline{H}_\mu$, and that $\mu(\overline{H}_\mu) = 1$. To simplify notation we thus assume $\overline{H}_\mu = E$, and it is easy to see this is no loss of generality.

Thus if C is a Borel subset of E, and I_C its indicator function, we define

$$I_{C,n}(x) = \int_E I_C(\Pi_n(x) + y)\, d\mu^{Q_n}(y)$$

where $\mu^{Q_n}(F) = \mu(Q_n^{-1}(F))$ for any Borel set F. Since the $\{\alpha_j : j \geq 1\}$ are independent, $I_{C,n} = E(I_C | \mathcal{F}_n)$ and hence w.p.1 and in $L^1(\mu)$ we have $\lim_{n\to\infty} I_{C,n} = I_C$. Now assume A, B Borel sets with A and B^c convex in E. Then, for $0 \leq \lambda \leq 1$, we have

$$\{I_{\lambda A + (1-\lambda)B, n} > 1/4\} \supseteq \lambda\{I_{A,n} > 3/4\} + (1-\lambda)\{I_{B,n} > 3/4\}. \tag{3.16}$$

To check (3.16) take a, b such that $I_{A,n}(a) > 3/4$ and $I_{B,n}(b) > 3/4$. Then $I_A(\Pi_n(a) + y) = 1$ and $I_B(\Pi_n(b) + y) = 1$ with μ^{Q_n} probability greater than 3/4, and hence

$$I_A(\Pi_n(a) + y) \cdot I_B(\Pi_n(b) + y) = 1 \tag{3.17}$$

with μ^{Q_n} probability greater than 1/4. Now

$$I_{\lambda A+(1-\lambda)B}(\lambda u + (1-\lambda)v) \geq I_A(u) \cdot I_B(v)$$

regardless of A and B, and hence

$$\begin{aligned} I_{\lambda A+(1-\lambda)B,n}(\lambda a + (1-\lambda)b) &= \int I_{\lambda A+(1-\lambda)B}(\Pi_n(\lambda a + (1-\lambda)b) + y) d\mu^{Q_n}(y) \\ &\geq \int I_A(\Pi_n(a) + y) \cdot I_B(\Pi_n(b) + y) d\mu^{Q_n}(y) \\ &> 1/4 \end{aligned}$$

since (3.17) holds with μ^{Q_n} probability greater than 1/4. Hence (3.16) holds. Using log-concavity of μ^{Q_n} as below, we can strengthen (3.16) in that 1/4 and 3/4 can all be taken to be 1/2. However, the argument used above is valid for any measure, and is all that is needed for (3.19) and the completion of the proof.

The next step of the proof is to show $\tilde{A}_n = \{I_{A,n} > 3/4\}$ is convex in E, and $\tilde{B}_n = \{I_{B,n} > 3/4\}$ is the complement of a convex subset of E. To see this first observe that $I_{A,n}(x) = \mu^{Q_n}(A - \Pi_n(x))$, and hence for $0 \leq \lambda \leq 1$ we have

$$\begin{aligned} I_{A,n}(\lambda x_1 + (1-\lambda)x_2) &= \mu^{Q_n}(A - \Pi_n(\lambda x_1 + (1-\lambda)x_2)) \\ &= \mu^{Q_n}(\lambda(A - \Pi_n(x_1)) + (1-\lambda)(A - \Pi_n(x_2))) \\ &\geq (\mu^{Q_n}(A - \Pi_n(x_1)))^{\lambda}(\mu^{Q_n}(A - \Pi_n(x_2)))^{1-\lambda} \end{aligned} \tag{3.18}$$

where the equality holds since A is convex and Π_n is linear, and the inequality holds since μ^{Q_n} is Gaussian, and hence log-concave. Hence (3.18) implies $\tilde{A}_n$ is convex and similarly, since B^c is convex, $\{I_{B^c,n} > 1/4\}$ is also convex in E. Now $I_{B,n} = 1 - I_{B^c,n}$, so $\tilde{B}_n = \{I_{B^c,n} \leq 1/4\}$, and hence $\tilde{B}_n$ is the complement of a convex subset of E.

Now $\tilde{A}_n$ and $\tilde{B}_n$ are both determined by x only through $\Pi_n(x)$, so $\tilde{A}_n$ and $\tilde{B}_n$ are $\mathcal{F}_n$-measurable with $\mu^{\Pi_n}(\tilde{A}_n) = \mu(\tilde{A}_n)$, and $\mu^{\Pi_n}(\tilde{B}_n) = \mu(\tilde{B}_n)$. Thus, since (E, μ^{Π_n}) is linearly isomorphic to $(\mathbb{R}^n, \gamma_n)$ via the map $x \to (\alpha_1(x), \ldots, \alpha_n(x))$, applying Proposition 1 to $\tilde{A}_n$ and $\tilde{B}_n$ we have by integrating (3.16) with respect to μ that

$$\begin{aligned} &\Phi^{-1} \circ \mu(x : I_{\lambda A+(1-\lambda)B,n}(x) > 1/4) \\ &\geq \Phi^{-1} \circ \mu(\lambda \tilde{A}_n + (1-\lambda)\tilde{B}_n) \\ &= \Phi^{-1} \circ \mu^{\Pi_n}(\lambda \tilde{A}_n + (1-\lambda)\tilde{B}_n) \\ &\geq \lambda \Phi^{-1} \circ \mu^{\Pi_n}(\tilde{A}_n) + (1-\lambda)\Phi^{-1} \circ \mu^{\Pi_n}(\tilde{B}_n) \\ &= \lambda \Phi^{-1} \circ \mu(\tilde{A}_n) + (1-\lambda)\Phi^{-1} \circ \mu(B_n). \end{aligned} \tag{3.19}$$

Letting $n \to \infty$ in (3.19) we see that

$$\begin{aligned} &\Phi^{-1} \circ \mu(x : I_{\lambda A+(1-\lambda)B}(x) > 1/4) \\ &\quad \geq \lambda \Phi^{-1} \circ \mu(x : I_A(x) > 3/4) + (1-\lambda)\Phi^{-1} \circ \mu(x : I_B(x) > 3/4), \end{aligned}$$

and hence the theorem is proven.

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