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Probability Theory and Related Fields

The Gaussian measure of shifted balls

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Summary. Let μ be a centered Gaussian measure on a Hilbert space H and let $B_R \subseteq H$ be the centered ball of radius $R>0$. For $a \in H$ and $\lim_{t\to\infty} R(t)/t < \|a\|$, we give the exact asymptotics of $\mu(B_{R(t)}+t\cdot a)$ as $t\to\infty$. Also, upper and lower bounds are given when μ is defined on an arbitrary separable Banach space. Our results range from small deviation estimates to large deviation estimates.

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1 Introduction

Let $(E, \|\cdot\|)$ be a separable Banach space and let μ be a centered Gaussian measure on E. For the closed ball $B_R=\{x\in E: \|x\| \leqq R\}$ centered at zero, the function $\mu(B_R+b)$ from $(0,\infty)\times E$ into R^1 is of special interest in the theory of Gaussian measures or processes, and has been investigated extensively in the literature.

The aim of this paper is to describe the exact asymptotic behavior of

$$\mu(B_{R(t)}+t\cdot a) \quad \text{as} \quad t\to\infty \tag{1.1}$$

where $a\in E$ is a fixed element and $\lim_{t\to\infty} R(t)/t<\|a\|$. The results range from small deviations to large deviations. We do so for $E=H$ (Hilbert space). Also, some related upper and lower bounds are given for μ defined on an arbitrary separable Banach space E.

Now we turn to some of the basic questions and related references that connect with our paper. There are three cases that are covered in the study of (1.1) in this paper.

The first case is when the radius is a constant, i.e. $R(t)=R=$constant. A general result of Borell [2] implies

$$\lim_{t\to\infty} \frac{1}{t^2} \log \mu(B_R+t\cdot a) = -\tfrac{1}{2}\|a\|_\mu^2 \tag{1.2}$$

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where $\|\cdot\|_\mu$ is the norm in the reproducing kernel Hilbert space $H_\mu \subseteq E$ of μ (with the convention $\|a\|_\mu = \infty$ if $a \notin H_\mu$). Thus, if $a \in H_\mu$, then $\mu(B_R + t \cdot a)$ is 'almost' $\exp(-\|a\|_\mu^2 t^2/2)$. We want to make this 'almost' more precise, and, more important, if $a \notin H_\mu$ we try to see how fast $t^{-2} \log \mu(B_R + t \cdot a)$ tends to $-\infty$ as $t \to \infty$.

The second case is when the radius is shrinking to zero, i.e. $\lim_{t \to \infty} R(t) = 0$. The problems of this type are more delicate and useful in general. When $a \in H_\mu \subseteq E$, de Acosta [1] gives the sharp estimates at the log level for Gaussian measure μ on Banach space E. For Wiener measure on $C[0, 1]$ and $a \in H_\mu$, Grill [4] recently provides the estimates that gives the second order term (up to a constant) at the log level. Very recently, Kuelbs et al. [9] gives the exact estimates up to the second order term for an important dense subset of H_μ when μ is Wiener measure on Hilbert space. This type of estimate is related to the problem of finding the convergence rate and constant in the functional form of Chung's law of the iterated logarithm.

The third case is when the radius is growing to infinity but not too fast, i.e. $\lim_{t \to \infty} R(t) = \infty$ and $\lim_{t \to \infty} R(t)/t < \|a\|$. These estimates are in the domain of large deviation theory and differ from what has been done previously in that we are able to obtain results which study the asymptotics of the probabilities rather than the logarithm of the probabilities. Unfortunately, our results are applicable to only a limited number of sets, and are restricted to the Hilbert space setting, so they compete with large deviation theory in only a limited way. They do, however, suggest more general questions of interest, which are beyond what can be done now.

The rest of this paper is arranged as follows. In Theorem 1 of Sect. 2, we give the exact asymptotics of (1.1) in the Hilbert space case. The main tool is to analyze the inversion formula of the characteristic function with a modified 'Saddle point method' for integrals of complex functions. This method has been used in [12] and [9] for special types of problems. In Sect. 3, we give some important corollaries of Theorem 1. In particular, the results related to (1.2) and the improved large deviation estimates are discussed. We recall some basic results about general Gaussian random vectors in Sect. 4, and in Theorem 2 we use them to give an upper and lower bound for (1.1) which are refinements of Grill [4]. The bounds are sharp at the log level (up to some constants). In Sect. 5, we briefly indicate an application of the results we have obtained. In particular, we discuss the convergence rate to a point outside H_μ for Gaussian samples in Hilbert space.

Throughout we write $a_n \sim b_n$ when $\lim_{n \to \infty} a_n/b_n = 1$ and $a_n \approx b_n$ if there is a constant C, $1 < C < \infty$, such that

$$1/C \leqq \varliminf_{n \to \infty} a_n/b_n \leqq \varlimsup_{n \to \infty} a_n/b_n \leqq C.$$

We also use C to denote various positive constants whose values might change from line to line.

2 Exact behavior

Let $\mu=\mathfrak{L}(X)$, $X=\sum_{k\geq 1}\lambda_k^{1/2}\xi_k e_k$ be a centered Gaussian measure on a separable Hilbert space H with $\sum_{k\geq 1}\lambda_k<\infty$ and $\lambda_k>0$, non-increasing. Here $\{\xi_k: n\geq 1\}$ are independent $N(0,1)$ and $\{e_k: k\geq 1\}$ is a orthonormal sequence in H. Then we have the following asymptotic estimates.

Theorem 1 *For any* $a=\sum_{k\geq 1}a_k e_k\in H$ *(i.e.* $\sum_{k\geq 1}a_k^2<\infty$*) and* $\lim_{t\to\infty}R(t)/t<\|a\|$*, we have as* $t\to\infty$,

$$P(\|X-t\cdot a\|^2\leq R^2(t)) \sim\frac{1}{\sqrt{2\pi}}\cdot\left(t^2\sum_{k\geq 1}\frac{a_k^2(2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3}+2\sum_{k\geq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right)^{-1/2} \cdot\exp\left\{R^2(t)\gamma-\sum_{k\geq 1}\frac{t^2a_k^2\gamma}{1+2\lambda_k\gamma}-\frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\gamma)\right\} \tag{2.1}$$

where $\gamma>0$ *is the unique solution of the following equation for t large*

$$R^2(t)=\frac{1}{\gamma}+t^2\sum_{k\geq 1}\frac{a_k^2}{(1+2\lambda_k\gamma)^2}+\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}. \tag{2.2}$$

Proof. The Laplace transform of the random variable $\|X-t\cdot a\|^2$ is given by

$$\int_H e^{-s\|x-ta\|^2}\mu(dx)=\exp\left\{-s\sum_{k\geq 1}\frac{t^2a_k^2}{1+2\lambda_k s}-\frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k s)\right\}.$$

Using the inversion formula (see, e.g. Theorem 27.1 in [3]), we have for every $\gamma>0$

$$P(\|X-ta\|^2\leq R^2(t))=\frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\Phi(s)}d\sigma \tag{2.3}$$

where $s=\gamma+i\sigma$ and

$$\Phi(s)=-\log s+R^2(t)s-s\sum_{k\geq 1}\frac{t^2a_k^2}{1+2\lambda_k s}-\frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k s).$$

Note that the function $\Phi'(s)$ has a zero at $(\gamma,0)$ where $\gamma=\gamma(t)$ is given by

$$R^2(t)=\frac{1}{\gamma}+t^2\sum_{k\geq 1}\frac{a_k^2}{(1+2\lambda_k\gamma)^2}+\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}. \tag{2.4}$$

Hence we take $\gamma>0$ in (2.3) to be the unique solution of (2.4) for t large.

Now we rewrite (2.3) as a sum

$$P(\|X-t\cdot a\|^2\leq R^2(t))=I_1+I_2+I_3+I_4 \tag{2.5}$$

where

$$I_1=\frac{1}{2\pi}\int_{\gamma\geqq|\sigma|>\gamma\beta^{-2/5}} e^{\Phi(s)}\,d\sigma$$

$$I_2=\frac{1}{2\pi}\int_{|\sigma|>\gamma} e^{\Phi(s)}\,d\sigma$$

$$I_3=\frac{1}{2\pi}\int_{|\sigma|<\gamma\beta^{-2/5}} e^{\operatorname{Re}\Phi(s)}\times(e^{i\operatorname{Im}\Phi(s)}-1)\,d\sigma$$

$$I_4=\frac{1}{2\pi}\int_{|\sigma|<\gamma\beta^{-2/5}} e^{\operatorname{Re}\Phi(s)}\,d\sigma$$

$$\beta=\beta(\gamma)=1+t^2\sum_{k\geqq 1}\frac{a_k^2(2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3}+2T$$

$$T=\sum_{k\geqq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2.$$

Note that $\beta\to\infty$ as $t\to\infty$. We will show I_4 is the dominating term.

Let us rewrite $\operatorname{Re}\Phi(s)=A(\gamma)+B(\gamma,\sigma)$ by using $\log(a+ib)=\log\sqrt{a^2+b^2}+i\arctan(b/a)$. Here

(2.6) $$A(\gamma)=R^2(t)\,\gamma-\log\gamma-\tfrac{1}{2}\sum_{k\geqq 1}\log(1+2\lambda_k\gamma)-\sum_{k\geqq 1}\frac{t^2a_k^2\gamma}{1+2\lambda_k\gamma}$$

and

(2.7) $$B(\gamma,\sigma)=-\tfrac{1}{2}\log\left(1+\left(\frac{\sigma}{\gamma}\right)^2\right)-\tfrac{1}{4}\sum_{k\geqq 1}\log\left(1+\left(\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}\right)^2\right)$$
$$-\left(\frac{\sigma}{\gamma}\right)^2\sum_{k\geqq 1}\frac{2t^2a_k^2\lambda_k\gamma^2}{(1+2\lambda_k\gamma)^3+(2\lambda_k\sigma)^2(1+2\lambda_k\gamma)}.$$

Then

(2.8) $$|I_1|\leqq\frac{1}{2\pi}e^{A(\gamma)}\int_{\gamma\geqq|\sigma|>\gamma\beta^{-2/5}} e^{B(\gamma,\sigma)}\,d\sigma$$

(2.9) $$|I_2|\leqq\frac{1}{2\pi}e^{A(\gamma)}\int_{|\sigma|>\gamma} e^{B(\gamma,\sigma)}\,d\sigma.$$

Since λ_k is non-increasing, we have for $\gamma\geqq|\sigma|>\gamma\beta^{-2/5}$

(2.10) $$\exp\left\{-\tfrac{1}{4}\sum_{k\geqq 1}\log\left(1+\left(\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}\right)^2\right)\right\}$$
$$\leqq\exp\left\{-\tfrac{1}{4}\sum_{k\geqq 5}\log\left(1+\beta^{-4/5}\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right)\right\}\cdot\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}$$
$$\leqq\exp\left\{-\tfrac{1}{2}\beta^{-4/5}\sum_{k\geqq 5}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right\}\cdot\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}$$
$$\leqq C\cdot\exp\{-\tfrac{1}{2}\beta^{-4/5}T\}\cdot\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}$$

where the second inequality is by $\log(1+x) \geqq x/2$ for $1>x>0$. For $\gamma \geqq |\sigma| > \gamma\beta^{-2/5}$, we also have

$$\exp\left\{-\left(\frac{\sigma}{\gamma}\right)^2 \sum_{k\geqq 1} \frac{2t^2 a_k^2 \lambda_k \gamma^2}{(1+2\lambda_k\gamma)^3+(2\lambda_k\sigma)^2(1+2\lambda_k\gamma)}\right\} \tag{2.11}$$
$$\leqq \exp\left\{-\beta^{-4/5} \sum_{k\geqq 1} \frac{t^2 a_k^2 \lambda_k \gamma^2}{(1+2\lambda_k\gamma)^3}\right\}$$
$$\leqq \exp\{-\beta^{-4/5}\cdot\tfrac{1}{4}(\beta-1-2T)\}.$$

Therefore by combining (2.8), (2.7), (2.10) and (2.11), we obtain

$$|I_1| \leqq C\cdot e^{A(\gamma)}\cdot\exp\{-\tfrac{1}{2}\beta^{-4/5}T\}\cdot\exp\{-\beta^{-4/5}\cdot\tfrac{1}{4}(\beta-1-2T)\} \tag{2.12}$$
$$\int_{\gamma\geqq|\sigma|>\gamma\beta^{-2/5}} \left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1} d\sigma$$
$$\leqq C\, e^{A(\gamma)}\cdot\exp\{-\tfrac{1}{4}\beta^{1/5}-\tfrac{1}{4}\beta^{-4/5}\} \int_{-\infty}^{+\infty} \left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1} d\sigma$$
$$\leqq C\cdot\gamma\cdot e^{A(\gamma)}\cdot\exp\{-\tfrac{1}{4}\beta^{1/5}\}.$$

Turning to I_2 and using similar estimates as in (2.10), we have for $|\sigma|>\gamma$

$$\exp\left\{-\tfrac{1}{4}\sum_{k\geqq 1}\log\left(1+\left(\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}\right)^2\right)\right\} \tag{2.13}$$
$$\leqq \exp\left\{-\tfrac{1}{4}\sum_{k\geqq 5}\log\left(1+\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right)\right\}\cdot\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}$$
$$\leqq \exp\left\{-\tfrac{1}{2}\sum_{k\geqq 5}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right\}\cdot\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}$$
$$\leqq C\cdot\exp\{-\tfrac{1}{2}T\}\cdot\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}.$$

For $|\sigma|>\gamma$, we also have by noting that the third term in (2.7) is an increasing function of σ (without the negative sign)

$$\exp\left\{-\left(\frac{\sigma}{\gamma}\right)^2 \sum_{k\geqq 1} \frac{2t^2 a_k^2 \lambda_k \gamma^2}{(1+2\lambda_k\gamma)^3+(2\lambda_k\sigma)^2(1+2\lambda_k\gamma)}\right\} \tag{2.14}$$
$$\leqq \exp\left\{-\sum_{k\geqq 1} \frac{t^2 a_k^2 \lambda_k \gamma^2}{(1+2\lambda_k\gamma)^3}\right\}$$
$$= \exp\{-\tfrac{1}{4}(\beta-1-2T)\}.$$

Therefore by combining (2.9), (2.7), (2.13) and (2.14), we obtain

$$(2.15)\qquad \begin{aligned}|I_2| &\leqq C\cdot e^{A(\gamma)}\cdot\exp\{-\tfrac{1}{2}T\}\\ &\quad\cdot\exp\{-\tfrac{1}{4}(\beta-1-2T)\}\int\limits_{|\sigma|>\gamma}\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}d\sigma\\ &\leqq C\cdot e^{A(\gamma)}\cdot\exp\{-\tfrac{1}{4}\beta\}\int\limits_{-\infty}^{+\infty}\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}d\sigma\\ &\leqq C\cdot\gamma\cdot e^{A(\gamma)}\cdot\exp\{-\tfrac{1}{4}\beta\}.\end{aligned}$$

Now turning to I_3, we have $|e^{i\,\mathrm{Im}\,\Phi(s)}-1|\leqq|\mathrm{Im}\,\Phi(s)|$ and

$$\begin{aligned}\mathrm{Im}\,\Phi(s) &= R^2(t)\,\sigma-\arctan\frac{\sigma}{\gamma}-\frac{1}{2}\sum_{k\geqq 1}\arctan\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}-\sigma\sum_{k\geqq 1}\frac{t^2a_k^2}{(1+2\lambda_k\gamma)^2+(2\lambda_k\sigma)^2}\\ &=\left(\frac{\sigma}{\gamma}-\arctan\frac{\sigma}{\gamma}\right)+\tfrac{1}{2}\sum_{k\geqq 1}\left(\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}-\arctan\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}\right)\\ &\quad+\sigma\sum_{k\geqq 1}\left(\frac{t^2a_k^2}{(1+2\lambda_k\gamma)^2}-\frac{t^2a_k^2}{(1+2\lambda_k\gamma)^2+(2\lambda_k\sigma)^2}\right)\end{aligned}$$

by plugging in (2.4). Thus we have by using the inequality $x-\arctan x\leqq x^3/3$, $x>0$ and $1/x^2-1/(x^2+y^2)\leqq y^2/x^4$

$$\begin{aligned}|\mathrm{Im}\,\Phi(s)| &\leqq \frac{1}{3}\left|\frac{\sigma}{\gamma}\right|^3+\frac{1}{6}\left|\frac{\sigma}{\gamma}\right|^3\sum_{k\geqq 1}\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^3+\left|\frac{\sigma}{\gamma}\right|^3\cdot t^2\sum_{k\geqq 1}\frac{a_k^2(2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3}\cdot\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\\ &\leqq \tfrac{1}{3}\beta^{-6/5}+\tfrac{4}{3}\beta^{-6/5}\sum_{k\geqq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2+\beta^{-6/5}\cdot t^2\sum_{k\geqq 1}\frac{a_k^2(2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3}\\ &\leqq C\cdot\beta^{-1/5}\end{aligned}$$

for $|\sigma|\leqq\gamma\beta^{-2/5}$. Hence

$$(2.16)\qquad |I_3|\leqq C\cdot\beta^{-1/5}\int\limits_{|\sigma|<\gamma\beta^{-2/5}}e^{\mathrm{Re}\,\Phi(s)}\,d\sigma.$$

Next we turn to the dominating term I_4. Using the inequality $x-x^2/2\leqq\log(1+x)\leqq x$, we have from (2.7) that

$$-\frac{1}{2}\left(\frac{\sigma}{\gamma}\right)^2\beta(\gamma)\leqq B(\gamma,\sigma)\leqq-\frac{1}{2}\left(\frac{\sigma}{\gamma}\right)^2\left(1-\left(\frac{\sigma}{\gamma}\right)^2\right)\beta(\gamma).$$

Hence by the change of variables,

$$\begin{aligned}\frac{\gamma}{\sqrt{\beta}}\int\limits_{|u|<\beta^{1/10}}e^{-u^2/2}\,du &\leqq \int\limits_{|\sigma|<\gamma\beta^{-2/5}}e^{B(\gamma,\sigma)}\,d\sigma\\ &\leqq\frac{\gamma}{\sqrt{\beta}}\int\limits_{|u|<\beta^{1/10}}e^{-u^2/2}\,du\times\exp\{\tfrac{1}{2}\beta^{-3/5}\}.\end{aligned}$$

Hence

$$\int_{|\sigma|<\gamma\beta^{-2/5}} e^{B(\gamma,\sigma)}\,d\sigma \sim \frac{\gamma}{\sqrt{\beta}} \int_{|u|<\beta^{1/10}} e^{-u^2/2}\,du \sim \frac{\sqrt{2\pi}\,\gamma}{\sqrt{\beta}}.$$

Thus

$$I_4 = \frac{1}{2\pi} e^{A(\gamma)} \int_{|\sigma|<\gamma\beta^{-2/5}} e^{B(\gamma,\sigma)}\,d\sigma \sim \frac{\gamma}{\sqrt{2\pi\beta}} e^{A(\gamma)}. \tag{2.17}$$

Combining (2.5), (2.12), (2.15), (2.16) and (2.17), we have as $t \to \infty$

$$\begin{aligned} &P(\|X - t\cdot a\|^2 \leqq R^2(t)) \\ &\sim \frac{\gamma}{\sqrt{2\pi\beta}} e^{A(\gamma)} \\ &= \frac{1}{\sqrt{2\pi\beta}} \exp\left\{R^2(t)\,\gamma - \sum_{k\geqq 1} \frac{t^2 a_k^2 \gamma}{1+2\lambda_k\gamma} - \frac{1}{2}\sum_{k\geqq 1} \log(1+2\lambda_k\gamma)\right\}. \end{aligned}$$

This completes the proof of the theorem by substituting β in.

Remark. The asymptotic expression given in our theorem is still implicit in terms of t and there does not seem to be an immediate explicit form. However, with a little bit of extra work, we can obtain some useful information in many interesting cases.

3 Corollaries

The following corollary is Borell's result [2] on Hilbert space.

Corollary 1 *If* $R(t)=R$, $0<R<\infty$ *and* $a\in H_\mu$ (*i.e.* $\|a\|_\mu^2 = \sum_{k\geqq 1} a_k^2/\lambda_k < \infty$), *then as* $t\to\infty$

$$\log P(\|X - t\cdot a\| \leqq R) \sim -\tfrac{1}{2}\|a\|_\mu^2 \cdot t^2.$$

Proof. Note that from (2.2), $\gamma>0$ is given by

$$R^2 = \frac{1}{\gamma} + \frac{t^2}{\gamma} \sum_{k\geqq 1} \frac{a_k^2}{\lambda_k} \cdot \frac{\lambda_k\gamma}{(1+2\lambda_k\gamma)^2} + \sum_{k\geqq 1} \frac{\lambda_k}{1+2\lambda_k\gamma}. \tag{3.1}$$

Hence it follows that $\gamma=\gamma(t)\to\infty$, $\gamma=o(t^2)$ as $t\to\infty$ since the two summations on the right side of (3.1) tend to zero as $t\to\infty$ by the D.C.T. Observing that

$$1 \leqq t^2 \sum_{k\geqq 1} \frac{a_k^2 (2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3} + 2\sum_{k\geqq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2 \leqq t^2 \sum_{k\geqq 1} \frac{a_k^2}{\lambda_k} + \gamma$$

for t large, we have by substituting (3.1) into (2.1)

$$\begin{aligned}
&\log P(\|X-t\cdot a\|^2 \leqq R^2)\\
&\sim R^2\gamma-\sum_{k\geqq 1}\frac{t^2 a_k^2\gamma}{1+2\lambda_k\gamma}-\frac{1}{2}\sum_{k\geqq 1}\log(1+2\lambda_k\gamma)\\
&=1-\frac{t^2}{2}\sum_{k\geqq 1}\frac{a_k^2}{\lambda_k}\cdot\frac{(2\lambda_k\gamma)^2}{(1+2\lambda_k\gamma)^2}+\sum_{k\geqq 1}\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}-\frac{1}{2}\sum_{k\geqq 1}\log(1+2\lambda_k\gamma)\\
&\sim-\frac{t^2}{2}\sum_{k\geqq 1}\frac{a_k^2}{\lambda_k}+o(\gamma)\\
&\sim-\frac{t^2}{2}\sum_{k\geqq 1}\frac{a_k^2}{\lambda_k}=-\frac{t^2}{2}\|a\|_\mu^2
\end{aligned}$$

which proves Corollary 1.

The next corollary gives the second order term for the above result when the point a comes from an important dense subset of H_μ. This can be viewed as a refinement of Borell's result.

Corollary 2 *If* $R(t)=R=constant$ *and* $a\in SH^*$ *(i.e.* $\|S^{-1}a\|_{H^*}^2=\sum_{k\geqq 1} a_k^2/\lambda_k^2<\infty$), *then as* $t\to\infty$

$$\log(\exp\{\tfrac{1}{2}\|a\|_\mu^2\cdot t^2\}\cdot P(\|X-t\cdot a\|\leqq R))\sim R\cdot\|S^{-1}a\|_{H^*}\cdot t.$$

Proof. From (2.2), $\gamma>0$ is given by

$$R^2=\frac{1}{\gamma}+\frac{t^2}{4\gamma^2}\sum_{k\geqq 1}\frac{a_k^2}{\lambda_k^2}\cdot\frac{(2\lambda_k\gamma)^2}{(1+2\lambda_k\gamma)^2}+\sum_{k\geqq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}. \tag{3.2}$$

It is easy to see that $\gamma=\gamma(t)\to\infty$ as $t\to\infty$, and

$$\lim_{\gamma\to\infty}\sum_{k\geqq 1}\frac{a_k^2}{\lambda_k^2}\cdot\frac{(2\lambda_k\gamma)^2}{(1+2\lambda_k\gamma)^2}=\sum_{k\geqq 1}\frac{a_k^2}{\lambda_k^2}\quad\text{and}\quad\lim_{\gamma\to\infty}\sum_{k\geqq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}=0$$

by the D.C.T. Hence we have as $t\to\infty$

$$R^2\sim\frac{t^2}{4\gamma^2}\cdot\sum_{k\geqq 1}\frac{a_k^2}{\lambda_k^2}=\frac{t^2}{4\gamma^2}\cdot\|S^{-1}a\|_{H^*}^2\quad\text{or}\quad\gamma\sim\frac{\|S^{-1}a\|_{H^*}}{2R}\cdot t. \tag{3.3}$$

Note that as $t\to\infty$

$$\begin{aligned}
&t^2\sum_{k\geqq 1}\frac{a_k^2(2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3}+2\sum_{k\geqq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\\
&=(t^2/2\gamma)\cdot\sum_{k\geqq 1}(a_k^2/\lambda_k^2)\cdot(2\lambda_k\gamma/(1+2\lambda_k\gamma))^3+o(\gamma)\\
&\sim(t^2/2\gamma)\cdot\sum_{k\geqq 1}(a_k^2/\lambda_k^2)+o(\gamma)\\
&=R\|S^{-1}a\|_{H^*}\cdot t+o(t)
\end{aligned} \tag{3.4}$$

and

$$\sum_{k\geq 1} \lambda_k \gamma/(1+2\lambda_k \gamma) = o(\gamma), \qquad \sum_{k\geq 1} \log(1+2\lambda_k \gamma) = o(\gamma).$$

Hence from Theorem 1, by substituting R in (3.2) into (2.1), we have

$$\begin{aligned}
&\log(\exp\{\tfrac{1}{2}\|a\|_\mu^2 \cdot t^2\} \cdot P(\|X - t\cdot a\| \leq R)) \\
&\sim \frac{t^2}{2} \sum_{k\geq 1} \frac{a_k^2}{\lambda_k} + t^2 \sum_{k\geq 1} \frac{a_k^2 \gamma}{(1+2\lambda_k \gamma)^2} - t^2 \sum_{k\geq 1} \frac{a_k^2 \gamma}{1+2\lambda_k \gamma} + o(\gamma) \\
&= \frac{t^2}{2\gamma} \cdot \sum_{k\geq 1} \frac{a_k^2}{\lambda_k^2} \cdot \frac{(2\lambda_k \gamma)^2}{(1+2\lambda_k \gamma)^2} + \frac{t^2}{2} \cdot \sum_{k\geq 1} \frac{a_k^2}{(1+2\lambda_k \gamma)^2 \lambda_k} + o(\gamma) \\
&\sim (t^2/2\gamma) \cdot \sum_{k\geq 1} a_k^2/\lambda_k^2 + o(\gamma) \\
&\sim R \cdot \|S^{-1} a\|_{H^*} \cdot t
\end{aligned}$$

which proves the claim.

The following corollary shows that under certain conditions, we do have an explicit asymptotic expression for $P(\|X - ta\| \leq R)$ as $t \to \infty$.

Corollary 3 *Let $a = \sum_{k\geq 1} a_k e_k$ be in H with $\sum_{k\geq 1} a_k^2/\lambda_k^3 < \infty$ and suppose that*

$$\lim_{s\to\infty} \sum_{k\geq 1} \frac{\lambda_k \sqrt{s}}{1+2\lambda_k s} = 0. \tag{3.5}$$

Then as $t \to \infty$

$$\begin{aligned}
P(\|X - ta\| \leq R) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{R\sqrt{2\rho t}} \prod_{k\geq 1} (1+2\lambda_k \rho t)^{-1/2} \\
\cdot \exp\left\{ -\frac{t^2}{2} \sum_{k\geq 1} \frac{a_k^2}{\lambda_k} + tR \left(\sum_{k\geq 1} \frac{a_k^2}{\lambda_k^2} \right)^{1/2} \right. \\
\left. -\frac{R^2}{2} \left(\sum_{k\geq 1} \frac{a_k^2}{\lambda_k^3} \right) \left(\sum_{k\geq 1} \frac{a_k^2}{\lambda_k^2} \right)^{-1} \right\}
\end{aligned}$$

where

$$\rho^2 = (4R^2)^{-1} \sum_{k\geq 1} a_k^2/\lambda_k^2 = (4R^2)^{-1} \|S^{-1} a\|_{H^*}^2.$$

Proof. For $\gamma \to \infty$ we have

$$\sum_{k\geq 1} \frac{a_k^2}{(1+2\lambda_k \gamma)^2} = \frac{1}{4\gamma^2} \sum_{k\geq 1} \frac{a_k^2}{\lambda_k^2} - \frac{1}{4\gamma^3} \sum_{k\geq 1} \frac{a_k^2}{\lambda_k^3} + o(\gamma^{-3}) \tag{3.6}$$

and

$$\sum_{k\geq 1} \frac{a_k^2 \gamma}{1+2\lambda_k \gamma} = \frac{1}{2} \sum_{k\geq 1} \frac{a_k^2}{\lambda_k} - \frac{1}{4\gamma} \sum_{k\geq 1} \frac{a_k^2}{\lambda_k^2} + \frac{1}{8\gamma^2} \sum_{k\geq 1} \frac{a_k^2}{\lambda_k^3} + o(\gamma^{-2}). \tag{3.7}$$

Let now γ be defined as in (2.2) with $R(t)=R$. Then from (2.2) and (3.6)

$$R^2=\frac{t^2}{\gamma^2}\left(\frac{1}{4}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^2}-\frac{1}{4\gamma}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^3}+o(\gamma^{-1})\right)+\frac{1}{\gamma}+\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}. \tag{3.8}$$

Hence

$$\gamma\cdot t^{-1}=(2R)^{-1}\Big(\sum_{k\geq 1}a_k^2/\lambda_k^2\Big)^{1/2}+o(1)=\rho+o(1) \tag{3.9}$$

and it follows by (3.8) that

$$R^2\gamma\left(1-\frac{\rho^2t^2}{\gamma^2}\right)=1-\frac{1}{4\rho^2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^3}+\gamma\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}+o(1). \tag{3.10}$$

Dividing (3.10) by $1+\rho t\gamma^{-1}$ and using $\rho t\gamma^{-1}\to 1$ as $t\to\infty$, we have

$$R^2(\gamma-\rho t)=-\frac{1}{8\rho^2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^3}+\frac{1}{2}+\frac{\gamma}{1+\rho t\gamma^{-1}}\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}+o(1) \tag{3.11}$$

as $t\to\infty$. This implies $\gamma>\rho t$ for t large, and

$$\begin{aligned}
&\frac{2\rho^2R^2t^2}{\gamma}+\gamma\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}\\
&=2\rho R^2t-\frac{2\rho tR^2}{\gamma}(\gamma-\rho t)+\gamma\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}\\
&=2\rho R^2t-1+\frac{1}{4\rho^2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^3}+\left(\gamma-\frac{2\rho t}{1+\rho t\gamma^{-1}}\right)\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}+o(1)\\
&=2\rho R^2t-1+\frac{1}{4\rho^2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^3}+\frac{\gamma}{R^2(1+\rho t\gamma^{-1})^2}\left(\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}\right)^2+o(1)\\
&=2\rho R^2t-1+\frac{1}{4\rho^2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^3}+o(1)
\end{aligned} \tag{3.12}$$

where we use (3.11) in the second and third equality, and (3.5) in the forth equality. Using (3.11), $\log(1+x)\leqq x$ for $x>0$ and (3.5), we have as $t\to\infty$ (since $\gamma>\rho t$ for t large)

$$\begin{aligned}
0&\leqq\tfrac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\gamma)-\tfrac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\rho t)\\
&=\tfrac{1}{2}\sum_{k\geq 1}\log\left(1+\frac{2\lambda_k(\gamma-\rho t)}{1+2\lambda_k\rho t}\right)\\
&\leqq(\gamma-\rho t)\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\rho t}\\
&\leqq\frac{\gamma}{2R^2}\cdot\left(\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\rho t}\right)\cdot\left(\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}\right)+o(1)\\
&=o(1).
\end{aligned} \tag{3.13}$$

Hence we have by (3.6), (3.7), (3.9), (3.12) and (3.13)

$$R^2\gamma-\sum_{k\geq 1}\frac{t^2a_k^2\gamma}{1+2\lambda_k\gamma}-\frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\gamma)$$

$$=1+t^2\left(\sum_{k\geq 1}\frac{\gamma a_k^2}{(1+2\lambda_k\gamma)^2}-\sum_{k\geq 1}\frac{a_k^2\gamma}{1+2\lambda_k\gamma}\right)$$

$$+\gamma\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}-\frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\gamma)$$

$$=1-\frac{t^2}{2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k}+\frac{2\rho^2R^2t^2}{\gamma}-\frac{3}{8\rho^2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^3}$$

$$+\gamma\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}-\frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\rho t)+o(1)$$

$$=1-\frac{t^2}{2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k}+2\rho R^2t-\frac{1}{8\rho^2}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^3}-\tfrac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\rho t)+o(1).$$

This provides the desired estimates for the exponential term in (2.1). The estimates for the other terms in (2.1) are given in (3.4). Thus the corollary is proved.

By using Corollary 3, one can prove the following result for the finite dimensional setting.

Corollary 4 *If* $\lambda_1,\ldots,\lambda_n>0$ *are real,* $a_1,\ldots,a_n\in R^1$ *and* $R(t)=R$, $0<R<\infty$, *then*

$$P\left(\sum_{k=1}^{n}|\lambda_k\xi_k-t\cdot a_k|^2\leq R^2\right)$$

$$\sim(2\pi)^{-1/2}\cdot\left(\sum_{k=1}^{n}\frac{a_k^2}{\lambda_k^2}\right)^{-(n+1)/4}\cdot\left(\prod_{k=1}^{n}\lambda_k\right)^{-1/2}\cdot R^{(n-1)/2}\cdot t^{-(n+1)/2}$$

$$\cdot\exp\left\{-\frac{t^2}{2}\sum_{k=1}^{n}\frac{a_k^2}{\lambda_k}+t\cdot R\left(\sum_{k=1}^{n}\frac{a_k^2}{\lambda_k^2}\right)^{1/2}-\frac{R^2}{2}\cdot\sum_{k=1}^{n}\frac{a_k^2}{\lambda_k^3}\cdot\left(\sum_{k=1}^{n}\frac{a_k^2}{\lambda_k^2}\right)^{-1}\right\}.$$

Our next two corollaries are a refinement of some large deviation results on Hilbert space. The large deviation estimates state that as $t\to\infty$

$$\log P\left(\left\|\frac{X}{t}-a\right\|\leq R\right)\sim-\inf_{x\in B_R(a)}I(x)\cdot t^2$$

where $B_R(a)=\{x\colon\|x-a\|\leq R\}$ and $I(x)=\|x\|_\mu^2/2=(1/2)\sum_{k\geq 1}x_k^2/\lambda_k$.

Corollary 5 *If* $R(t)=R\cdot t$ *where* R *is a constant and* $0<R<(\sum_{k\geq 1}a_k^2)^{1/2}$, *then as* $t\to\infty$

$$P(\|X-t\cdot a\|\leq R\cdot t)\sim K_{a,R}\cdot\frac{1}{t}\cdot\exp\{-\inf_{x\in B_R(a)}I(x)\cdot t^2\}\tag{3.14}$$

where

$$K_{a,R}=\frac{1}{\sqrt{2\pi}}\cdot\left(\sum_{k\geqq 1}\frac{a_k^2\cdot 4\lambda_k\gamma_0^2}{(1+2\lambda_k\gamma_0)^3}\cdot\prod_{k\geqq 1}(1+2\lambda_k\gamma_0)\right)^{-1/2}$$

and $\gamma_0>0$ *satisfies*

$$R^2=\sum_{k\geqq 1}\frac{a_k^2}{(1+2\lambda_k\gamma_0)^2}. \tag{3.15}$$

Proof. In the setting here, we have from Theorem 1,

$$\begin{aligned}&P(\|X-t\cdot a\|^2\leqq R^2\cdot t^2)\\ &\sim\frac{1}{\sqrt{2\pi}}\cdot\left(t^2\sum_{k\geqq 1}\frac{a_k^2(2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3}+2\sum_{k\geqq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right)^{-1/2}\\ &\cdot\exp\left\{R^2t^2\gamma-\tfrac{1}{2}\sum_{k\geqq 1}\log(1+2\lambda_k\gamma)-\sum_{k\geqq 1}\frac{t^2a_k^2\gamma}{1+2\lambda_k\gamma}\right\}\end{aligned} \tag{3.16}$$

where $\gamma>0$ is the unique solution of the following equation for t large

$$R^2t^2=\frac{1}{\gamma}+t^2\sum_{k\geqq 1}\frac{a_k^2}{(1+2\lambda_k\gamma)^2}+\sum_{k\geqq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}. \tag{3.17}$$

Note that from (3.17), $\gamma=\gamma(t)$ is a decreasing function of t by implicit differentiation, and $\gamma=\gamma(t)\to\gamma_0$ as $t\to\infty$ where γ_0, given in (3.15), is independent of t. Hence by substituting R in (3.17) into (3.16), we have

$$\begin{aligned}&P(\|X-t\cdot a\|^2\leqq R^2\cdot t^2)\\ &\sim\frac{1}{\sqrt{2\pi}}\cdot\left(t^2\sum_{k\geqq 1}\frac{a_k^2(2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3}\cdot\prod_{k\geqq 1}(1+2\lambda_k\gamma_0)\right)^{-1/2}\\ &\cdot\exp\left\{1-t^2\sum_{k\geqq 1}\frac{a_k^2\cdot 2\lambda_k\gamma^2}{(1+2\lambda_k\gamma)^2}+\sum_{k\geqq 1}\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right\}.\end{aligned} \tag{3.18}$$

Observing that by substituting R in (3.15) into (3.17), we have

$$t^2(\gamma-\gamma_0)\sum_{k\geqq 1}\frac{a_k^2\cdot 4\lambda_k(1+\lambda_k\gamma_0+\lambda_k\gamma)}{(1+2\lambda_k\gamma_0)^2(1+2\lambda_k\gamma)^2}=\frac{1}{\gamma}+\sum_{k\geqq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}. \tag{3.19}$$

Letting $t\to\infty$ in (3.19) gives us

$$\left(\lim_{t\to\infty}t^2(\gamma-\gamma_0)\right)\cdot\sum_{k\geqq 1}\frac{a_k^2\cdot 4\lambda_k}{(1+2\lambda_k\gamma_0)^3}=\frac{1}{\gamma_0}+\sum_{k\geqq 1}\frac{\lambda_k}{1+2\lambda_k\gamma_0}. \tag{3.20}$$

Now

$$\lim_{t\to\infty} t^2\left(\sum_{k\geq 1}\frac{a_k^2\cdot 2\lambda_k\gamma^2}{(1+2\lambda_k\gamma)^2}-\sum_{k\geq 1}\frac{a_k^2\cdot 2\lambda_k\gamma_0^2}{(1+2\lambda_k\gamma_0)^2}\right) \tag{3.21}$$
$$=\lim_{t\to\infty}\left(t^2(\gamma-\gamma_0)\sum_{k\geq 1}\frac{2a_k^2\lambda_k(\gamma+\gamma_0+4\lambda_k\gamma_0\gamma)}{(1+2\lambda_k\gamma_0)^2(1+2\lambda_k\gamma)^2}\right)$$
$$=\lim_{t\to\infty}(t^2(\gamma-\gamma_0))\cdot\sum_{k\geq 1}\frac{a_k^2\lambda_k\cdot 4\gamma_0}{(1+2\lambda_k\gamma_0)^3}$$
$$=1+\sum_{k\geq 1}\frac{\lambda_k\gamma_0}{1+2\lambda_k\gamma_0}.$$

From the results in [9], we have

$$\inf_{x\in B_R(a)} I(x)=\sum_{k\geq 1}\frac{a_k^2\cdot 2\lambda_k\gamma_0^2}{(1+2\lambda_k\gamma_0)^2}. \tag{3.22}$$

Combining (3.18), (3.21) and (3.22), we obtain (3.14) and hence finish the proof.

Before we state the next corollary we need to establish the following lemma. It is well known from results in large deviation theory.

Lemma 1 *Let V be a convex open subset of H such that $V\cap \bar{H}_\mu$ is non-empty and the zero vector is not in V. Then there is a unique point b on the boundary of V such that*

$$I(b)=\inf_{x\in V} I(x)=\inf_{x\in\bar{V}} I(x)<\infty. \tag{3.23}$$

Corollary 6 *Let U be a convex open subset of the Hilbert space H such that $U\cap\bar{H}_\mu\neq\emptyset$ and $0\notin V$. Take $x_0\in H_\mu$ to be the unique point on the boundary of U such that*

$$I(x_0)=\inf_{x\in U} I(x). \tag{3.24}$$

If $I(x_0)>0$, and for some $R>0$ and $a\in U\cap\bar{H}_\mu$, the interior of the ball $B_R(a)$ is a subset of U, and x_0 is on the boundary of $B_R(a)$, then, as $t\to\infty$

$$P(X/t\in U)\approx t^{-1}\cdot\exp\{-\inf_{x\in U} I(x)\cdot t^2\}. \tag{3.25}$$

Proof. Since the interior of the ball $B_R(a)$ is a subset of U, we have

$$P(X/t\in U)\geqq P(\|X/t-a\|<R). \tag{3.26}$$

Also, since $a\in U\cap\bar{H}_\mu$, Proposition 5 in [5] implies

$$P(\|X/t-a\|\leqq R)=P(\|X/t-a\|<R). \tag{3.27}$$

Furthermore, by the lemma $\inf_{\|x-a\|<R} I(x) = \inf_{x\in B_R(a)} I(x)$, and since (3.24) holds with the interior of $B_R(a)$ in U and $x_0 \in B_R(a)$, we get

$$I(x_0) = \inf_{x\in B_R(a)} I(x). \tag{3.28}$$

Thus for each $\delta>0$ and t sufficiently large

$$P(X/t\in U) \geqq (1-\delta)\, K_{a,R}\, t^{-1}\cdot \exp\{-I(x_0)\cdot t^2\}. \tag{3.29}$$

Now let $C=\{x\in H_\mu: \|x\|_\mu \leqq (2I(x_0))^{1/2}\}$. Then C is compact and convex in H, and $C\cap\{x: \|x-a\|<R\}=\emptyset$, or we would have a contradiction to (3.28). Hence by the Hahn-Banach theorem there exists a continuous linear functional $f\in H^*$ such that $\{x: f(x)\leqq f(x_0)\}\supseteq C$ and $V=\{x: f(x)>f(x_0)\}\supseteq U$. Thus

$$\begin{aligned} P(X/t\in U) &\leqq P(X/t\in V) \\ &= P(f(X)>t\cdot f(x_0)) \\ &\leqq (2\pi)^{-1/2}\sigma_f\, f(x_0)^{-1}\cdot t^{-1}\exp\{-t^2 f^2(x_0)/(2\sigma_f^2)\} \end{aligned} \tag{3.30}$$

where

$$\sigma_f^2 = E(f^2(X)). \tag{3.31}$$

Now from [7, Lemma 2.1],

$$\sigma_f^2 = \sup_{\|x\|_\mu\leqq 1} f^2(x), \tag{3.32}$$

and hence

$$2\sigma_f^2 I(x_0) = \sup_{x\in C} f^2(x) \geqq f^2(x_0) \tag{3.33}$$

as $x_0\in C$. On the other hand, since $-C=C$ and $\{x: f(x)\leqq f(x_0)\}\supseteq C$ we have $C\subseteq\{x: |f(x)|\leqq f(x_0)\}$. Thus $\sup_{x\in C} f^2(x)\leqq f^2(x_0)$, so (3.33) implies

$$f^2(x_0)/(2\sigma_f^2) = I(x_0). \tag{3.34}$$

Combining (3.30) and (3.34) we thus have

$$P(X/t\in U)\leqq (4\pi I(x_0))^{-1/2}\cdot t^{-1}\exp\{-I(x_0)\cdot t^2\}.$$

Hence the corollary is proved.

Remark. If $I(x_0)=0$ the conclusion of the corollary need not be true. For example, if $U=\{x: f(x)>0\}$ where f is a continous linear functional with $f\neq 0$, then $P(X/t\in U)=P(X\in U)=1/2$, for all $t>0$. If $I(x_0)>0$ and U can be separated from C by finitely many half spaces with C in the interior of all but one of them which contains x_0 as in the proof above, then (3.25) holds as $t\to\infty$.

4 Some upper and lower bounds in the Banach space setting

If E is a separable Banach space, we have not been able to obtain the precision of the results given in Theorem 1. However, extending the approach of Grill in [4], we do obtain some upper and lower bounds for translated balls which have interesting applications to the functional form of Chung's LIL, see, for example, [8] and [9]. We include these results in anticipation that they will have further uses as well. Now we need some additional notation.

Let E denote a separable Banach space with norm $\|\cdot\|$ and topological dual E^*. If X is a centered Gaussian random vector with values in E and $\mu=\mathfrak{L}(X)$, then it is well known that there is a unique Hilbert space $H_\mu \subseteq E$ such that μ is determined by considering the pair (E, H_μ) as an abstract Wiener space (see [6]). The Hilbert space H_μ can be described as the completion of the range of the mapping $S\colon E^* \to E$ defined by the Bochner integral

(4.1) $$Sf=\int_E x f(x)\, d\mu(x) \qquad f\in E^*,$$

where the completion is in the inner product norm

(4.2) $$\langle Sf, Sg\rangle_\mu=\int_E f(x)\, g(x)\, d\mu(x) \qquad f, g\in E^*.$$

Lemma 2.1 in [7] presents the details of this construction along with various properties of the relationship between H_μ and E. In particular, we will use the continuous linear maps

(4.3) $$\Pi_d(x)=\sum_{k=1}^{d} \alpha_k(x)\, S\alpha_k \quad \text{and} \quad Q_d(x)=x-\Pi_d(x) \qquad d\geqq 1$$

taking E to E. In (4.3), $\{\alpha_k\colon k\geqq 1\}$ is a sequence in E^* orthonormal in $L^2(\mu)$ such that $\{S\alpha_k\colon k\geqq 1\}$ is a CONS in $H_\mu\subseteq E$, and when restricted to H_μ, Π_d and Q_d are orthogonal projections onto their ranges. It is also well known that $\lim_{d\to\infty} \|Q_d(x)\|=0$ with μ-probability one when μ is centered Gaussian measure, and that for $f\in H_\mu$ we can define the stochastic inner product for μ almost all x in E by

(4.4) $$\langle x, f\rangle^\sim = \lim_{d\to\infty} \sum_{k=1}^{d} \alpha_k(x)\, \langle f, S\alpha_k\rangle_\mu = \lim_{d\to\infty} \sum_{k=1}^{d} \alpha_k(x)\, \alpha_k(f).$$

Then $\langle\cdot, f\rangle^\sim$ is $N(0,\sigma^2)$ where $\sigma^2=\langle f, f\rangle_\mu$, and if $f=Sh$ for some $h\in E^*$, we have

(4.5) $$\langle x, f\rangle^\sim = h(x).$$

Finally, if $f\in H_\mu$ the Cameron-Martin formula for the centered Gaussian measure μ takes the form

(4.6) $$\mu(A+f)=\int_A \exp\{-\tfrac{1}{2}\|f\|_\mu^2-\langle x, f\rangle^\sim\}\, d\mu(x)$$

for Borel subsets E of B. This is well known, but a particularly nice proof is contained in Proposition 2.1 of [1].

In this setting, the I-function of large deviations is

$$I(x)=\begin{cases}\|x\|_\mu^2/2 & x\in H_\mu\\ +\infty & \text{otherwise.}\end{cases} \tag{4.7}$$

Furthermore, defining

$$I(f,\delta)=\inf_{\|f-x\|\leqq\delta} I(x), \tag{4.8}$$

we see $I(f,\delta)<\infty$ for all $f\in\bar{H}_\mu$, the support of μ in E. It is also the case that all of the properties established for the function $I(x,\delta)$ in Lemma 1 of [4], when μ is Wiener measure on $C_0[0,1]$, have analogues for μ an arbitrary centered Gaussian measure on E. In particular, if $f\in E$ and $\delta>0$, then there is a unique element, call it $h_{f,\delta}$, such that $\|h_{f,\delta}-f\|\leqq\delta$ and

$$I(f,\delta)=I(h_{f,\delta}). \tag{4.9}$$

We will use these properties freely, and now we establish a lemma which provides a slight refinement of (5) in [4].

Lemma 2 *Let* $U(f,\delta)=\{x\in E\colon \|x-f\|\leqq\delta\}$. *If* $f\in H_\mu$, $\delta>0$, $\alpha>0$, *and* $h=h_{f,\alpha\delta}$, *then for almost all* $x\in U(f,\delta)$

$$\langle x,h\rangle^\sim\geqq(\alpha-1)\,\alpha^{-1}\langle f,h\rangle_\mu+\alpha^{-1}\langle h,h\rangle_\mu. \tag{4.10}$$

In particular, if $\alpha=1$, *then for almost all* $x\in U(f,\delta)$

$$\langle x,h\rangle^\sim\geqq\langle h,h\rangle_\mu. \tag{4.11}$$

Proof. Take $g\in U(f,\alpha)\cap H_\mu$. Then $h\in U(f,\alpha\delta)\cap H_\mu$, $\alpha g-(\alpha-1)f\in U(f,\alpha\delta)\cap H_\mu$, and hence for $0\leqq\lambda\leqq 1$,

$$(1-\lambda)\,h+\lambda(\alpha g-(\alpha-1)\,f)\in U(f,\alpha\delta).$$

Thus for $0\leqq\lambda\leqq1$

$$I((1-\lambda)\,h+\lambda(\alpha g-(\alpha-1)\,f))\geqq I(h),$$

and recalling the definition of $I(\cdot)$, this implies

$$\lambda^2\,\|\alpha g-(\alpha-1)\,f-h\|_\mu^2+2\lambda\langle\alpha g-(\alpha-1)\,f-h,h\rangle_\mu\geqq 0.$$

Since $0\leqq\lambda\leqq1$, this gives $\langle\alpha g-(\alpha-1)\,f-h,h\rangle_\mu\geqq 0$, which implies

$$\langle g,h\rangle_\mu\geqq(\alpha-1)\,\alpha^{-1}\langle f,h\rangle_\mu+\alpha^{-1}\langle h,h\rangle_\mu \tag{4.12}$$

for all $\alpha>0$ and $g\in H_\mu\cap U(f,\delta)$.

Recalling (4.3) and (4.4) we see for almost all $x\in E$

$$\lim_{d\to\infty}\|x-\Pi_d\,x\|=0 \tag{4.13}$$

and

(4.14) $$\langle x, h\rangle^{\sim}=\lim_{d\to\infty}\sum_{k=1}^{d}\alpha_k(x)\,\alpha_k(h)=\lim_{d\to\infty}\langle \Pi_d\, x, h\rangle_\mu.$$

Since the boundary of $U(f,\delta)$ has μ measure zero, (4.13) implies that for almost all $x\in U(f,\delta)$, that $\Pi_d\, x\in H_\mu\cap U(f,\delta)$ for d sufficiently large. Hence (4.14) and (4.12) combine to imply for almost all $x\in U(f,\delta)$

$$\langle x, h\rangle^{\sim}=\lim_{d\to\infty}\langle \Pi_d\, x, h\rangle_\mu \geqq (\alpha-1)\,\alpha^{-1}\langle f, h\rangle_\mu+\alpha^{-1}\langle h, h\rangle_\mu.$$

Hence (4.10) holds and the lemma is proved.

Theorem 2 *Let μ be a centered Gaussian measure on E. For $f\in H_\mu$, $\delta>0$, and $h=h_{f,\alpha\delta}$*

(4.15) $$\mu(x\colon \|x-f\|\leqq\delta) \leqq \exp\{-\sup_{\alpha>0}((\alpha-1)\,\alpha^{-1}\langle f,h\rangle_\mu-(\alpha-2)\,\alpha^{-1}I(h))\}\,\mu(x\colon \|x\|\leqq\delta),$$

and for $0\leqq\alpha\leqq 1$

(4.16) $$\mu(x\colon \|x-f\|\leqq\delta)\geqq\exp\{-I(h)\}\,\mu(x\colon \|x\|\leqq(1-\alpha)\,\delta).$$

Remark. If $\alpha=1$, then (4.15) implies

(4.17) $$\mu(x\colon \|x-f\|\leqq\delta)\leqq\exp\{-I(h_{f,\delta})\}\,\mu(x\colon \|x\|\leqq\delta).$$

Also it is easy to check that the sup in (4.15) is obtained at $\alpha=1$ when $\delta\leqq\|f\|$ and E is a Hilbert space, since then $I(f,\delta)$ can be computed explicitly as given in (3.22).

Proof. To proof (4.16), take $A=\{x\colon \|x\|\leqq r\}$, apply the Cameron-Martin translation formula in (4.6), Jensen's inequality, and the symmetry of $\langle x, f\rangle^{\sim}$ to obtain

(4.18) $$\mu(A-f)\geqq\exp\{-I(f)\}\,\mu(A).$$

Now let $h=h_{f,\alpha\delta}$ and applying (4.18) we obtain

$$\begin{aligned}\mu(x\colon \|x-f\|\leqq\delta)&\geqq\mu(x\colon \|x-h\|\leqq(1-\alpha)\,\delta)\\ &\geqq\exp\{-I(h)\}\,\mu(x\colon \|x\|\leqq(1-\alpha)\,\delta).\end{aligned}$$

Hence (4.16) holds.

To prove (4.15) apply the Cameron-Martin formula to obtain for $\alpha>0$ and $h=h_{f,\alpha\delta}$ that

$$\begin{aligned}\mu(x\colon \|x-f\|\leqq\delta)&=\exp\{-I(h)\}\int_{\|x-(f-h)\|\leqq\delta}\exp\{-\langle x, h\rangle^{\sim}\}\,d\mu(x)\\ &=\exp\{I(h)\}\int_{\|(x+h)-f\|\leqq\delta}\exp\{-\langle x+h, h\rangle^{\sim}\}\,d\mu(x)\end{aligned}$$

since $\langle x+h, h\rangle^{\sim} = \langle x, h\rangle^{\sim} + 2I(h)$. Applying (4.10) implies

$$(4.20)\qquad \int_{\|(x+h)-f\|\leqq\delta} \exp\{-\langle x+h, h\rangle^{\sim}\}\, d\mu(x) \leqq \exp\{-(\alpha-1)\,\alpha^{-1}\langle f, h\rangle_\mu - 2\alpha^{-1} I(h)\}\,\mu(x\colon \|x-(f-h)\|\leqq\delta),$$

and combining (4.19) and (4.20) yields

$$(4.21)\qquad \mu(x\colon \|x-f\|\leqq\delta) \leqq \exp\{-((\alpha-1)\,\alpha^{-1}\langle f, h\rangle_\mu - (\alpha-2)\,\alpha^{-1} I(h))\}\,\mu(x\colon \|x\|\leqq\delta)$$

since $\mu(x\colon \|x-g\|\leqq\delta)\leqq\mu(x\colon \|x\|\leqq\delta)$ for all $g\in E$ is well known. Since (4.21) holds for all $\alpha>0$, we thus have (4.15) and the theorem is proved.

Remark. If $\alpha\geqq\|f\|\,\delta^{-1}$, then $h_{f,\alpha\delta}=0$. So the sup in (4.15) is really on $0<\alpha\leqq\|f\|\,\delta^{-1}$.

Corollary 7 *For any* $f\in\bar{H}_\mu$ *and* $R(t)<(1-\varepsilon)\,\|f\|\,t$, $1>\varepsilon>0$, *we have*

$$(4.22)\qquad \mu(x\colon \|x-t\cdot f\|\leqq R(t))\geqq\exp\{-t^2 I(g_t)\}\,\mu(x\colon \|x\|\leqq(1-\varepsilon)\,R(t))$$

and

$$(4.23)\qquad \mu(x\colon \|x-t\cdot f\|\leqq R(t)) \leqq \exp\{-t^2 I(g_t, (1+\varepsilon)\,R(t)/t)\}\,\mu(x\colon \|x\|\leqq(1+\varepsilon)\,R(t))$$

where $g_t\in SE^*$ *satisfies*

$$(4.24)\qquad \|f-g_t\|\leqq\varepsilon R(t)/t.$$

Remark. Since SE^* is dense in H_μ and the E norm satisfies

$$\|x\|_E\leqq(E\,\|X\|^2)^{1/2}\,\|x\|_\mu \qquad x\in H_\mu,$$

we have SE^* is dense in $\bar{H}_\mu$. Thus g_t satisfying (4.24) exists.

Proof. To prove (4.22) observe that

$$\mu(x\colon \|x-t\cdot f\|\leqq R(t))\geqq\mu(x\colon \|x-t\cdot g_t\|\leqq(1-\varepsilon)\,R(t)) \geqq\exp\{-t^2 I(g_t)\}\,\mu(x\colon \|x\|\leqq(1-\varepsilon)\,R(t))$$

by (4.16) with $\alpha=0$. For (4.23) apply (4.15) with $\alpha=1$ to obtain

$$\begin{aligned}\mu(x\colon \|x-t\cdot f\|\leqq R(t))&\leqq\mu(x\colon \|x-t\cdot g_t\|\leqq(1+\varepsilon)\,R(t))\\ &\leqq\exp\{-I(t g_t, (1+\varepsilon)\,R(t))\}\,\mu(x\colon \|x\|\leqq(1+\varepsilon)\,R(t))\\ &\leqq\exp\{-t^2 I(g_t, (1+\varepsilon)\,R(t)/t)\}\,\mu(x\colon \|x\|\leqq(1+\varepsilon)\,R(t)).\end{aligned}$$

Thus the corollary is proved.

Remark. If $f\in H_\mu$ and $R(t)=R$, $0<R<\infty$, then taking $g_t=\Pi_{m_t} f$ where

$$m_t=\min\{m\geqq 1\colon \|f-\Pi_m f\|\leqq\varepsilon R/t\}$$

we have $g_t \in SE^*$, and (4.22) and (4.23) combine to give

$$\lim_{t\to\infty} t^{-2} \log \mu(x: \|x - t \cdot f\| \leqq R) = -\|f\|_\mu^2/2.$$

Returning to the Hilbert space setting, we can say more since the function $I(f, \delta)$ can be computed and g_t takes a special form. Let $(e_k)_{k\geqq 1}$ be an orthonormal bases in H and $\lambda_k > 0$ with $\sum_{k\geqq 1} \lambda_k < \infty$ such that $\mu(B) = P(\sum_{k\geqq 1} \lambda_k \xi_k e_k \in B)$.

Theorem 3 *Let* $a = \sum_{k\geqq 1} a_k e_k \in H$ *and*

(4.25) $$m_t = \min\left\{m \geqq 1: \sum_{k=m+1}^{\infty} a_k^2 \leqq (1-\varepsilon)(R(t)/t)^2\right\}$$

for $0 < \varepsilon < 1$. *Then*

(4.26) $$P(\|X - t \cdot a\| \leqq R(t)) \geqq \exp\left\{-\frac{t^2}{2} \sum_{k=1}^{m_t} \frac{a_k^2}{\lambda_k}\right\} \cdot P(\|X\| \leqq (1-\varepsilon) R(t))$$

and

(4.27) $$P(\|X - t \cdot a\| \leqq R(t)) \leqq \exp\left\{-\frac{t^2}{2} \sum_{k=1}^{m_t} a_k^2 \cdot \frac{\lambda_k x_t^2}{(1+\lambda_k x_t)^2}\right\} \cdot P(\|X\| \leqq (1+\varepsilon) R(t))$$

where x_t *is defined by*

(4.28) $$\sum_{k=1}^{m_t} \frac{a_k^2}{(1+\lambda_k x_t)^2} = (1+\varepsilon)^2 \cdot \frac{R^2(t)}{t^2}.$$

Proof. Let $g_t = \sum_{k=1}^{m_t} a_k e_k$ in Corollary 7. Then the claim follows from Corollary 7, (3.22) and (3.15).

5 An application

As mentioned in the introduction, the estimates we have in this paper can be used to give the convergence rate and constant in the functional form of Chung's LIL. These only depend on the shift being in H_μ. Since the estimates we have also work for the shift not in H_μ, we can formalize the problem for points outside H_μ.

Let $X, X_1, X_2, \ldots$ be iid H-valued centered Gaussian vectors where X is defined as in Theorem 1 with $\lambda_k = k^{-2}$. If $a_k = k^{-1}$ and $a = (a_k)$, then

(5.1) $$\varliminf_{n\to\infty} \|X_n - 2^{5/4} \pi^{-1/2} R^{1/2} \cdot (\log n)^{1/4} \cdot a\| = R \quad \text{a.s.}$$

where $0<R<\infty$ is a constant. This follows from Borel-Cantelli Lemma and the estimate

$$\log P(\|X-t\cdot a\|\leqq R)\sim -2^{-5}\pi^2 R^{-2}\cdot t^4$$

obtained by Theorem 1.

Remark. Strassen's LIL asserts that the limit points of $\{X_n/\sqrt{2\log n}\}$ are $K=\{a: \|a\|_\mu\leqq 1\}$. Results studying how close $\{X_n/\sqrt{2\log n}\}$ can approximate a fixed point inside K (see [1]) and on the boundary of K (see [9]) have been obtained. In particular, in this situation we have

$$\underline{\lim}_{n\to\infty} \log n \,\|X_n/\sqrt{2\log n}\| = \pi/4 \quad \text{a.s.}$$

which tells how close $\{X_n/\sqrt{2\log n}\}$ can be to zero. On the other hand, (5.1) can be rewritten as

$$\underline{\lim}_{n\to\infty} \sqrt{\log n}\,\|X_n/\sqrt{2\log n} - 2^{3/4}\pi^{-1/2}R^{1/2}(\log n)^{-1/4}\cdot a\| = R/\sqrt{2} \quad \text{a.s.}$$

This tells us how closely $\{X_n/\sqrt{2\log n}\}$ can approximate a sequences of points that are not in H_μ but which converge to zero. Of course, (5.1) is just for a particular Gaussian measure on l_2 and a special point, but the general case can be handled provided the asymptotic behavior at the log level can be calculated.

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