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Small Ball Probabilities for Gaussian Processes with Stationary Increments Under Hölder Norms[1]

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Small ball probabilities are estimated for Gaussian processes with stationary increments when the small balls are given by various Hölder norms. As an application we establish results related to Chung's functional law of the iterated logarithm for fractional Brownian motion under Hölder norms. In particular, we identify the points approached slowest in the functional law of the iterated logarithm.

KEY WORDS: Gaussian processes; small ball probabilities; the law of the iterated logarithm.

1. INTRODUCTION

Let $\{X(t): 0 \leqslant t \leqslant 1\}$ be a separable centered Gaussian process with stationary increments and assume $X(0) = 0$. If

$$\sigma^2(t) = E(X^2(t)) \qquad (t > 0) \tag{1.1}$$

then stationarity of increments and $X(0) = 0$ imply $E((X(t+h) - X(t))^2) = \sigma^2(h)$ for $t \geqslant 0$, $h \geqslant 0$. Throughout $C_0[0, 1]$ denotes the continuous functions on $[0, 1]$ with value zero at the origin, and for $x \in C_0[0, 1]$, and f a nondecreasing strictly positive function on $(0, 1]$ satisfying $f(0) = 0$, we define

$$\lambda_f(x) = \sup_{0 \leqslant s \neq t \leqslant 1} |x(t) - x(s)|/f(|t-s|) \tag{1.2}$$

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Of course, if $f(t) = t^q$, $0 \leqslant q \leqslant 1$, then λ_f is a classical Hölder norm, and we will abbreviate this by writing $\lambda_q(\cdot)$ instead of $\lambda_f(\cdot)$. In particular, $\lambda_0(x) = \max_{0 \leqslant t \leqslant 1} x(t) - \min_{0 \leqslant t \leqslant 1} x(t)$ and perhaps it should be mentioned explicitly that these quantities are norms on $C_0[0, 1]$, but not on $C[0, 1]$.

In Section 2 we consider small ball probabilities of the form

$$P(\lambda_f(X) \leqslant \sigma(x)/f(x)) \qquad \text{as} \quad x \to 0 \tag{1.3}$$

When X is standard Brownian motion, results for λ_q, $0 < q < 1/2$, have been obtained in Refs. 1 and 6. However, what we prove here applies easily to fractional Brownian motion as well, and to norms other than the classical Hölder norms. When $q = 0$, results of this type are related to those obtained previously in Refs. 10 and 11, where small ball probabilities for Gaussian processes are studied under the sup-norm.

In Section 3, we present a detailed version of Chung's functional law of the iterated logarithm for fractional Brownian motions using general Hölder norms. In particular, we generalize the classical results for Brownian motion,[2, 3] and present a fairly complete picture of what happens when the limiting function is a boundary point of the limit set in Strassen's FLIL. Results of this type were studied in the sup-norm case for Brownian motion by Grill,[5] and in terms of lim-inf results for general Gaussian samples in Ref. 8. These later results were employed in Ref. 6, to study similar problems for Hölder norms applied to Brownian motion and the Brownian sheet, and in Monrad and Rootzén[10] to obtain results for fractional Brownian motion under the sup-norm. Some further comments will be included when these results are stated in Section 3, but for now we mention that FLIL results of Monrad and Rootzén[10] for fractional Brownian motion follow from those later by setting $\lambda_f = \lambda_0$.

2. SMALL BALL PROBABILITIES FOR $\lambda_f(X)$

The main results for small ball probabilities are the following two theorems.

Theorem 2.1. Let $\{X(t): 0 \leqslant t \leqslant 1\}$ be a separable centered Gaussian process with $X(0) = 0$, and having stationary increments. Let $\sigma(\cdot)$ be given by (1.1), and assume λ_f is as in (1.2) where f is a nondecreasing, strictly positive function on $(0, 1]$ satisfying $f(0) = 0$. Then:

(i) $\sigma^2(h)$ concave on $[0, 1]$ implies that

$$P(\lambda_f(X) \leqslant \sigma(x)/f(x)) \leqslant \exp\{-0.17[1/x]\} \tag{2.1}$$

where $[\cdot]$ denotes the greatest integer function.

(ii) If $\sigma^2(2x) \leqslant \theta\sigma^2(x)$ for $0 \leqslant x \leqslant 1/2$, and some $\theta \in (0, 4]$, and

$$6\sigma^2(jx) + \sigma^2((j+2)x) + \sigma^2((j-2)x) \geqslant 4\sigma^2((j+1)x) + 4\sigma^2((j-1)x) \quad (2.2)$$

for $0 < x < 1$ and $2 \leqslant j \leqslant 1/x - 2$, then

$$P(\lambda_f(X) \leqslant \sigma(x)/f(x)) \leqslant \exp\{-[1/2x]\, ln(1/\Phi(2/\sqrt{4-\theta}))\} \quad (2.3)$$

where $\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2}\, du$.

Theorem 2.2 presents the companion lower bounds for the upper bounds of Theorem 2.1.

Theorem 2.2. Assume $\{X(t): 0 \leqslant t \leqslant 1\}$ and λ_f are as in Theorem 2.1, and that $\sigma(x)/(x^\beta f(x))$ is nondecreasing on $(0, 1]$ for some $\beta > 0$. Then

$$P(\lambda_f(X) \leqslant \sigma(x)/f(x)) \geqslant \exp\{-c(\beta)/x\} \quad (2.4)$$

where $c(\beta) > 0$ is an absolute constant depending only on β.

Remarks. (1) The proofs will show that some improvement for various constants is possible, but exact constants are unknown. Hence we stated things with simplicity in mind. It also follows from the proofs that estimates analogous to those in Theorems 2.1 and 2.2 will hold for

$$P(\sup_{0 < s \leqslant 1} |(X(s)|/f(s) \leqslant \sigma(x))$$

but they are not included.

(2) The application of Theorems 2.1 and 2.2 to fractional Brownian motion will be discussed in Section 3.

Proof of Theorem 2.1. The proof of Theorem 2.1 depends on Slepian's lemma which can be found, for example, in [Tong,(13) p. 10], or the recent book by Ledoux and Talagrand.(9)

It is easy to see that

$$P(\lambda_f(X) \leqslant \sigma(x)/f(x)) \leqslant P(\max_{1 \leqslant i \leqslant 1/x} |X(ix) - X((i-1)x)| \leqslant \sigma(x)) \quad (2.5)$$

Hence put

$$\xi_i = X(ix) - X((i-1)x), \qquad i \geqslant 1$$

Then $E(\xi_i^2) = \sigma^2(x)$ for $i \geqslant 1$, and if $\sigma^2(\cdot)$ is concave it follows fairly easily that $E(\xi_i \xi_j) \leqslant 0$ for all $i \neq j$. Therefore,

$$\begin{aligned}
P(\max_{1 \leqslant i \leqslant 1/x} &|X(ix) - X((i-1)x)| \leqslant \sigma(x)) \\
&\leqslant P(\max_{1 \leqslant i \leqslant 1/x} \xi_i \leqslant \sigma(x)) \\
&\leqslant \prod_{i=1}^{[1/x]} P(\xi_i \leqslant \sigma(x)) \\
&= (\Phi(1))^{[1/x]} \\
&\leqslant \exp\{-0.17[1/x]\}
\end{aligned} \tag{2.6}$$

where the second inequality above follows from Slepian's lemma. Combining (2.5) and (2.6) yields (2.1), so now we turn to (2.3).

If $\sigma^2(2x) \leqslant \theta\sigma^2(x)$ for $0 \leqslant x \leqslant 1/2$ and $\theta \in (0, 4)$ and (2.2) is satisfied, we let

$$\eta_i = \xi_{2i} - \xi_{2i-1}, \qquad 1 \leqslant i \leqslant 1/(2x)$$

Here ξ_i is as before. Then a direct calculation shows that

$$E(\eta_i^2) = 4\sigma^2(x) - \sigma^2(2x) \geqslant (4-\theta)\, \sigma^2(x), \qquad 1 \leqslant i \leqslant 1/(2x)$$

and

$$\begin{aligned}
E(\eta_i\eta_j) = &-1/2(6\sigma^2(2\,|j-i|\,x) + \sigma^2((2\,|j-i|+2)x) + \sigma^2((2\,|j-i|-2)x) \\
&-4\sigma^2((2\,|j-i|+1)x) - 4\sigma^2((2\,|j-i|-i)x)) \\
\leqslant{}& 0
\end{aligned}$$

for every $1 \leqslant i \neq j \leqslant 1/(2x)$. Hence Slepian's lemma can be applied again, and we obtain

$$\begin{aligned}
P(\max_{1 \leqslant i \leqslant 1/x} &|X(ix) - X((i-1)x)| \leqslant \sigma(x)) \\
&\leqslant P(\max_{1 \leqslant i \leqslant 1/(2x)} |\eta_i| \leqslant 2\sigma(x)) \\
&\leqslant P(\max_{1 \leqslant i \leqslant 1/(2x)} \eta_i \leqslant 2\sigma(x)) \\
&= \prod_{i=1}^{[1/2x]} P(\eta_i \leqslant 2\sigma(x)) \\
&= \prod_{i=1}^{[1/2x]} \Phi(2\sigma(x)/(E(\eta_i^2))^{1/2}) \\
&\leqslant \prod_{i=1}^{[1/2x]} \Phi(2\sigma(x)/((4-\theta)\,\sigma^2(x))^{1/2}) \\
&= \exp\{-[1/2x] \ln \Phi(2/(4-\theta)^{1/2})\}
\end{aligned} \tag{2.7}$$

Combining (2.5) and (2.7) yields (2.3), and hence Theorem 2.1 is proved.

Proof of Theorem 2.2. First we observe that

$$\lambda_f(X) \leqslant \max\{ \sup_{\substack{0 \leqslant s \neq t \leqslant 1 \\ |s-t| \geqslant x}} |X(t)-X(s)|/f\,|t-s|),$$

$$\sup_{\substack{0 \leqslant s \neq t \leqslant 1 \\ |s-t| \leqslant x}} |X(t)-X(s)|/f\,|t-s|)\}$$

$$\leqslant \max\{2 \sup_{0 \leqslant s \leqslant 1} |X(s)-X(0)|/f(x),$$

$$\sup_{\substack{0 \leqslant s \leqslant 1 \\ 0 \leqslant t \leqslant x \\ s+t \leqslant 1}} |X(t+s)-X(s)|/f(t)\} \tag{2.8}$$

and understanding henceforth that $s+t \leqslant 1$ we have

$$\sup_{\substack{0 \leqslant s \leqslant 1 \\ 0 \leqslant t \leqslant x}} |X(t+s)-X(s)|/f(t)$$

$$\leqslant \sup_{0 \leqslant s \leqslant 1} \max_{j \geqslant 0} \sup_{x2^{-j-1} \leqslant t \leqslant x2^{-j}} |X(t+s)-X(s)|/f(t)$$

$$\leqslant \max_{j \geqslant 0} \sup_{0 \leqslant s \leqslant 1} \sup_{0 \leqslant t \leqslant x2^{-j}} |X(t+s)-X(s)|/f(x2^{-j-1})$$

$$\leqslant 3 \max_{j \geqslant 0} \max_{0 \leqslant i \leqslant 2^j x} \sup_{0 \leqslant t \leqslant x2^{-j}} |X(t+ix2^{-j})$$

$$-X(ix2^{-j})|/f(x2^{-j-1}) \tag{2.9}$$

Now for each $0<s<1$ we can write

$$s = \sum_{l=1}^{\infty} \varepsilon_l 2^{-l}$$

where $\varepsilon_l = 0$ or 1. Hence

$$\sup_{0 \leqslant s \leqslant 1} |X(s)-X(0)| \leqslant \sum_{l=1}^{\infty} \max_{1 \leqslant i \leqslant 2^l} |X(i2^{-l})-X((i-1)\,2^{-l})| \equiv I_1(X) \tag{2.10}$$

and

$$\max_{j \geqslant 0} \max_{0 \leqslant i \leqslant 2^j/x} \sup_{0 \leqslant t \leqslant x2^{-j}} |X(t+ix2^{-j})-X(ix2^{-j})|/f(x2^{-j-1})$$

$$\leqslant \max_{j \geqslant 0} \max_{0 \leqslant i \leqslant 2^j/x} \sum_{l=j+1}^{\infty} \max_{0 \leqslant m \leqslant 2^{l-j}} |X((m+1)\,x2^{-l}+ix2^{-j})$$

$$-X(mx2^{-l}+ix2^{-j})|/f(x2^{-j-1})$$

$$\equiv I_2(X) \tag{2.11}$$

From (2.8)–(2.11) we obtain

$$P(\lambda_f(X) \leqslant \sigma(x)/f(x)) \geqslant P(I_1(X) \leqslant \sigma(x)/2 \text{ and } I_2(X) \leqslant \sigma(x)/(3f(x))) \quad (2.12)$$

Let n_0 be an integer such that

$$1/x \leqslant 2^{n_0} \leqslant 2/x$$

Define

$$x_l = \sigma((3/2)^{-|l-n_0|}\, x)(1-2^{-\beta/2})/4 \qquad l = 1, 2, \ldots$$

$$y_{j,l} = \tfrac{1}{3}\sigma(x2^{-l}) \times 2^{\beta(l-j)/2} 2^{j\beta}(1-2^{-\beta/2})/f(x2^{-j-1}) \qquad j \geqslant 0,\ l \geqslant j+1$$

Since $\sigma(x)/(x^\beta f(x))$ is nondecreasing, we have for $0 < a < 1$ that $\sigma(ax)/f(ax) \leqslant a^\beta \sigma(x)/f(x)$. Hence

$$\begin{aligned}\sum_{l=1}^{\infty} x_l &\leqslant \sum_{l=1}^{\infty} (3/2)^{-|l-n_0|\beta}\sigma(x)(1-2^{-\beta/2})/4 \\ &\leqslant \sum_{l=0}^{\infty} (3/2)^{-l\beta}\, \sigma(x)(1-2^{-\beta/2})/2 \\ &= \sigma(x)(1-2^{-\beta/2})/(2(1-(2/3)^\beta)) \\ &\leqslant \sigma(x)/2 \end{aligned} \quad (2.13)$$

and

$$\begin{aligned}\sum_{l=j+1}^{\infty} y_{j,l} &= \frac{1}{3}\sum_{l=j+1}^{\infty} \sigma(x2^{-(l-j-1)}2^{-j-1})\, 2^{\beta(l-j)/2+j\beta}(1-2^{-\beta/2})/f(x2^{-j-1}) \\ &\leqslant \frac{1}{3}\sum_{l=j+1}^{\infty} \frac{\sigma(x2^{-j-1})}{f(x2^{-j-1})}(1-2^{-\beta/2})\, 2^{-\beta(l-j-1)+\beta(l-j)/2+j\beta} \\ &= 1/3(1-2^{-\beta/2})\frac{\sigma(x2^{-j-1})}{f(x2^{-j-1})} 2^{(j+1)\beta}2^{-\beta/2}/(1-2^{-\beta/2}) \\ &\leqslant \sigma(x)/(3f(x)) \end{aligned} \quad (2.14)$$

Combining (2.10)–(2.14) we thus have

$$\begin{aligned}
&P(\lambda_f(X) \leqslant \sigma(x)/f(x)) \\
&\quad \geqslant P\Bigg(\max_{1 \leqslant i \leqslant 2^l} |X(i2^{-l}) - X((i-1)\,2^{-l})| \leqslant x_l,\ l \geqslant 1, \text{ and} \\
&\quad \max_{0 \leqslant i \leqslant 2^j/x} \ \max_{1 \leqslant m \leqslant 2^{l-j}} \frac{|X((m+1)\,2^{-l}x + ix2^{-j}) - X(m2^{-l}x + ix2^{-j})|}{f(x2^{-j-1})} \\
&\quad \leqslant y_{j,l} \text{ for all } l \geqslant j+1,\ j \geqslant 0 \Bigg) \\
&\quad \geqslant A \cdot B
\end{aligned} \tag{2.15}$$

where

$$A = \prod_{l=1}^{\infty} \prod_{i \leqslant i \leqslant 2^l} P(|X(i2^{-l}) - X((i-1)\,2^{-l}| \leqslant x_l)$$

and

$$\begin{aligned}
B = \prod_{j=0}^{\infty} \prod_{l=j+1}^{\infty} \prod_{m=1}^{2^{l-j}} \prod_{i=0}^{2^j/x} & P(|X((m+1)\,2^{-l}x + ix2^{-j}) \\
& - X(m2^{-l}x + ix2^{-j})|/f(x2^{-j-1}) \leqslant y_{i,l})
\end{aligned}$$

by Sidak's theorem (Šidák,[12] Corollary 3). If Z is $N(0, 1)$ then

$$A = \prod_{l=1}^{\infty} (P(|Z| \leqslant \sigma((3/2)^{-|l-n_0|}\,x)(1 - 2^{-\beta/2})/(4\sigma(2^{-l}))))^{2^l}$$

and

$$B = \prod_{j=0}^{\infty} \prod_{l=j+1}^{\infty} (P(|Z| \leqslant (1/3)\,2^{\beta(l-j)/2 + j\beta}(1 - 2^{-\alpha/2}))^{2^l/x}$$

To estimate A and B we note that

$$\begin{aligned}
&\text{(i)} \quad P(|Z| \leqslant t) \geqslant t/2 \qquad \text{if} \quad 0 \leqslant t \leqslant 1 \\
&\text{(ii)} \quad P(|Z| \leqslant st) \geqslant \exp\{-\theta(s)\, e^{-(st)^2/2}\} \qquad \text{if} \quad s > 0,\ t \geqslant 1
\end{aligned} \tag{2.16}$$

where $\theta(s) = (1 - e^{-s^2/2})^{-1}$.

Thus by rewriting A we obtain

$$A = \prod_{l=1}^{n_0} (P(|Z| \leqslant \sigma((3/2)^{-(n_0-l)}\,x)(1 - 2^{-\beta/2})/(4\sigma(2^{-l}))))^{2^l} \cdot C \tag{2.17}$$

where

$$C=\prod_{l=n_0+1}^{\infty}(P(|Z|\leqslant\sigma((3/2)^{-(l-n_0)}x)(1-2^{-\beta/2})/(4\sigma(2^{-l}))))^{2^l} \tag{2.18}$$

Thus, recalling $2^{-n_0}\leqslant x$ and $\sigma(\cdot)$ is nondecreasing,

$$A\geqslant C\prod_{l=1}^{n_0}(P(|Z|\leqslant\sigma((3/2)^{-(n_0-l)}2^{-n_0})(1-2^{\beta/2})/(4\sigma(2^{-l}))))^{2^l}$$

$$\geqslant C\prod_{l=1}^{n_0}(P(|Z|\leqslant\tfrac{1}{4}(\tfrac{1}{3})^{n_0-l}(1-2^{\beta/2})))^{2^l}$$

since Minkowski's inequality implies $3^{n_0-l}\sigma((3/2)^{-(n_0-l)}2^{-n_0})\geqslant\sigma(2^{-l})$

$$\geqslant C\prod_{l=1}^{n_0}(\tfrac{1}{8}(\tfrac{1}{3})^{n_0-l}(1-2^{-\beta/2}))^{2^l}$$

by (2.16-i)

$$=C\exp\left\{-\sum_{l=1}^{n_0}2^l(\ln(8/(1-2^{\beta/2}))+(n_0-l)\ln 3)\right\}$$

$$=C\exp\left\{-2^{n_0}\sum_{l=1}^{n_0}2^{-(n_0-l)}(\ln(8/(1-2^{-\beta/2}))+(n_0-l)\ln 3)\right\}$$

$$=C\exp\{-2^{n_0}c_1(\beta)\} \tag{2.19}$$

where $c_1(\beta)>0$ is an absolute constant depending only an $\beta>0$. Now (2.18), $\sigma(\cdot)$ nondecreasing, and $x2^{n_0}\geqslant 1$, together imply

$$C\geqslant\prod_{l=n_0+1}^{\infty}(P(|Z|\leqslant\sigma((2/3)^{l-n_0}x)(1-2^{\beta/2})/(4\sigma((1/2)^{l-n_0}x))))^{2^l}$$

$$\geqslant\prod_{l=n_0+1}^{\infty}P(|Z|\leqslant\tfrac{1}{4}(1-2^{-\beta/2})(4/3)^{\beta(l-n_0)})^{2^l}$$

where the second inequality results from $\sigma(ax)\leqslant a^{\beta}\sigma(x)$ when $0<a<1$ and $x>0$. Hence by (2.16-ii)

$$C\geqslant\exp\left\{-\sum_{l=n_0+1}^{\infty}2^l\theta(1/4(1-2^{-\beta/2}))\,e^{-(1-2^{-\beta/2})^2(4/3)^{2\beta(l-n_0)}/32}\right\}$$

Letting $c_2(\beta)=\theta(1/4(1-2^{-\beta/2}))$ we thus have

$$
\begin{aligned}
C &\geqslant \exp\left\{-2^{n_0}c_2(\beta)\sum_{k=1}^{\infty} 2^k \exp\{-(1-2^{-\beta/2})^2\,(4/3)^{2\beta k}/32\}\right\}\\
&\geqslant \exp\{-c_3(\beta)\,2^{n_0}\}
\end{aligned}
$$

where $c_3(\beta)>0$ is an absolute constant depending only on $\beta>0$. Hence, with $c_4(\beta)=c_1(\beta)+c_3(\beta)$, we have

$$
A \geqslant \exp\{-c_4(\beta)\,2^{n_0}\} \tag{2.20}
$$

Now we turn to estimating B by again using (2.16-ii). This yields

$$
\begin{aligned}
B &\geqslant \prod_{j=0}^{\infty}\prod_{l=j+1}^{\infty} \exp\left\{-\frac{2^l}{x}\,\theta((1-2^{-\beta/2})/3)\right.\\
&\qquad\left.\times \exp\left\{-\frac{1}{18}(1-2^{-\beta/2})^2\,2^{\beta(l-j)+2\beta j}\right\}\right\}\\
&= \exp\left\{-\theta((1-2^{-\beta/2})/3)\Big/x \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} 2^{k+j}\right.\\
&\qquad\left.\times \exp\left\{-\frac{1}{18}(1-2^{-\beta/2})^2\,2^{\beta k+2\beta j}\right\}\right\}\\
&= \exp\{-c_5(\beta)/x\}
\end{aligned}
$$

Combining (2.15), (2.20) and (2.21), and taking $c(\beta)=2c_4(\beta)+c_5(\beta)$ we have (2.4), and the theorem is proved.

3. FRACTIONAL BROWNIAN MOTION AND HÖLDER NORMS

Throughout this section $\{X(t)\colon t\geqslant 0\}$ denotes α-fractional Brownian motion with $X(0)=0$ and $0<\alpha<1$. Then $\{X(t)\colon t\geqslant 0\}$ has covariance function

$$
E(X(s)\,X(t))=\tfrac{1}{2}(s^{2\alpha}+t^{2\alpha}-|s-t|^{2\alpha}) \tag{3.1}
$$

for $s, t\geqslant 0$, and representation

$$
X(t)=\int_{\mathbb{R}^1}\frac{1}{k_\alpha}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}\,dB(x) \tag{3.2}
$$

where

$$\text{(i)}\quad k_\alpha^2 = \int_{\mathbb{R}^1} (|x-1|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2})^2\, dx$$

$$\text{(ii)}\quad (B(t)\colon -\infty < t < \infty) \text{ is Brownian motion, and} \tag{3.3}$$

$$\text{(iii)}\quad \frac{1}{k_\alpha} (|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}) \text{ is interpreted to be } I_{(0,t]} \text{ when } \alpha = 1/2.$$

In particular, $\{X(t)\colon t \geqslant 0\}$ has stationary increments with

$$E(X^2(t)) = \sigma^2(t) = t^{2\alpha}, \qquad t \geqslant 0 \tag{3.4}$$

and is standard Brownian motion when $\alpha = 1/2$. The limit set associated with functional laws of the iterated logarithm for $\{(X(t)\colon t \geqslant 0)\}$ is K_α, the subset of functions in $C_0[0, 1]$ of the form

$$f(t) = \int_{\mathbb{R}^1} \frac{1}{k_\alpha} (|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2})\, g(x)\, dx, \qquad 0 \leqslant t \leqslant 1 \tag{3.5}$$

In (3.5) the function $g(\cdot)$ ranges over the unit ball of $L^2(\mathbb{R}^1)$, and hence

$$\int_{\mathbb{R}^1} g^2(x)\, dx \leqslant 1 \tag{3.6}$$

Take $0 \leqslant q < \alpha$ and set

$$H_{q,0} = \{f \in C_0[0, 1]\colon \lim_{\delta \to 0} \sup_{\substack{s,t \in [0,1] \\ 0 \leqslant |s-t| \leqslant \delta}} |f(t) - f(s)|/|t-s|^q = 0\} \tag{3.7}$$

If $f \in K_\alpha$, then the Cauchy-Schwartz inequality and a change of variables easily implies $|f(t) - f(s)| \leqslant |t-s|^\alpha$ for all $s, t \in [0, 1]$, and hence for $0 \leqslant q < \alpha$ we see $K_\alpha \subset H_{q,0}$. Furthermore, the set $K = K_\alpha$ is the unit ball of the Hilbert space H_μ which generates the Gaussian measure $\mu = \mathscr{L}(X)$ on the separable Banach space $C_0[0, 1]$ under the norm λ_0 or the sup-norm. The next lemma yields even more, and shows H_μ actually generates μ on the real separable Banach space $(H_{q,0}, \lambda_q)$ provided $0 \leqslant q < \alpha$. It also gives the small ball probability for $\lambda_q(X)$.

Lemma 3.1. If $\{X(t)\colon 0 \leqslant t \leqslant 1\}$ is α-fractional Brownian motion and $0 \leqslant q < \alpha$, then

$$P(X \in H_{q,0}) = 1 \tag{3.8}$$

and there exist constants $0<c<C<\infty$ such that as $\varepsilon \downarrow 0$

$$-C\varepsilon^{-1/(\alpha-q)} \leqslant \log P((\lambda_q(X) \leqslant \varepsilon) \leqslant -c\varepsilon^{-1/(\alpha-q)} \tag{3.9}$$

Remark. We abbreviate (3.9) by writing $\log P(\lambda_f(X) \leqslant \varepsilon) \approx -\varepsilon^{-1/(\alpha-q)}$.

Proof. Take $q' \in (q, \alpha)$. Since α-fractional Brownian motion has stationary increments with $\sigma^2(t)=t^{2\alpha}$ both Theorems 2.1 and 2.2 apply to α-fractional Brownian motion. In particular, Theorem 2.2 implies $P(\lambda_{q'}(X)<\infty)>0$, and hence by the zero-one law for Gaussian norms [Fernique[4]] we have

$$P(\lambda_{q'}(X)<\infty)=1$$

Now $q'>q$ and an easy calculation yields (3.8). Furthermore, when $f(x)=x^q$ with $0 \leqslant q<\alpha$, then $\sigma(x)/f(x)=x^{\alpha-q}$, and Theorems 2.1 and 2.2 combine to imply (3.9). Hence the lemma is proved.

If $f \in H_\mu$, the Hilbert space generating $\mu=\mathscr{L}(X)$ on $H_{q,0}$, then $\|f\|_\mu$ denotes the H_μ-norm of f, and we point out the well known fact that H_μ is a subspace of $H_{q,0}$ with $\lambda_{q(f)} \leqslant c\,\|f\|_\mu$ for some $c<\infty$ and all $f \in H_\mu$. The next lemma describes the behavior of translates of small balls by elements in H_μ.

Lemma 3.2. If $f \in H_\mu$, $0 \leqslant q<\alpha$, and X is α-fractional Brownian motion, then

$$\begin{aligned}-\tfrac{1}{2}\|f\|_\mu^2+\log P(\lambda_q(X) \leqslant \varepsilon) &\leqslant \log P(\lambda_q(X-f) \leqslant \varepsilon)\\ &\leqslant \log P(\lambda_q(X) \leqslant \varepsilon)-\tfrac{1}{2}\|f_\varepsilon\|_\mu^2\end{aligned} \tag{3.10}$$

where f_ε is the unique element of H_μ such that

$$\|f_\varepsilon\|_\mu=\inf\{\|g\|_\mu : \lambda_q(f-g) \leqslant \varepsilon\} \tag{3.11}$$

Furthermore, we have

$$\lim_{\varepsilon \to 0} \|f_\varepsilon\|_\mu=\|f\|_\mu$$

and if c, C are the constants in (3.9), then for $f \in H_\mu$, $r>0$

$$\varliminf_{t \to \infty} t^{-2} \log P(\lambda_q(X-tf) \leqslant t^{-2(\alpha-q)}r) \geqslant -\tfrac{1}{2}\|f\|_\mu^2-Cr^{-(1/(\alpha-q))} \tag{3.12}$$

and

$$\varlimsup_{t \to \infty} t^{-2} \log P(\lambda_q(X-tf) \leqslant t^{-2(\alpha-q)}r) \leqslant -\tfrac{1}{2}\|f\|_\mu^2-cr^{-(1/(\alpha-q))} \tag{3.13}$$

Proof. The inequality in (3.10) appears in [Kuelbs *et al.*,[17] Th. 2] and (3.11) follows easily by adapting the ideas in Lemma 1 of Grill[5] to arbitrary centered Gaussian measures. The inequalities in (3.12) and (3.13) follow from Lemma 3.1, and an argument adapted from Theorem 3.3 of de Acosta.[3] Lemma 3 of Kuelbs and Li[6] adapted these arguments to the case of Hölder norms for μ Wiener measure, but with some slight changes they work equally well for (3.12) and (3.13). Hence the lemma holds.

The following contains functional LIL results related to Chung's LIL, and except for constants is quite precise even when $\|f\|_\mu = 1$. We write Lx to denote $\max(1, \log_e x)$ and $L_2 x$ to denote $L(Lx)$.

Theorem 3.1. Let $\{X(t): t \geqslant 0\}$ be α-fractional Brownian motion with $X(0) = 0$ and $0 < \alpha < 1$. Let $K = K_\alpha$ denote the unit ball of H_μ, the Hilbert space which generates $\mu = \mathscr{L}(X)$ on $(H_{q,0}, \lambda_q)$ where $0 \leqslant q < \alpha$. Let

$$\eta_n(t) = X(nt)/(2n^{2\alpha} L_2 n)^{1/2} \qquad (0 \leqslant t \leqslant 1, n \geqslant\ ) \tag{3.14}$$

Then the following hold:

A: If $f \in K$, $\|f\|_\mu < 1$, then w.p.1

$$0 < \varliminf_n\ (L_2 n)^{(2(\alpha - q) + 1)/2}\ \lambda_q(\eta_n - f) < \infty \tag{3.15}$$

and if $f \in H_\mu$, $\|f\|_\mu \geqslant 1$, then w.p.1.

$$\varliminf_n\ (L_2 n)^{(2(\alpha - q) + 1)/2}\ \lambda_q(\eta_n - f) = \infty \tag{3.16}$$

B: If $f \in K$, $\|f\|_\mu = 1, f = Sh$ where h is a continuous linear functional on the real separable Banach space $(H_{q,0}, \lambda_q)$ with Sh denoting the Bochner integral $E(Xh(X))$, then w.p.1.

$$0 < \varliminf_n\ (L_2 n)^{(2(\alpha - q) + 1)/(2(\alpha - q + 1))}\ \lambda_q(\eta_n - f) < \infty \tag{3.17}$$

C: If $f \in K$, $\|f\|_\mu = 1$, but $f \neq Sh$ for some h in the dual of $(H_{q,0}, \lambda_q)$, then w.p.1.

$$\varliminf_n\ (Ln)^{(2(\alpha - q) + 1)/(2(\alpha - q + 1))}\ \lambda_q(\eta_n - f) = 0 \tag{3.18}$$

Remarks. (1) If $\|f\|_\infty$ denotes the sup-norm on $C_0[0, 1]$, then $\|f\|_\infty \leqslant \lambda_0(f) \leqslant 2\ \|f\|_\infty$ for all $f \in C_0[0, 1]$. Hence when $q = 0$, Theorem 3.1 implies Theorem 4.3 (except for the estimate on the constant) and Theorem 4.4 of Monrad and Rootzén.[10] Of course, these theorems also yield Theorem 3.1, when $q = 0$.

(2) The results in (3.15) and (3.16) are analogues of classical results for standard Brownian motion obtained by Csaki[2] and de Acosta[3] for the sup-norm on $C_0[0,1]$. The inequalities in (3.17) and (3.18) are motivated, as are those in (4.10) and (4.11) of Monrad and Rootzén,[10] by the results obtainable from Kuelbs *et al.*[8] for i.i.d. samples of α-fractional Brownian motion. In fact, the relevant power of $L_2 n$ in (3.17) and (3.18), and also in (4.10) and (4.11) of Monrad and Rootzén,[10] is derived via Theorem 1 in Kuelbs *et al.*[8] In Kuelbs and Li,[6] we showed how sample results obtained from Kuelbs *et al.*,[8] combined with scaling arguments, apply to the analogues of A, B, C in Theorem 3.1 for Brownian motion and the Brownian sheet. Much of that approach would also apply directly here, but those parts of the argument where independence is required need some modification. Hence we proceed directly to the problems at hand. In the parts of the proof where independence is involved we now use the method employed in Monrad and Rootzén[10] suitably modified.

The proof of Theorem 3.1 will proceed via a sequence of Lemmas, the first of which are Lemmas 3.1 and 3.2. The next is a analogue of standard results when applied to λ_0 of the sup-norm, see, for example, Fernique.[4] However, for the Hölder norms we are unaware of a result which applies directly, so we include a complete proof.

Lemma 3.3. Let $\{Y(t): 0 \leqslant t \leqslant 1\}$ be a separable, centered, real-valued Gaussian process with incremental variance satisfying

$$(E((Y(t+h)-Y(t))^2))^{1/2} \leqslant \psi(h) \leqslant c_\psi h^\beta (\beta > 0) \tag{3.19}$$

Then, for $(c_\psi)^{-1} x > 1$ and $0 \leqslant q < \beta$

$$P(\lambda_f(Y) \geqslant x) \leqslant \frac{1}{\theta} \exp\{-\theta((c_\psi)^{-1} x)^2\}$$

where θ is a positive constant independent of c_ψ and x.

Proof. Understanding that $s+t$ is always taken to be less than or equal to one, we first observe

$$\begin{aligned}\lambda_q(Y) &\leqslant \sup_{0 \leqslant s \leqslant 1} \sup_{0 < t \leqslant 1} |Y(s+t)-Y(s)|/t^q \\ &\leqslant \sup_{0 \leqslant s \leqslant 1} \sup_{j \geqslant 0} \sup_{2^{-j-1} \leqslant t \leqslant 2^{-j}} |Y(s+t)-Y(s)|/2^{-(j+1)q}\end{aligned}$$

$$\leqslant 3 \sup_{j\geqslant 0} \sup_{0\leqslant i\leqslant 2^j} \sup_{0\leqslant t\leqslant 2^{-j}} |Y(t+i2^{-j})-Y(i2^{-j})|/2^{-(j+1)q}$$

$$\leqslant 3 \sup_{j\geqslant 0} \sup_{0\leqslant i\leqslant 2^j} \sum_{l=j+1}^{\infty} \max_{0\leqslant m\leqslant 2^{l-j}} |Y((m+1)\,2^{-l}+i2^{-j})$$

$$-Y(m2^{-l}+i2^{-j})|/2^{-(j+1)q} \tag{3.20}$$

Hence

$$P(\lambda_q(Y)\geqslant x)\leqslant \sum_{j\geqslant 0}\sum_{i=0}^{2^j} P\left(\sum_{l=j+1}^{\infty} \max_{0\leqslant m\leqslant 2^{l-j}} |Y((m+1)\,2^{-l}+i2^{-j})\right.$$

$$\left.-Y(m2^{-l}+i2^{-j})|/2^{-(j+1)q}\geqslant \frac{x}{3}\right)$$

$$\leqslant \sum_{j\geqslant 0}\sum_{i=0}^{2^j}\sum_{l=j+1}^{\infty}\sum_{m=0}^{2^{l-j}} P(|Y((m+1)\,2^{-l}+i2^{-j})$$

$$-Y(m2^{-l}+i2^{-j})|/2^{-(j+1)q}\geqslant y_{j,l}x) \tag{3.21}$$

where

$$y_{j,l}=2^{-(\beta-\gamma)(l-j)}k_{\beta,\gamma}, \qquad 0<\gamma<\beta,\ k_{\beta,\gamma}=\frac{2^{\beta-\gamma}-1}{3}$$

and

$$\sum_{l=j+1}^{\infty} y_{j,l}=1/3$$

Since (3.19) hold, and letting Z be a $N(0, 1)$ random variable, we have

$$P(|Y((m+1)\,2^{-l}+i2^{-j})-Y(m2^{-l}+i2^{-j})|/2^{-(j+1)q}\geqslant y_{j,l}\lambda)$$

$$\leqslant P(|Z|\geqslant y_{j,l}2^{-(j+1)q}x/\psi(2^{-l}))$$

$$\leqslant \int_{a_{j,l}}^{\infty} e^{-u^2/2}\,du$$

where

$$a_{j,l}=y_{j,l}x2^{-(j+1)q}/\psi(2^{-l})\geqslant c_{\psi}^{-1}k_{\beta,\gamma}2^{(\beta-\gamma)(l-j)}x2^{-(j+1)q}\cdot 2^{\beta l}$$

$$=k_{\beta,\gamma}2^{-q}c_{\psi}^{-1}x2^{\gamma(l-j)+(\beta-q)j}$$

Hence

$$
\begin{aligned}
P(\lambda_q(Y) \geqslant x) &\leqslant \sum_{j \geqslant 0} \sum_{l=j+1}^{\infty} 2^l \int_{a_{j,l}} e^{-u^2/2}\, du \\
&\leqslant \sum_{j \geqslant 0} \sum_{l=j+1}^{\infty} 2^l/(k_{\beta,\gamma} 2^{-q} c_\psi^{-1} x 2^{\gamma(l-j)+(\beta-q)j}) \\
&\quad \cdot \exp\{-1/2(k_{\beta,\gamma} 2^{-q})^2 (c_\psi^{-1} x)^2 2^{2\gamma(l-j)+2(\beta-q)j}\} \\
&\leqslant (k_{\beta,\gamma} 2^{-2q})^{-1} \cdot \sum_{j \geqslant 0} \sum_{l \geqslant 1} 2^{l+j} \\
&\quad \times \left(\exp\left\{-\frac{1}{4}(k_{\beta,\gamma} 2^{-q})^2 (c_\psi^{-1} x)^2 2^{2\gamma l+2(\beta-q)j}\right\}\right)^2 \\
&\leqslant (k_{\beta,\gamma} 2^{-q})^{-1} \exp\left\{-\frac{1}{4}(k_{\beta,\gamma} 2^{-q})^2 (c_\psi^{-1} x)^2\right\} \\
&\quad \cdot \sum_{j=0} \sum_{l \geqslant 1} 2^{l+j} \exp\left\{-\frac{1}{4}(k_{\beta,\gamma} 2^{-q})^2 2^{2\gamma l+2(\beta-q)j}\right\} \\
&\leqslant \frac{1}{\theta} \exp\{-\theta (c_\psi^{-1} x)^2\}
\end{aligned}
$$

where θ is a positive constant independent of c_ψ and x. Hence the lemma is proved.

The next lemma modifies a result we learned from an early version of Monrad and Rootzén.[10]

Lemma 3.4. Let $0 < \alpha < 1$ and fix $0 < q < q' < \alpha$. Let $d_r = r^{r+(1-\gamma)}$, $n_r = r^r$ for $r \geqslant 1$ and $0 < \gamma < 1$. Let

$$
Y_r(t) = \int_{|x| \notin I_r} \frac{1}{k_\alpha} (|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2})\, dB(x), \qquad 0 \leqslant t \leqslant 1 \tag{3.22}
$$

where $\{B(t)\colon -\infty < t < \infty\}$ is standard Brownian motion, and $I_r = (d_{r-1}/n_r, d_r/n_r)$. Let $0 < \beta < \gamma$. Then, for $\delta = \min(2\beta(\alpha - q'), \gamma - \beta, (1-\gamma)(2-2\alpha), (2\alpha - 2q')\gamma\}$ there is a constant $C \in (0, \infty)$ depending only on α such that uniformly in t, h, r.

$$
\sigma_r^2(t, h) = E((Y_r(t+h) - Y_r(t)^2) \leqslant C h^{2q'} r^{-\delta} \tag{3.23}
$$

Proof. If $\alpha = 1/2$, recall the kernel is interpreted to be $I_{[0,r]}(x)$. Then, for $h \geqslant 0$

$$\sigma_r^2(t, h) = \begin{cases} 0 & t \geqslant (r-1)^{r-\gamma}/r^r \\ (r-1)^{r-\gamma}/r^r - t & 0 \leqslant t \leqslant (r-1)^{r-\gamma}/r^r < t+h \\ h & 0 \leqslant t \leqslant t+h \leqslant (r-1)^{r-\gamma}/r^r \leqslant h \wedge r^{-\gamma} \end{cases}$$

$$= \begin{cases} h^{2q'} r^{-(1-2q')\gamma} & \text{if} \quad h \leqslant r^{-\gamma} \\ r^{-2q'\gamma} r^{-\gamma(1-2q')} \leqslant h^{2q'} r^{-(1-2q')\gamma} & \text{if} \quad h > r^{-\gamma} \end{cases} \tag{3.24}$$

Hence (3.23) holds for $C = 1$ when $\alpha = \frac{1}{2}$ and $\delta = (1-2q')\gamma$.

If $0 < \alpha < 1$, $\alpha \neq 1/2$, let

$$f^2(y) = 1/k_\alpha^2(|y - 1/2|^{(2\alpha-1)/2} - |y + 1/2|^{(2\alpha-1)/2}) \qquad -\infty < y < \infty$$

Then $\int_{\mathbb{R}} f^2(y)\, dy = 1$ by definition of k_α and for:

(a) $0 < \alpha < 1/2$, differentiation shows $f^2(y)$ is increasing for $0 < y < 1/2$ and decreasing for $y > 1/2$ with $f(y) = f(-y)$, and hence $f^2(y) \leqslant 1/k_\alpha^2$ on $[0, \infty)$.

(b) $1/2 < \alpha < 1$, the function $x^{(2\alpha-1)/2}$ is concave on $[0, \infty)$, so $f^2(y) \leqslant 1/k_\alpha^2$ on $[0, \infty)$.

Hence, changing variables implies

$$\sigma_r^2(t, h) = h^{2\alpha} \int_{|y + 1/h + 1/2| \notin I_r/h} f^2(y)\, dy$$

and for $0 \leqslant q < q' < \alpha$, this implies

$$\sigma_r^2(t, h) = h^{2q'} h^{2(\alpha - q')} \int_{|y + t/h + 1/2| \notin I_r/h} f^2(y)\, dy \tag{3.25}$$

If $0 \leqslant h \leqslant r^{-\beta}$, then (3.25) and $\int_{\mathbb{R}} f^2(y)\, dy \leqslant 1$ implies

$$\sigma_r^2(t, h) \leqslant h^{2q'} r^{-2\beta(\alpha - q')} \tag{3.26}$$

Hence assume $h \geqslant r^{-\beta}$. Then

$$|y + t/h + 1/2| \notin I_r/h$$

implies

$$|y + t/h + 1/2| \geqslant \frac{r^{1-\gamma}}{r}$$

or

$$0 \leqslant |y + t/h + 1/2| \leqslant \frac{r^{-\gamma}}{h}$$

For $|y + t/h + 1/2| \geqslant r^{1-\gamma}/h$ we have (since $0 \leqslant t \leqslant 1$)

$$y + 1/2 \geqslant (r^{1-\gamma} - 1)/h$$

or

$$y + 1/2 \leqslant -\frac{r^{1-\gamma}}{h}$$

Thus for $r > 2^{\gamma-1}$

$$\begin{aligned} \int_{|y+t/h+1/2| \geqslant r^{1-\gamma}/h} f^2(y)\,dy &\leqslant 2\int_{(r^{1-\gamma}-1)/h}^{\infty} f^2(y)\,dy \\ &\leqslant 2\int_{r^{1-\gamma}/(2h)} f^2(y)\,dy \\ &\leqslant C\int_{r^{1-\gamma}/(2h)}^{\infty} y^{2\alpha-3}\,dy \\ &\leqslant Ch^{2-2\alpha}r^{-(1-\gamma)(2-2\alpha)} \end{aligned} \tag{3.27}$$

where $C \in (0, \infty)$ depends only on α but differs from line to line.

Now

$$0 \leqslant |y + t/h + 1/2| \leqslant \frac{r^{-\gamma}}{h}$$

implies

$$-(r^{-\gamma} + t)/h \leqslant y + 1/2 \leqslant (r^{-\gamma} - t)/h$$

Since $f^2(y) \leqslant 1/k_\alpha^2$,

$$\int_{-1/2-(r^{-\gamma}+t)/h}^{-1/2+(r^{-\gamma}-t)/h} f^2(y)\,dy \leqslant 2r^{-\gamma}/(hk_\alpha)^2 \tag{3.28}$$

Combining (3.27) and (3.28) we have for $r \geqslant 2^{1/(1-\gamma)}$

$$\sigma_r^2(t, h) \leqslant h^{2\alpha}\{Ch^{2-2\alpha}r^{-(1-\gamma)(2-2\alpha)} + 2r^{-\gamma}/(hk_\alpha)^2\}$$

Since we are assuming $h \geqslant r^{-\beta}$ and $\gamma > \beta$, then for $0 < h \leqslant 1$ we have for some new C, depending only on α, that

$$\begin{aligned}\sigma_r^2(t, h) &\leqslant Ch^{2\alpha}\{r^{-(1-\gamma)(2-2\alpha)} + r^{-(\gamma-\beta)}\}\\ &\leqslant Ch^{2\alpha}r^{-\delta}\end{aligned} \tag{3.29}$$

for $\delta > 0$ as in the hypothesis. Putting (3.26) and (3.29) together now yields (3.23), so the lemma is proved.

Lemma 3.5. If $f \in K = K_\alpha$ and $g(\cdot) = f(\lambda(\cdot))$ on $[0, 1]$ with $0 < \lambda < 1$, then for $0 \leqslant q < \alpha$

$$\lambda_q(f - g) \leqslant 2\,|1 - \lambda|^{\alpha - q} \tag{3.30}$$

Proof. Since $g(\cdot) = f(\lambda(\cdot))$

$$\lambda_q(f-g) = \sup_{0 \leqslant s < t \leqslant 1} |(f(t) - f(\lambda t)) - (f(s) - f(\lambda s))|\,|t-s|^{-q}$$

If $0 \leqslant s < t \leqslant 1$, then we have two cases:

(a) $0 \leqslant \lambda s < \lambda t < t \leqslant 1$ and

(b) $0 \leqslant \lambda s < \lambda t \leqslant s < t \leqslant 1$.

If (a) holds, then (3.5) implies there exists h such that $\int_{\mathbb{R}} h^2(x)\,dx \leqslant 1$ and

$$\begin{aligned}&|(f(t) - f(\lambda t)) - f(s) - f(\lambda s))|\\ &\quad= \left|\int_{\mathbb{R}} \frac{1}{k_\alpha}\left(|x-t|^{(2\alpha-1)/2} - |x-\lambda t|^{(2\alpha-1)/2}\right) h(x)\,dx\right.\\ &\qquad \left.- \int_{\mathbb{R}} \frac{1}{k_\alpha}\left(|x-s|^{(2\alpha-1)/2} - |x-\lambda s|^{(2\alpha-1)/2}\right) h(x)\,dx\right|\\ &\quad\leqslant |t-\lambda t|^\alpha + |s - \lambda s|^\alpha = (s^\alpha + t^\alpha)(1-\lambda)^\alpha\end{aligned}$$

where the last inequality follows by the Cauchy-Schwartz inequality applied to each of the integrals and then a change of variables. However, for (a) holding we have $s/t < \lambda$ and hence

$$|t-s|^{-q} = t^{-q}\,|1 - s/t|^{-q} \leqslant t^{-q}\,|1-\lambda|^{-q}$$

Thus if (a) holds

$$\begin{aligned} |(f(t)-f(\lambda t))-(f(s)-f(\lambda s))|\,|t-s|^{-q} &\leqslant t^{-q}(s^\alpha+t^\alpha)\,|1-\lambda|^{\alpha-q} \\ &\leqslant 2^2 t^{\alpha-q}\,|1-\lambda|^{\alpha-q} \\ &\leqslant 2\,|1-\lambda|^{\alpha-q} \end{aligned}$$

since $0\leqslant t\leqslant 1$.

If (b) holds, then by a similar argument

$$\begin{aligned} |(f(t)-f(\lambda t))-(f(s)-f(\lambda s))| &= |(f(t)-f(s))-(f(\lambda t)-f(\lambda s))| \\ &\leqslant |t-s|^\alpha+\lambda^\alpha\,|t-s|^\alpha \\ &= (1+\lambda^\alpha)\,|t-s|^\alpha \end{aligned}$$

However, when (b) holds

$$|t-s|^{\alpha-q}=t^{\alpha-q}\,|1-s/t|^{\alpha-q}\leqslant |1-\lambda|^{\alpha-q}$$

so putting these inequalities together we get

$$\lambda_q(f-g)<2\,|1-\lambda|^{\alpha-q}$$

and (3.30) holds.

Lemma 3.6 is adapted from A. de Acosta's Lemma 5.3.[3]

Lemma 3.6. Let m, n, r be positive integers with $m\leqslant n\leqslant r$ and $a_n=(2n^{2\alpha}L_2 n)^{1/2}$ for $n\geqslant 1$, and $0\leqslant q<\alpha<1$. Then for $f\in H_\mu$, μ the law of α-fractional Brownian motion on $H_{q,0}$, and $1/2<p\leqslant(2(\alpha-q)+1)/2$

$$\begin{aligned} &(L_2 n)^p\,\lambda_q(X(n(\cdot))/a_n-f) \\ &\quad\geqslant (n/m)^q\,(m/r)^\alpha\,(L_2 m)^p\,\lambda_q(X(m(\cdot))/a_m-f) \\ &\qquad-2(r/m)^q\,(L_2 r)^p\,|1-m/r|^{\alpha-q}\,\|f\|_\mu \\ &\qquad-(r/m)^q\,(L_2 r)^p\,|1-a_m/a_r|\,\lambda_q(f) \end{aligned} \tag{3.31}$$

Proof. Since $X(n((m/n)(\cdot)))=X(m(\cdot))$, by rescaling we have

$$\begin{aligned} &(L_2 n)^p\,\lambda_q(X(n(\cdot))/a_n-f) \\ &\quad=\left(\frac{L_2 n}{a_n}\right)^p\lambda_q(X(n(\cdot))-a_n f) \\ &\quad\geqslant\frac{(L_2 n)^p}{a_n}\sup_{0\leqslant(n/m)s<(n/m)t\leqslant 1}|(X(nt)-a_n f(t)) \\ &\qquad-(X(ns)-a_n f(s))|/|t-s|^q \end{aligned}$$

$$=\frac{(L_2n)^p}{a_n}\sup_{0\leqslant u<v\leqslant 1}\frac{|(X(mv)-a_nf((m/n)r))-(X(mu)-a_nf((m/n)u))|}{|m/n|^q\,|u-v|^q}$$

$$=\frac{(L_2n)^p}{a_n}\left(\frac{n}{m}\right)^q\lambda_q\left(X(m(\cdot))-a_nf\left(\frac{m}{n}(\cdot)\right)\right)$$

$$\geqslant\left(\frac{n}{m}\right)^q\frac{(L_2r)^p}{a_r}\lambda_f(X(m(\cdot))-a_n\,g(\cdot)) \tag{3.32}$$

where $g(\cdot)=f((m/n)(\cdot))$ (since $((L_2r)^p/a_r)\leqslant((L_2m)^p/a_m)$ when $Lx=\max(1,\log_e x)$ and $1/2<p<(2(\alpha-q)+1)/2$. Since $a_n\nearrow\infty$, we thus have

$$\lambda_q(X(m(\cdot))-a_n\,g(\cdot))\geqslant\lambda_q(X(m(\cdot)-a_mf)-a_r\lambda_q(f-g)$$
$$-(a_r-a_m)\,\lambda_q(f) \tag{3.33}$$

and since $f\in H_\mu$, Lemma 3.5 implies

$$\lambda_q(f-g)\leqslant 2\,|1-m/r|^{\alpha-q}\,\|f\|_\mu \tag{3.34}$$

Combining (3.32)–(3.34) we have (3.31) and the lemma holds.

Lemma 3.7. Under the assumptions of Theorem 3.1, (3.15) and (3.16) hold.

Proof. Let $n_r=\exp\{r/(Lr)^\phi\}$ where $\phi(\alpha-q)>(2(\alpha-q)+1)/2$. Then by (3.13) for any $\delta>0$ and r sufficiently large

$$p_r=P((2L_2n_r)^{(2(\alpha-q)-1)/2}\,\lambda_q(\eta_{n_r}-f)\leqslant b)$$
$$\leqslant\exp\{-(L_2n_r)(\|f\|_\mu^2+\sqrt{2}\,cb^{-1/(\alpha-q)}-\sqrt{2}\,\delta)\} \tag{3.35}$$

Hence by taking $b_0=b>0$ sufficiently small we have $\sum_r p_r<\infty$. Hence by the Borel-Cantelli lemma

$$\varliminf_r\,(L_2n_r)^{(2(\alpha-q)+1)/2}\,\lambda_q(\eta_{n_r}-f)\geqslant b_0>0 \tag{3.36}$$

Furthermore, applying Lemma 3.6 for $n_{r-1}\leqslant n\leqslant n_r$ it follows that

$$\varliminf_n\,(L_2n)^{(2(\alpha-q)+1)/2}\,\lambda_q(\eta_n-f)$$
$$\geqslant b_0-2\varlimsup_r\,(L_2n_r)^{(2(\alpha-q)+1)/2}\left|1-\frac{n_{r-1}}{n_r}\right|^{\alpha-q}\|f\|_\mu$$
$$-\varlimsup_r\,(L_2n_r)^{(2(\alpha-q)+1)/2}\,|1-a_{n_{r-1}}/a_{n_r}|\,\lambda_q(f)$$
$$=b_0>0 \tag{3.37}$$

since $n_r = \exp\{r/(Lr)^\phi\}$ with $\phi(\alpha - q) > (2(\alpha - q) + 1)/2$. Hence the left-hand side of (3.15) holds.

If $\|f\|_\mu > 1$ and p_r is as in (3.35), then no matter how large b is, we have $\sum_r p_r < \infty$ provided $\sqrt{2}\,\delta < \|f\|_\mu^2 - 1$. Hence the Borel-Cantelli lemma and Lemma 3.6 yield (3.16) when $\|f\|_\mu > 1$.

If $\|f\|_\mu = 1$ we fix b as large as we like, and then choose $\delta > 0$ sufficiently small so that

$$cb^{-1/(\alpha - q)} - \delta > 0$$

Again $\sum_r p_r < \infty$, so Lemma 3.6 implies that (3.16) holds as b is arbitrarily large.

Thus it remains to show the right-hand side of (3.15) if $\|f\|_\mu < 1$. For this we define

$$Z_r(t) = \int_{\{d_{r-1} \leqslant |x| \leqslant d_r\}} \frac{1}{k_\alpha} \left(|x - t|^{(2\alpha - 1)/2} - |x|^{(2\alpha - 1)/2}\right) dB(x) \tag{3.38}$$

$$X_r(t) = X(t) - Z_r(t) \tag{3.39}$$

for $t \geqslant 0$, $d_r = r^{r + (1 - \gamma)}$, $0 < \gamma < 1$. Then the Z_r's are independent, and the right-hand side of (3.15) will follow if we show with probability one that

$$\varliminf_r \, (L_2 n_r)^{(2(\alpha - q) + 1)/2} \, \lambda_q(Z_r(n_r(\cdot))/a_{n_r} - f) < \infty \tag{3.40}$$

and

$$\varlimsup_r \, (L_2 n_r)^{(2(\alpha - q) + 1)/2} \, \lambda_q(X_r(n_r(\cdot))/a_{n_r}) = 0 \tag{3.41}$$

for some subsequence $n_r \nearrow \infty$.

For $i = 1, 2$, $\varepsilon > 0$, $b > \varepsilon$, let

$$\begin{aligned} A_r(i) &= \{(2L_2 n_r)^{\beta(\alpha, q, i)} \, \lambda_q(Z_r(n_r(\cdot))/a_{n_r} - f) \leqslant b\} \\ B_r(i) &= \{(2L_2 n_r)^{\beta(\alpha, q, i)} \, \lambda_q(X_r(n_r(\cdot)))/a_{n_r} > \varepsilon\} \\ C_r(i) &= \{(2L_2 n_r)^{\beta(\alpha, q, i)} \, \lambda_q(\eta_{n_r} - f) \leqslant b - \varepsilon\} \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} \beta(\alpha, q, 1) &= (2(\alpha - q) + 1)/2 \\ \beta(\alpha, q, 2) &= (2(\alpha - q) + 1)/(2(\alpha - q + 1)) \end{aligned} \tag{3.43}$$

Then for $i=1,2$

$$C_r(i) \subset A_r(i) \cup B_r(i) \tag{3.44}$$

Now (3.12) implies that for all $\delta>0$ and r sufficiently large

$$P(C_r(1)) \geqslant \exp\{-L_2 n_r(\|f\|_\mu^2 + \sqrt{2}\, C(b-\varepsilon)^{-1/(\alpha-q)} + \sqrt{2}\,\delta)\} \tag{3.45}$$

Hence, if $n_r = r^r$ and $\delta>0$ is such that $\sqrt{2}\,\delta + \|f\|_\mu^2 < 1$, then by taking b sufficiently large we have

$$\sum_r P(C_r(1)) = \infty \tag{3.46}$$

Thus (3.45) and (3.46) combine to imply

$$\sum_r P(A_r(1)) = \infty \tag{3.47}$$

if for all $\varepsilon>0$

$$\sum_r P(B_r(1)) < \infty \tag{3.48}$$

Now (3.48) for all $\varepsilon>0$ implies (3.41), and (3.47) is exactly that

$$\sum_r P((2L_2 n_r)^{(2(\alpha-q)+1)/2}\, \lambda_q(Z_r(n_r(\cdot))/a_{n_r} - f) \leqslant b) = \infty \tag{3.49}$$

Hence, since the Z_r's are independent, the Borel-Cantelli lemma yields (3.40) once (3.49) holds for some $b<\infty$. Thus it remains to show (3.48) for all $\varepsilon>0$.

To verify (3.48) recall $\{Y_r(t): 0 \leqslant t \leqslant 1\}$ of Lemma 3.4. Since $d_r = r^{r+(1-\gamma)}$, $n_r = r^r$, we see

$$\{X_r(t): 0 \leqslant t \leqslant n_r\} \stackrel{\mathscr{L}}{=} \{n_r^\alpha Y_r(t/n_r): 0 \leqslant t \leqslant n_r\} \tag{3.50}$$

Hence by (3.50), and Lemmas 3.3 and 3.4, for all r sufficiently large

$$\begin{aligned} P(B_r(1)) &= P(\lambda_q(X_r(n_r(\cdot))) > \varepsilon a_{n_r}(2L_2 n_r)^{-(2(\alpha-q)+1)/2}) \\ &= P(\lambda_q(Y_r((\cdot))) > \varepsilon(2L_2 n_r)^{-(\alpha-q)}) \\ &\leqslant \frac{1}{\theta} \exp\{-\theta(cr^{-\delta})^{-1}\, \varepsilon^2 (2L_2 n_r)^{-2(\alpha-q)}\} \end{aligned} \tag{3.51}$$

Thus (3.48) holds, and the lemma is proved.

Lemma 3.8. Under the assumptions of Theorem 3.1, (3.17) holds.

Proof. If $\|f_\mu\|=1$ and $f=Sh$ where h is a continuous linear functional on $(H_{q,0}, \lambda_q)$, then by Proposition 1 of Kuelbs *et al.*[7]

$$\lim_{\delta \to 0} (1-\|f_\delta\|_\mu)/\delta = 2\,\|h\|_* \tag{3.52}$$

where $\|h\|_*$ is the norm in the dual of $(H_{q,0}, \lambda_q)$, and f_δ is as in (3.11) of Lemma 3.2. An easy computation also yields that for $a>0$, $\|(af)_\delta\|_\mu = a\,\|(f)_{\delta/a}\|_\mu$. Hence by Lemma 3.1 and the right-hand side of (3.10) there is a constant c such that for $n_r=\exp\{r/(Lr)^\phi\}$ we have that

$$\begin{aligned} p_r &= P((2L_2n_r)^{(2(\alpha-q)+1)/(2(\alpha-q)+2)}\,\lambda_q(\eta_{n_r}/a_{n_r}-f)\leqslant b)\\ &= P(\lambda_q(X-(2L_2n_r)^{1/2}f)\leqslant b(2L_2n_r)^{-(\alpha-q)/(2(\alpha-q)+2)})\\ &\leqslant \exp\{-(L_2n_r)\,\|(\delta/(2L_2n_r)^{1/2})f\|_\mu^2\\ &\quad -cb^{-1/(\alpha-q)}(2L_2n_r)^{-1/(2(\alpha-q)+2)}\} \end{aligned} \tag{3.53}$$

with $\delta=b(2L_2n_r)^{-(\alpha-q)/(2(\alpha-q)+2)}$. Hence by (3.52) for each $\gamma>0$ and r sufficiently large

$$\begin{aligned} p_r &\leqslant \exp\{-L_2n_r(1-2\,\|h\|_*\,(1+\gamma)\,b(2L_2n_r)^{-(\alpha-q)/(2(\alpha-q)+2)-1/2})^2\\ &\quad -cb^{-1/(\alpha-q)}(2L_2n_r)^{1/(2(\alpha-q)+2)}\}\\ &\leqslant \exp\{-L_2n_r+(2\,\|h\|_*\,(1+\gamma)b-cb^{-1/(\alpha-q)})(2L_2n_r)^{1/(2(\alpha-q)+2)}\} \end{aligned} \tag{3.54}$$

Hence for $b>0$ sufficiently small, $\sum_r p_r<\infty$ and the left-hand side of (3.17) holds by applying Lemma 3.6 as in Lemma 3.7 with $\phi(\alpha-q)>(2(\alpha-q)+1)/2$.

To prove the right-hand side of (3.17) we need some independence, and recall the events in (3.42) with $n_r=r^r$, $i=2$, and $\beta(\alpha, q, 2)$ as in (3.43). If $f_r=f-f(L_2n_r)^{-(2(\alpha-q)+1)/(2(\alpha-q)+2)}$, then for r sufficiently large

$$P(C_r(2))\geqslant P(\lambda_q(\eta_{n_r}-f_r)\leqslant\frac{(b-\varepsilon)}{2}(2L_2n_r)^{-(2(\alpha-q)+1)/(2(\alpha-q)+2)})$$

provided $(b-\varepsilon)/2>\lambda_q(f)$. Hence Lemma 3.1 and Lemma 3.2 and $\|f\|_\mu=1$ imply there is a constant $C>0$ such that for r sufficiently large

$$P(C_r(2)) \geqslant 2^{-1} \exp \left\{ -L_2 n_r \|f_\mu\|_\mu^2 \right.$$

$$\left. - C\left(\frac{(b-\varepsilon)}{2} (2L_2 n_r)^{-((2(\alpha-q)+1)/(2(\alpha-q)+2))+1/2}\right)^{-1/(\alpha-q)} \right\}$$

$$= 2^{-1} \exp \left\{ -L_2 n_r + (L_2 n_r)^{1/(2(\alpha-q)+2)} \right.$$

$$\left. \times \left(2 - C\left(\frac{b-\varepsilon}{2}\right)^{-1/(\alpha-q)} 2^{1/(2(\alpha-q)+2)} \right) \right\} \tag{3.55}$$

Thus for $b > 0$ sufficiently large

$$\sum_r P(C_r(2)) = \infty \tag{3.56}$$

and in view of (3.44) with $i = 2$ we have

$$\sum_r P(A_r(2)) = \infty \tag{3.57}$$

provided

$$\sum_r P(B_r(2)) < \infty \tag{3.58}$$

for all $\varepsilon > 0$. Now the events $\{A_r(2),\ r \geqslant 1\}$ are independent, so (3.57) and the Borel-Cantelli lemma imply that $P(A_r(2)$ i.o.$) = 1$. Given (3.58), this then implies $P(C_r(2)$ i.o.$) = 1$ for sufficiently large b, and hence the right-hand side of (3.17) follows. Hence it remains to verify (3.58).

Recalling $\{Y_r(t)\colon 0 \leqslant t \leqslant 1\}$ of Lemma 3.4, since $n_r = r^r$, $d_r = r^{r+(1-\gamma)}$, $0 < \gamma < 1$, then (3.50) holds. Thus we see as in (3.51), by applying Lemmas 3.3 and 3.4, that

$$P(B_r(2)) \leqslant P(\lambda_q(Y_r(\cdot)) > \varepsilon(2L_2 n_r)^{1/2-((2(\alpha-q)+1)/(2(\alpha-q)+2))})$$

$$\leqslant \frac{1}{\theta} \exp\{-\theta(cr^{-\delta})^{-1} \varepsilon^2 (2L_2 n_r)^{-(\alpha-q)/(2(\alpha-q)+2)}\} \tag{3.59}$$

Then (3.58) holds, and the lemma is proved.

Lemma 3.9. Under the assumptions of Theorem 3.1, (3.18) holds.

Proof. The proof of (3.18) follows as in the second part of Lemma 3.8. Here, since $f \neq Sh$ for some h in the dual of $(H_{q,0}, \lambda_q)$, we have by Proposition 2 of Kuelbs *et al.*[7] that

$$\lim_{\delta \to 0} (1 - \|f_\delta\|)/\delta = \infty \tag{3.60}$$

Hence by defining

$$C_r = \{(2L_2 n_r)^{(2(\alpha-q)+1)/(2(\alpha-q)+2)} \lambda_q(\eta_{n_r} - f) < 2b)\}$$

we have

$$\begin{aligned} P(C_r) &= P(\lambda_q(X - (2L_2 n_r)^{1/2} f) \leqslant 2b(2L_2 n_r)^{-(\alpha-q)/(2(\alpha-q)+2)}) \\ &\geqslant P(\lambda_q(X - ((2L_2 n_r^{1/2}) f)_\delta) \leqslant b(2L_2 n_r)^{-(\alpha-q)/(2(\alpha-q)+2)}) \end{aligned} \tag{3.61}$$

with $\delta = b(2L_2 n_r)^{-(\alpha-q)/(2(\alpha-q)+2)}$. Again, since $(af)_\delta = a(f_{\delta/a})$, we have from (3.61) and Lemmas 3.1 and 3.2 a $C \in (0, \infty)$ such that

$$P(C_r) \geqslant \exp\{-L_2 n_r \|(f)_{\delta/(2L_2 n_r)^{1/2}}\|_\mu^2 - Cb^{-1/(\alpha-q)}(2L_2 n_r)^{1/(2(\alpha-q)+2)}\} \tag{3.62}$$

Since (3.60) holds, for every $M > 0$ we can take r sufficiently large (depending on M), such that

$$\|f_{\delta/(2L_2 n_r)^{1/2}}\|_\mu \leqslant 1 - M \cdot \delta/(2L_2 n_r)^{1/2} \tag{3.63}$$

Since $\delta = b(2L_2 n_r)^{(\alpha-q)/(2(\alpha-q)+2)}$ we have from (3.62) and (3.63) that for all r sufficiently large

$$P(C_r) \geqslant \tfrac{1}{2} \exp\{-L_2 n_r + (Mb - Cb^{-1/(\alpha-q)})(2L_2 n_r)^{1/(2(\alpha-q)+2)}\} \tag{3.64}$$

Thus for any $b > 0$, no matter how small, we can take M sufficiently large so that

$$Mb - Cb^{-1/(\alpha-q)} \geqslant 0$$

For such M we have (3.64) holding provided r is sufficiently large, and hence the probabilities in (3.64) diverge.

Hence define the analogous of $B_r(2)$ and $A_r(2)$ by

$$B_r = \{(2L_2 n_r)^{(2(\alpha-q)+1)/(2(\alpha-q)+2)} \lambda_q(X_r(n_r(\cdot))/a_{n_r}) > b\}$$

and

$$A_r = \{(2L_2 n_r)^{(2(\alpha-q)+1)/(2(\alpha-q)+2)} \lambda_q(Z_r(n_r(\cdot))/a_{n_r} - f) \leqslant 3b\}$$

Then the divergence of the probabilities in (3.64) for arbitrarily small $b > 0$ implies

$$\sum_r P(A_r) = \infty \tag{3.65}$$

provided

$$\sum_r P(B_r) < \infty \tag{3.66}$$

for such $b > 0$, since

$$C_r \subset A_r \cup B_r$$

Since the events $\{A_r\colon r \geqslant 1\}$ are independent, (3.65) and the Borel-Cantelli lemma implies $P(A_r \text{ i.o.}) = 1$, and (3.66) implies $P(B_r \text{ i.o.}) = 0$. Thus $P(C_r \text{ i.o.}) = 1$, and the lemma is proved, provided (3.66) holds for $b > 0$ arbitrarily small. Now (3.66) holds using $d_r = r^{r + (1-\gamma)}$, $n_r = r^n$, and (3.50) as in (3.59) with $\varepsilon = b$. Thus the lemma is proved.

Combining Lemmas 3.7–3.9 we have the theorem proved.

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