SOME LARGE DEVIATION RESULTS FOR GAUSSIAN MEASURES

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1. Introduction. Let B denote a real separable Banach space with norm $\|\cdot\|$ and topological dual B^*, and assume X is a centered B-valued Gaussian random vector with $\mu = \mathcal{L}(X)$. If B is a Hilbert space H, then in [5] we obtained the exact asymptotics of

$$P(\|X - ta\| < R(t)) \tag{1.1}$$

as $t \to \infty$, provided $\lim_t R(t)/t$ exists and it is strictly less than $\|a\|$, and a is in the support of the measure μ. We also provided some upper and lower bounds for general B, which extended (slightly) the fundamental ideas in [1]. However, our most intricate results were in the Hilbert space setting, and one of their consequences is the following result when $R(t) = tR$ (see Corollary 5 of [5]).

Theorem 1. Let X be an H valued centered Gaussian random vector and assume $a \in H$ is in the support of $\mu = \mathcal{L}(X)$, $a \neq 0$, and $0 < R < \|a\|$. Then, as $t \to \infty$,

$$P(\|X - ta\| < tR) \sim K_{a,R} t^{-1} \exp\left\{ - \inf_{\|x-a\|<R} I(x) t^2 \right\} \tag{1.2}$$

where $K_{a,R}$ is a given positive constant.

Remark. In [5], (1.2) was established for $P(\|X - ta\| \le tR)$, but since Proposition 5 in [2] implies $P(\|X - ta\| \le tR) = P(\|X - ta\| < tR)$, (1.2) stands as stated.

In (1.2), and hereafter, we write $a_t \sim b_t$ as $t \to \infty$ to denote that $\lim_{t\to\infty} a_t/b_t = 1$. For non-negative a_t and b_t we will also use $a_t << b_t$ to denote that $\overline{\lim}_{t\to\infty} a_t/b_t < \infty$,

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and $a_t \approx b_t$ if $a_t << b_t$ and $b_t << a_t$. The quantity $I(x)$ in (1.2) is the I-function for Gaussian measures in large deviation theory, and is given by

$$I(x) = \begin{cases} \frac{1}{2}\|x\|_\mu^2 & x \in H_\mu \\ +\infty & \text{otherwise}, \end{cases} \tag{1.3}$$

where H_μ is the closure of $SB^* = \left\{\int_B x f(x) d\mu(x) : f \in B^*\right\}$ in the inner product norm given in $S(B^*)$ by

$$\langle Sf, Sg \rangle_\mu = \int_B f(x) g(x) d\mu(x). \tag{1.4}$$

We write $\|\cdot\|_\mu$ to denote the induced inner product norm on H_μ. Finally, we point out the well known fact that the support of μ is $\overline{H}_\mu$, the B-closure of H_μ in B. Throughout we freely use many of the properties of H_μ and its relationship to B and μ. Lemma 2 1 of [4] provides the basic facts regarding these properties.

Of course, (1.2) is a precise large deviation result, and except for constants, it implies that the large deviation probabilities of a ball not containing zero decay at the same rate as those of a half-space not containing zero. A natural question to ask is whether a similar result holds in more generality, and this is the motivating question for all the results we establish.

Inspection of the proofs in [5] used to obtain the asymptotic result in (1.2) show that they depend heavily on Hilbert space. Hence, except for possibly independent coordinate centered Gaussian measures on ℓ^p, $1 \leq p < \infty$, it seems unlikely that the precision of (1.2) can be obtained in great generality. In fact, even the ℓ^p cases, $p \neq 2$, seems to be difficult, and they are not completely general since an arbitrary centered Gaussian measure on ℓ^p, $p \neq 2$, need not have independent coordinates. It is only in the ℓ^2 case where this is no loss of generality.

In Theorem 2 we obtain a general result, which can then be applied to obtain precise large deviation probabilities for balls in 2-smooth Banach spaces. We also include some upper and lower bounds on the rate at which balls increase to half-space in uniformly $(1+\alpha)$-smooth spaces and uniformly p-convex spaces.

2. Statement of results and some definitions. Our first result is related to the concept of a dominating point as defined in [6], and shows that every open convex set not containing the origin has a unique dominating point which is in SB^*. To make precise our notion of a dominating point, consider the following definition.

Definition. Let $D \subseteq B$ be such that there exists $b \in \partial D$, the boundary of D, satisfying

$$I(b) = \inf_{x \in D} I(x) = \inf_{x \in \overline{D}} I(x) < \infty, \tag{2.1}$$

and for some $f \in B^*$

$$D \subseteq \{x : f(x) \geq f(b)\}. \tag{2.2}$$

Then b is called a dominating point for D.

What we prove regarding dominating points is contained in the following proposition. If $B = \mathbb{R}^d$, it is a special case of the results in [6], but we are unaware of such a result in the infinite dimensional setting.

Proposition 1. Let D be an open convex subset of B such that $D \cap \overline{H}_\mu$ is non-empty, and assume $0 \notin D$. Then there exists a unique point b on the boundary of D and $f \in B^*$ such that (2.1) holds, $b \in SB^*$, and (2.2) improves to

$$D \subseteq \{x : f(x) > f(b)\} \tag{2.3}$$

Hence b is a unique dominating point for D, and if $b \neq 0$, then there exists $f \in B^*$ such that both $b = Sf$ and (2.3) holds.

The abstract result we prove requires the following definition.

Definition. Let D be a convex open subset of B with $0 \notin D$, and assume h is the unique dominating point of D. If $h \neq 0$, let $\alpha > 0$ and assume $h = Sf$ where $f \in B^*$ is such that $D \subseteq \{x : f(x) > f(h)\}$. Then we say D contains slices whose diameters near h dominate the power function $s^{1/(1+\alpha)}$ if there exists $a \in B$, $\delta > 0$, $\beta > 0$ such that $f(a) > f(h)$, and for $x_0 = a - h$, $0 \leq s \leq \delta$, and $M_s = \{x : f(x) = sf(x_0)\}$ we have

$$M_s \cap (D - h) \supseteq \{y + sx_0 : y \in M_0, \|y\| \leq \beta s^{1/(1+\alpha)}\}. \tag{2.4}$$

If the dominating point of D is zero, then we say D contains slices whose diameters near 0 dominate the power function $s^{1/(1+\alpha)}$ if there exists $f \in B^*$, $a \in B$, $\delta > 0$, $\beta > 0$ such that $D \subseteq \{x : f(x) > 0\}$, and (2.4) holds with $x_0 = a$ and $h = 0$.

Given the above definition, we can now formulate our first theorem.

Theorem 2. Let D be an open convex set in B such that $D \cap \overline{H}_\mu \neq \phi$, $0 \notin \overline{D}$, and assume h is the unique dominating point of D. Let $\alpha > 0$, and assume D contains slices whose diameters near h dominate the power function $s^{1/(1+\alpha)}$. Then, if $\alpha \geq 1$, we have as $t \to \infty$ that

$$P(X \in tD) \approx t^{-1} \exp\{-t^2 \|h\|_\mu^2 / 2\}.$$

Remark. If $0 \notin D$, but $0 \in \overline{D}$, then 0 is the unique dominating point of D. Furthermore, by [8, p. 38], we have for all $x \in D$, that the ray $L(0, x) = \{tx : 0 < t \leq 1\} \subseteq D$, and hence tD increases as t increases. Thus when $D \cap \overline{H}_\mu \neq \phi$, the probability $P(X \in tD)$ increases to a strictly positive constant as $t \nearrow \infty$. In addition, since $D \subseteq \{x : f(x) > 0\}$ for some $f \in B^*$ by Proposition 1, we see $\mu(tD) \leq \frac{1}{2}$, and it is natural to examine how fast the convergence happens. In order to describe these results, we introduce some further terminology regarding the geometry of Banach spaces.

Definition. Let B denote a Banach space with norm $\|\cdot\|$. Then the modulus of convexity of B with respect to $\|\cdot\|$ is the function defined for $0 \leq \epsilon \leq 2$ by

$$\delta_B(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}. \tag{2.5}$$

The modulus of smoothness of B with respect to $\|\cdot\|$ is defined for $0 \leq t < \infty$ by

$$\rho_B(t) = \sup_{\|x\|=1, \|y\|=1} \left\{\frac{1}{2}(\|x+ty\| + \|x-ty\| - 2)\right\}. \tag{2.6}$$

Then, a Banach space with norm $\|\cdot\|$ is uniformly $(1+\alpha)$-smooth, $0 < \alpha \leq 1$, if for some $C < \infty$ and all $t \in [0, \infty)$ the modulus of smoothness satisfies

$$\rho_B(t) \leq C|t|^{1+\alpha}. \tag{2.7}$$

We also say the Banach space B is uniformly p-convex, $2 \leq p < \infty$, if for some $\beta > 0$ and all ϵ in $[0, 2]$ the modulus of convexity satisfies

$$\delta_B(\epsilon) \geq \beta \epsilon^p. \tag{2.8}$$

In Theorem 2.4 of [3] it is shown that if $0 < \alpha \leq 1$, then B is uniformly $(1+\alpha)$-smooth with respect to $\|\cdot\|$ iff there exists a $C < \infty$ such that

$$\|x+y\|^{1+\alpha} + \|x-y\|^{1+\alpha} \leq 2\|x\|^{1+\alpha} + C\|y\|^{1+\alpha} \tag{2.9}$$

for all $x, y \in B$. The constant C in (2.9) is not the same as that in (2.7), but we only use (2.9) in our proofs, and hence there will be no confusion in this notation.

Our next result is an application of Theorem 2, which extends Theorem 1 balls in uniformly 2-smooth spaces. However, here we do not have the constant $K_{a,R}$ as in (1.2) of Theorem 1.

Theorem 3. Let X be a centered Gaussian random vector with values in a Banach space B which is 2-smooth with respect to the norm $\|\cdot\|$. Furthermore, assume $a \in B$, $0 < R < \|a\|$, and $\overline{H}_\mu \cap \{x : \|x - a\| < R\} \neq \phi$. Then, as $t \to \infty$,

$$P(\|X - ta\| < tR) \approx t^{-1} \exp\left\{- \inf_{\|x-a\|<R} I(x)t^2\right\}. \tag{2.10}$$

If $R = \|a\|$ in Theorem 3, and $D = \{x : \|x - a\| < \|a\|\}$, or more generally, if D is open, convex, and $0 \notin D$, but $0 \in \overline{D}$, then $\mu(tD) \nearrow$ to a constant less than or equal to 1/2 (see the remark following Theorem 2). Our next result considers the rate at which this happens.

Theorem 4. Let X be a centered Gaussian random vector with values in B, and assume D is an open convex subset of B such that $0 \notin D$, $0 \in \overline{D}$, and $D \cap \overline{H}_\mu \neq \phi$. Let $\alpha > 0$, and assume D contains slices whose diameters near 0 dominate the power function $s^{1/(1+\alpha)}$. Then $P(X \in tD) \leq \frac{1}{2}$, and as $t \to \infty$

$$\frac{1}{2} - P(X \in tD) << t^{-\alpha}. \tag{2.11}$$

Corollary 1. If $D = \{x : \|x - a\| < R\}$ where $R = \|a\| > 0$, $D \cap \overline{H}_\mu \neq \phi$, and B is uniformly $(1+\alpha)$-smooth with respect to the norm $\|\cdot\|$, then $P(X \in tD) \leq \frac{1}{2}$, and as $t \to \infty$

$$\frac{1}{2} - P(X \in tD) << t^{-\alpha}. \tag{2.12}$$

Our final result obtains a lower bound for these probabilities in uniformly p-convex spaces, $2 \leq p < \infty$.

Theorem 5. Let B be a uniformly p-convex Banach space with respect to the norm $\|\cdot\|$, and assume $D = \{x : \|x - a\| < R\}$ where $R = \|a\| > 0$ and $D \cap \overline{H}_\mu \neq \phi$. Then $P(X \in tD) \leq \frac{1}{2}$ and as $t \to \infty$

$$\frac{1}{2} - P(X \in tD) >> t^{-(p-1)}. \tag{2.13}$$

Remark. Let $B = \ell^q$ or L^q. If $1 < q \leq 2$, then B is uniformly 2-convex and q-smooth whereas if $2 \leq q < \infty$, then B is uniformly 2-smooth and q-convex. In particular, Hilbert space is 2-smooth and 2-convex, so in this setting (2.12) and (2.13) combine to yield

$$\frac{1}{2} - \mu(tD) \approx t^{-1}. \tag{2.14}$$

Here D is as in Corollary 1 and Theorem 5, and the following example in ℓ^q shows that (2.12) and (2.13) are, up to constants, best possible in many instances.

Example. Let $B = \ell^q$ with canonical basis $\{e_j : j \geq 1\}$ and take $\{g_j : j \geq 1\}$ independent centered Gaussian random variables such that $X = \sum_{j\geq 1} g_j e_j$ is ℓ^q valued. Let $1 < q < \infty$, and take $x_0 = re_1$. Then, as $t \to \infty$,

$$\begin{aligned}
&P(\|X - tx_0\| < tr) \\
&= P\left(\sum_{j\geq 2} |g_j|^q < (tr)^q - |g_1 - tr|^q\right) \\
&= \int_0^{t^{1/2}r} \left[1 - P\left(\sum_{j\geq 2} |g_j|^q \geq (tr)^q - |u - tr|^q\right)\right] dP_{g_1}(u)
\end{aligned}$$

$$+ \epsilon_1(t) \tag{2.15}$$

$$= \frac{1}{2} - \int_0^{t^{1/2}r} P\left(\left(\sum_{j\geq 2} |g_j|^q\right)^{1/q} \geq |(tr)^q - |u - tr|^q|^{1/q}\right) dP_{g_1}(u)$$
$$+ \epsilon_1(t) - \epsilon_2(t)$$

where $\epsilon_1(t)$ and $\epsilon_2(t)$ are non-negative and $\epsilon_1(t)+\epsilon_2(t) = O(e^{-t^{1/4}})$. If $0 \leq u \leq t^{1/2}r$, then as $t \to \infty$ the mean value theorem implies $|(tr)^q - |u - tr|^q| \sim q(tr)^{q-1}u$. Hence if $\|Z\|^q = \sum_{j\geq 2} |g_j|^q$ and

$$J_t = \int_0^{t^{1/2}r} P(\|Z\| > (tr)^q - |u - tr|^q)^{1/q})dP_{g_1}(u),$$

then for sufficiently large t we have

$$\int_0^{t^{1/2}r} P(\|Z\| > (2q(tr)^{q-1}u)^{1/q})dP_{g_1}(u) \leq J_t$$
$$\leq \int_0^{t^{1/2}r} P(\|Z\| > (q(tr)^{q-1}u/2)^{1/q})dP_{g_1}(u).$$

Using the change of variable $v = t^{(q-1)}u$ as at the end of the proof of Lemma 3.3, we see

$$J_t \approx t^{-(q-1)}.$$

Hence by (2.15) we have

$$\tfrac{1}{2} - P(\|X - tx_0\| < tr) \approx t^{-(q-1)}. \tag{2.16}$$

When $1 < q \leq 2$, then ℓ^q is q-smooth, so (2.16) shows (2.12) is best possible. However, since ℓ^q is 2-convex when $1 < q \leq 2$ we see (2.13) yields a lower bound which is sometimes too small. On the other hand, if $2 \leq q < \infty$, then ℓ^q is q-convex and 2-smooth, so (2.16) shows that the lower bound in (2.13) is best possible while the upper bound in (2.12) is sometimes too large. A similar example and computation in ℓ^q, $1 < q < 2$, shows Theorem 3 does not hold in uniformly $(1 + \alpha)$-smoooth spaces when $0 < \alpha < 1$.

3. Proof of Proposition 1 and a useful lemma. The proof of Proposition 1 will proceed with several lemmas.

Lemma 3.1. Under the conditions of Proposition 1, there exists a unique $b \in \partial D$ such that (2.1) holds.

Proof. Since $D \cap \overline{H}_\mu \neq \phi$ and D is open we have $D \cap H_\mu \neq \phi$. Let A denote the H_μ-closure of $D \cap H_\mu$. Then A is a closed, non-empty, convex subset of H_μ, so A has a unique element b of minimal H_μ norm, and $A \subseteq \overline{D} \cap H_\mu$ where $\overline{D}$ is the B-closure of D. Thus $b \in \overline{D} \cap H_\mu$ satisfies $I(b) = \inf_{x \in D} I(x)$, and b is unique in $\overline{D}$.

If $b = 0$, then $0 \notin D$ and b a limit point in the B-norm of $\{x_j\} \subseteq D$ implies $b \in \partial D$. Furthermore, (2.1) is trivial in this case.

Hence assume $b \neq 0$. If $b \in D$, then D open implies there exists a $\lambda \in (0,1)$ such that $\lambda b \in D$. Thus

$$\inf_{x \in D} I(x) \leq \lambda^2 I(b) < I(b) \tag{3.1}$$

which violates $I(b) = \inf_{x \in D} I(x)$. Thus again $b \in \partial D$, and it now remains to show $I(b) = \inf_{x \in \overline{D}} I(x)$ when $b \neq 0$.

To verify this let $a \in (\partial D) \cap H_\mu$, $d \in D \cap H_\mu$, and let $L(a,d) = \{ta + (1-t)d : 0 < t < 1\}$. Then $L(a,d) \subseteq D \cap H_\mu$ by [8, p. 38], and since $I(x)$ is convex on H_μ we have

$$\inf_{x \in L(a,d)} I(x) \leq \min(I(a), I(d)).$$

Thus $\inf_{a \in \partial D} I(a) \geq \inf_{x \in D} I(x)$, which implies $\inf_{x \in \overline{D}} I(x) = \inf_{x \in D} I(x)$. Hence (2.1) holds and the lemma is proved.

Lemma 3.2. Under the conditions of Proposition 1, the unique point b in Lemma 3.1 is in SB^* and (2.3) holds. Furthermore, if $b \neq 0$, there exists $f \in B^*$ such that both $b = Sf$ and (2.3) holds.

Proof. If $b = 0$, then $b \in SB^*$. Furthermore since D is open and $0 \notin D$, the Hahn-Banach separation theorem given in [8, p. 64] implies there exists non-zero $f \in B^*$ such that $D \subseteq \{x : f(x) \geq f(b) = 0\}$. Now D open implies D is a subset of the interior of $\{x : f(x) \geq 0\}$, but this is $\{x : f(x) > 0\}$. Hence (2.3) holds as well.

Now assume $b \neq 0$, so $I(b) > 0$, and define

$$C = \{x \in H_\mu : \|x\|_\mu \leq (2I(b))^{1/2}\}.$$

Then it is well known that C is compact and, of course, convex in B. We also have $D \cap C = \phi$, since $p \in D \cap C$ and D open implies $\lambda p \in D$ for some $\lambda \in (0,1)$. Now $I(p) \leq I(b)$ for $p \in C$, and hence $I(\lambda p) < I(b)$ which contradicts (2.1) if $\lambda p \in D$. Thus $D \cap C = \phi$, and since $b \in C$ is the unique point in ∂D such that (2.1) holds it follows that

$$\overline{D} \cap C = \{b\}.$$

Applying the Hahn-Banach separation theorem again, we obtain $f \in B^*$, $f \neq 0$, such that for some α

$$\sup_{x \in C} f(x) \leq \alpha \leq \inf_{x \in D} f(x). \tag{3.2}$$

Since $0 \in C$ we have $\alpha \geq 0$, and since D is open with $D \subseteq \{f(x) \geq \alpha\}$ we have D a subset of the interior of $\{x : f(x) \geq \alpha\}$, but this is $\{x : f(x) > \alpha\}$. Thus $f(x) > 0$ for all $x \in D$ and since $D \cap \overline{H}_\mu \neq \phi$ with D open we see

$$\sigma_f^2 = \int_B f^2(x) d\mu(x) > 0.$$

Now by Lemma 2.1 of [4] we have

$$\sup_{x \in C} f(x) = (2I(b))^{1/2} \sigma_f .$$

Hence α is strictly positive, and by rescaling f, if necessary, we assume $\sigma_f^2 = 2I(b)$. Then (3.2) and $C \cap \overline{D} = \{b\}$ implies

$$\sup_{x \in C} f(x) = f(b) = \inf_{x \in D} f(x). \tag{3.3}$$

Hence $f(b) > 0$ and (3.3) implies $D \subseteq \{x : f(x) \geq f(b)\}$. Hence (2.2) holds, and since D is open, D is a subset of $\{x : f(x) > f(b)\}$, the interior of $\{x : f(x) \geq f(b)\}$. Hence (2.3) holds, and it remains to show $b = Sf$.

Let $g = Sf$. Since $\langle x, g\rangle_\mu = f(x)$ for all $x \in H_\mu$, the Cauchy-Schwarz inequality implies

$$f(x) = \langle g, x\rangle_\mu \le \|Sf\|_\mu \|x\|_\mu = \|x\|_\mu (2I(b))^{1/2}$$

because $\|Sf\|_\mu^2 = \int_B f^2(x) d\mu(x) = 2I(b)$. Thus

$$0 < f(b) = \sup_{x\in C} f(x) \le 2I(b), \tag{3.4}$$

but since $C = \{x \in H_\mu : \|x\|_\mu \le (2I(b))^{1/2}\}$ with $\|Sf\|_\mu = (2I(b))^{1/2}$ we actually have

$$\sup_{x\in C} f(x) = \|Sf\|_\mu (2I(b))^{1/2} = 2I(b). \tag{3.5}$$

Combining (3.4) and (3.5) we thus see

$$f(b) = 2I(b), \tag{3.6}$$

and we claim $b = g = Sf$.

This follows since the above implies that $0 < f(b) = \langle g, b\rangle_\mu = \|b\|_\mu^2$, but equality holding in Cauchy-Schwarz implies $g = \lambda b$, and $\lambda = 1$ is now obvious. Hence $b = g = Sf$ and Lemma 3.2 is proven.

Combining Lemmas 3.1 and 3.2, Proposition 1 is established. Hence we turn to the proof of a useful lemma.

Lemma 3.3. Let X_1 be a centered real-valued Gaussian vector, and X_2 a centered Gaussian random vector with values in B. If X_1 and X_2 are independent and non-degenerate, then for $\beta > 0$, $\theta > 0$, as $t \to \infty$

$$\frac{1}{2} - P(0 \le X_1 \le c_1 t^{1/2}, \|X_2\| \le c_2 t^\beta X_1^\theta) \approx t^{-\beta/\theta}, \tag{3.7}$$

where c_1 and c_2 are positive constants.

Proof. Since X_1 is non-degenerate, $a^2 = E(X_1^2) > 0$. Hence independence of X_1 and X_2 imply

$$\begin{aligned} &P(0 \le X_1 \le c_1 t^{1/2}, \|X_2\| \le c_2 t^\beta x_1^\theta) \\ &= \int_0^{c_1 t^{1/2}} (1 - P(\|X_2\| > c_2 t^\beta u^\theta)) dP_{X_1}(u) \\ &= \frac{1}{2} - O(t^{-1/2} \exp\{-c_1^2 t/2a^2\}) - Q_t \end{aligned}$$

where

$$Q_t = \int_0^{c_1 t^{1/2}} P(\|X_2\| > c_2 t^\beta u^\theta) dP_{X_1}(u).$$

Using the change of variable $v = t^{\beta/\theta} u$, we have by the dominated convergence theorem that as $t \to \infty$

$$\begin{aligned} Q_t &= \frac{1}{\sqrt{2\pi a^2}} t^{-\beta/\theta} \int_0^{c_1 t^{(1/2)+(\beta/\theta)}} P(\|X_2\| > c_2 v^\theta) e^{-v^2/(2a^2 t^{2\beta/\theta})} dv \\ &\sim t^{-\beta/\theta} \int_0^\infty P(\|X_2\| > c_2 v^\theta) dv / \sqrt{2\pi a^2}. \end{aligned}$$

Here, of course, the dominating function is $P(\|X_2\| > c_2 v^\theta)$, which is integrable as $\|X_2\|$ has exponential moments. Hence (3.7) holds, and Lemma 3.3 is established.

4. Proof of Theorem 2. Since $0 \notin \overline{D}$, Proposition 1 implies the unique dominating point h of D is non-zero. In addition, $h \in \partial D$ and for some $f \in B^*$ we have $h = Sf$ and $D \subseteq \{x : f(x) > f(h)\}$. Thus

$$P(X \in tD) \leq P(f(X) > tf(h)) \approx t^{-1} \exp\{-t^2 f^2(h)/(2E(f^2(X)))\} \tag{4.1}$$

with $E(f^2(X)) = f(h) = \|h\|_\mu^2$. Hence we have as $t \to \infty$ that

$$P(X \in tD) << t^{-1} \exp\{-t^2 \|h\|_\mu^2 / 2\}, \tag{4.2}$$

and it suffices to prove a comparable lower bound for $P(X \in tD)$.

Applying the Cameron-Martin formula we obtain

$$\begin{aligned} P(X \in tD) &= P(X \in th + t(D-h)) \\ &= \exp\{-t^2 \|h\|_\mu^2/2\} \int_{t(D-h)} e^{-tf(x)} d\mu(x), \end{aligned} \tag{4.3}$$

so it suffices to prove that for $\alpha \geq 1$, as $t \to \infty$,

$$\int_{t(D-h)} e^{-tf(x)} d\mu(x) >> t^{-1}. \tag{4.4}$$

Since D contains slices whose diameter near $h = Sf$ dominate the power function $s^{1/(1+\alpha)}$, it follows that there exist $a \in B$, $\delta > 0$, $\beta > 0$ such that $f(a) > f(h)$ and for $x_0 = a - h$

$$t(D-h) \supseteq \{x = w + rx_0 : w \in M_0, 0 \le r \le t\delta, \|w\| \le \beta t^{\alpha/(1+\alpha)} r^{1/(1+\alpha)}\} \quad (4.4)$$

where

$$M_r = \{x : f(x) = rf(x_0)\} \qquad (r \ge 0).$$

That is, (4.4) follows by rescaling (2.4), since for $r = st$, $0 \le s \le \delta$

$$\begin{aligned} M_r \cap t(D-h) &= M_{st} \cap t(D-h) \\ &= t(M_s \cap (D-h)) \\ &\supseteq \{t(y + sx_0) : y \in M_0; \|y\| \le \beta s^{1/(1+\alpha)}\} \qquad (4.5) \\ &= \{w + rx_0 : \frac{w}{t} = y \in M_0, \|w/t\| \le \beta(\frac{r}{t})^{(1/(1+\alpha)}\} \\ &= \{w + rx_0 : w \in M_0, \|w\| \le \beta t^{\alpha/(1+\alpha)} r^{1/(1+\alpha)}\}. \end{aligned}$$

Furthermore, if $\pi_f(x) = f(x)/f(x_0)$, then since $x - \pi_f(x)x_0 \in M_0$ and the representation $x = w + rx_0$ for $w \in M_0$ is unique, (4.4) implies

$$\begin{aligned} t(D-h) \supseteq \\ \{x = (x - \pi_f(x)x_0) + \pi_f(x)x_0 : 0 \le \pi_f(x) \le t\delta, \qquad (4.6) \\ \|x - \pi_f(x)x_0\| \le \beta t^{\alpha/(1+\alpha)}(\pi_f(x))^{1/(1+\alpha)}\}. \end{aligned}$$

Hence we have

$$t(D-h) \supseteq V_t - \left\{x : \|f(x)h/\|h\|_\mu^2 - \pi_f(x)x_0\| > \frac{\beta}{2} t^{\alpha/(1+\alpha)} |\pi_f(x)|^{1/(1+\alpha)}\right\} \quad (4.7)$$

where

$$V_t = \left\{x : 0 \le \pi_f(x) \le t\delta, \|x - f(x)h/\|h\|_\mu^2\| \le \frac{\beta}{2} t^{\alpha/(1+\alpha)} (\pi_f(x))^{1/(1+\alpha)}\right\}.$$

If $\|h/\|h\|_\mu^2 - x_0/f(x_0)\| = 0$ the second term in the right hand side of (4.7) is empty, so assume the contrary. Thus, since $f(x) \geq 0$ on $D - h$, we have

$$\int_{t(D-h)} e^{-tf(x)} d\mu(x) \geq \int_{V_t} e^{-tf(x)} d\mu(x) - \mu(x : |f(x)| > \lambda t) \tag{4.8}$$

where

$$\lambda = \left(\beta(2\|h/\|h\|_\mu^2 - x_0/f(x_0)\|(f(x_0))^{1/(1+\alpha)})^{-1}\right)^{(1+\alpha)/\alpha} > 0.$$

Now $\mu(x : |f(x)| > \lambda t)$ decays exponentially fast as $t \to \infty$, so it suffices to show

$$\int_{V_t} e^{-tf(x)} d\mu(x) >> \frac{1}{t}. \tag{4.9}$$

Letting $\gamma = \beta/(2(f(x_0))^{1/(1+\alpha)})$, we have for large t that

$$V_t \supseteq \{x : 0 < f(x) \leq t^{1/2}, \|x - f(x)h/\|h\|_\mu^2\| \leq \gamma t^{\alpha/(1+\alpha)}(f(x))^{1/1+\alpha}\}, \tag{4.10}$$

and since $f(x)$ and $x - \frac{f(x)}{\|h\|_\mu^2} h$ are independent on B with respect to μ, we have

$$\int_{V_t} e^{-tf(x)} d\mu(x) \geq \int_0^{t^{1/2}} e^{-tu}(1 - P(\|G\| > \gamma t^{\alpha/(1+\alpha)} u^{1/(1+\alpha)}) \frac{e^{-u^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} du \tag{4.11}$$

where $\mathcal{L}(G)$ is the μ-distribution of $x - f(x)h/\|h\|_\mu^2$ and $\sigma^2 = E(f^2(X)) = \|h\|_\mu^2$. Thus for $t \to \infty$ and $\alpha \geq 1$

$$\int_{V_t} e^{-tf(x)} d\mu(x) \geq \int_{t^{-\alpha}}^1 e^{-tu} \rho du >> \frac{e^{-t^{(1-\alpha)}}}{t} >> t^{-1}$$

where

$$\begin{aligned}\rho &= \inf_{t^{-\alpha} \leq u \leq 1} \left(1 - P(\|G\| > \gamma t^{\alpha/(1+\alpha)} u^{1/(1+\alpha)})\right) \frac{e^{-u^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \\ &\geq (1 - P(\|G\| > \gamma)) \frac{e^{-1/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \\ &> 0\end{aligned}$$

since $\gamma > 0$. Hence Theorem 2 is proved.

5. Proof of Theorem 3. If $\dim(H_\mu) = 1$, the result is trivial, so assume $\dim(H_\mu) \geq 2$. Let $D = \{x : \|x - a\| < R\}$, and assume h is the unique dominating point of D. Then $h \in \partial D$, and since $0 \notin \overline{D}$, we have $h \neq 0$. Hence by Proposition 1 there exists $f \in B^*$ such that $h = Sf$ and $D \subseteq \{x : f(x) > f(h)\}$.

Letting $x_0 = a - h$ and $M_s = \{x : f(x) = sf(x_0)\}$ for $0 \le s < \infty$, we then have

$$D - h \subseteq \{x : f(x) > f(0)\}, \tag{5.1}$$

and $D - h$ is the open ball of radius $R = \|x_0\|$ centered at x_0. Furthermore, $0 \in \partial(D - h)$, and we easily see

$$\inf_{m \in M_0} \|x_0 - m\| = \|x_0\| = R. \tag{5.2}$$

Thus the closest point in M_0 to x_0 is the zero vector, and we define the distance from a point p to M_s by

$$d(p, M_s) = \inf_{m \in M_s} \|p - m\|. \tag{5.3}$$

Hence, since each $m \in M_s$ is of the form $sx_0 + m$ for $m \in M_0$, we have

$$d(p, M_s) = d(p - sx_0, M_0). \tag{5.4}$$

In particular, from (5.2) if $s, t \ge 0$, then (5.4) implies

$$d(tx_0, M_s) = d(|t - s|x_0, M_0) = |t - s|R. \tag{5.5}$$

Now take $y \in M_0$ such that $\|sx_0 + y - x_0\| = R$. Then $sx_0 + y$ is on the boundary of $D - h$ and $0 \le s \le 2$. Furthermore, since $y \in M_0$, $x_0 \in M_1$ we have $sx_0 - y \in M_s$, and the above implies

$$\begin{aligned} \|sx_0 - y - x_0\| &\ge d(sx_0 - y, M_1) \\ &= d(sx_0 - y - x_0, M_0) \\ &= d((s-1)x_0, M_0) \\ &= |1 - s|R. \end{aligned} \tag{5.6}$$

Letting $x = sx_0 - x_0$ and $y = y$ we have $\|x + y\| = R$ and $\|x - y\| \ge |1 - s|R$ under the above assumptions. Hence since B is uniformly 2-smooth, (2.9) implies

$$R^2 + ((1 - s)R)^2 \le 2((1 - s)R)^2 + C\|y\|^2.$$

Thus for such $y \in M_0$ we have

$$\|y\| \geq (C^{-1}R^2(2s - s^2))^{1/2}.$$

Hence for $\beta^2 = C^{-1}R^2$ and $0 \leq s \leq 1$ the above implies

$$\|y\| \geq \beta s^{1/2}, \tag{5.7}$$

and consequently

$$M_s \cap (D - h) \supseteq \{y + sx_0 : y \in M_0, \|y\| < \beta s^{1/2}\}. \tag{5.8}$$

Indeed, if (5.8) fails, then there exists y such that $\|y\| < \beta s^{1/2}$ and $y + sx_0 \notin M_s \cap (D - h)$ for some $s \in (0, 1]$ and $\beta^2 = C^{-1}R^2$. Hence $y + sx_0 \in M_s$, but is outside the ball $D - h$ of radius R. Thus

$$\|(y + sx_0) - x_0\| \geq R,$$

so for some $\lambda \in (0, 1]$ we have $\|\lambda y + sx_0 - x_0\| = R$. Now the above argument applied to λy implies

$$\|\lambda y\| \geq \beta s^{1/2}$$

with $0 < \lambda \leq 1$. This contradicts $\|y\| < \beta s^{1/2}$. Hence (5.8) holds and D contains slices whose diameters near h dominate the power function $s^{1/2}$. Thus, Theorem 2 applies and yields (2.10) since by Proposition 1

$$\|h\|_\mu^2 = \inf_{\|x - a\| < R} \|x\|_\mu^2. \tag{5.9}$$

6. Proof of Theorem 4 and Corollary 1. First we observe that 0 is the unique dominating point for D. Since D contains slices whose diameters near zero dominate the power function $s^{1/(1+\alpha)}$, there exists $f \in B^*$, $a \in B$, $\delta > 0$, $\beta > 0$ such that $D \subseteq \{x : f(x) > 0\}$, $f(a) > 0$, and for $x_0 = a$, $0 \leq s \leq \delta$ we have

$$M_s \cap D \supseteq \{y + sx_0 : y \in M_0, \|y\| \leq \beta s^{1/(1+\alpha)}\},$$

where

$$M_s = \{x : f(x) = sf(x_0)\}.$$

Hence by rescaling as in the proof of Theorem 2 we have

$$tD \supseteq \{x = w + rx_0 : w \in M_0, 0 \leq r \leq t\delta, \|w\| \leq \beta t^{\alpha/(1+\alpha)} r^{1/(1+\alpha)}\}. \tag{6.1}$$

Again, since $x - \pi_f(x)x_0 \in M_0$ and the representation $x = w + rx_0$ for $w \in M_0$ is unique, (6.1) and the argument used in Theorem 2 implies

$$\begin{aligned} tD \supseteq \{x = (x - \pi_f(x)x_0) + \pi_f(x)x_0 : 0 \leq \pi_f(x) \leq t\delta, \\ \|x - \pi_f(x)x_0\| \leq \beta t^{\alpha/(1+\alpha)}(\pi_f(x))^{1/(1+\alpha)}\Big\} \\ \supseteq V_t - \left\{x : \|f(x)g/\|g\|_\mu^2 - \pi_f(x)x_0\| > \frac{\beta}{2} t^{\alpha/(1+\alpha)} |\pi_f(x)|^{1/(1+\alpha)}\right\} \end{aligned} \tag{6.2}$$

where $g = Sf$ and

$$V_t = \left\{x : 0 \leq \pi_f(x) \leq t\delta, \|x - f(x)g/\|g\|_\mu^2\| \leq \frac{\beta}{2} t^{\alpha/(1+\alpha)} (\pi_f(x))^{1/(1+\alpha)}\right\}.$$

Now $g = Sf \neq 0$ since $D \cap \overline{H}_\mu \neq \phi$ and $D \subseteq \{x : f(x) > 0\}$ implies $f(g) = \|g\|_\mu^2 = \int_B f^2(x) d\mu(x) > 0$. Hence V_t is well defined, and for t sufficiently large

$$V_t \supseteq \left\{x : 0 \leq f(x) \leq t^{1/2}, \|x - f(x)g/\|g\|_\mu^2\| \leq \frac{\beta}{2} t^{\alpha/(1+\alpha)} (\pi_f(x))^{1/(1+\alpha)}\right\}. \tag{6.3}$$

Also, $f(x)$ and $x - f(x)g/\|g\|_\mu^2$ are independent, so by Lemma 3.3, as $t \to \infty$

$$\frac{1}{2} - \mu(V_t) << t^{-\alpha}. \tag{6.4}$$

Furthermore, the μ-measure of the set

$$\{x : \|f(x)g/\|g\|_\mu^2 - \pi_f(x)x_0\| > \frac{\beta}{2} t^{\alpha/1+\alpha} |\pi_f(x)|^{1/(1+\alpha)}\}$$

is zero, or at least decays exponentially fast, so combining (6.2), (6.4), we get

$$\frac{1}{2} - \mu(tD) << t^{-\alpha},$$

and Theorem 4 is proved.

Proof of Corollary 1. As in Theorem 4, 0 is the unique dominating point of D and there exists $f \in B^*$ such that $D \subseteq \{x : f(x) > 0\}$. Since $D \cap \overline{H}_\mu \neq \phi$ and D is an open ball, it suffices, as in the proof of Theorem 3, to show D contains slices whose diameter near zero dominate the power function $s^{1/1+\alpha}$.

To do this let $x_0 = a$ and $M_s = \{x : f(x) = sf(x_0)\}$ for $s \geq 0$. Then $D \subseteq \{x : f(x) > 0\}$ and $D = \{x : \|x - x_0\| \leq R\}$ where $R = \|a\|$, so the closest point to x_0 in M_0 is the zero vector. Repeating the argument from (5.3)–(5.6) and using B is uniformly $(1+\alpha)$-smooth with $\alpha > 0$, (2.9) implies

$$R^{1+\alpha} + (|1-s|R)^{1+\alpha} \leq 2(|1-s|R)^{1+\alpha} + C\|y\|^{1+\alpha}. \tag{6.5}$$

for $y \in M_0$ such that $\|sx_0 + y - x_0\| = R$ and $0 \leq s \leq 2$. Thus for such $y \in M_0$ we have

$$\|y\| \geq (C^{-1}R^{1+\alpha}(1 - |1-s|^{1+\alpha}))^{\frac{1}{1+\alpha}}, \tag{6.6}$$

and for $(2\beta)^{(1+\alpha)} = C^{-1}R^{(1+\alpha)}$ and $0 \leq s \leq \delta$, $\delta > 0$ sufficiently small, the mean value theorem implies

$$\|y\| \geq \beta s^{\frac{1}{(1+\alpha)}}. \tag{6.7}$$

Hence, arguing as in the proof of Theorem 3, we have

$$M_s \cap D \supseteq \{y + sx_0 : y \in M_0, \|y\| < \beta s^{1/(1+\alpha)}\}, \tag{6.8}$$

and (6.8) implies D contains slices whose diameters near 0 dominate the power function $s^{1/(1+\alpha)}$. Thus Theorem 4 implies Corollary 1 holds.

7. Proof of Theorem 5. As in Theorem 4 and Corollary 1, 0 is the unique dominating point of $D = \{x : \|x - a\| < R\}$ where $R = \|a\|$. Hence there exists $f \in B^*$ such that $D \subseteq \{x : f(x) > 0\}$ and since $D \cap \overline{H}_\mu \neq \phi$ we also have $g = Sf \neq 0$ with $f(g) > 0$.

To verify (2.13) first observe that as $t \to \infty$ we easily have

$$\begin{aligned}\mu(tD) &\leq \mu(tD \cap \{x : 0 \leq f(x) \leq t^{1/2}\}) + \mu(x : f(x) > t^{1/2}) \\ &= \mu(tD \cap \{x : 0 \leq f(x) \leq t^{1/2}\}) + o(t^{-p}).\end{aligned} \tag{7.1}$$

Letting $x_0 = a$ and $M_s = \{x : f(x) = sf(x_0)\}$ for $s \geq 0$, we define

$$A_s = \{y \in M_0 : \|sx_0 + y - tx_0\| \leq tR\}, \tag{7.2}$$

and

$$N(x) = \inf_{m \in M_t} \|x - m\| = d(x, M_t) \tag{7.3}$$

Then $N(sx_0) = |t - s|\,\|x_0\| = |t - s|R$, and the following lemma is a minor perturbation of a result in [7].

Lemma 7.1. If $y \in A_s$ and $0 \leq s \leq t^{1/2}$, then

$$\|y\| \leq \left(\frac{1}{\beta}\right)^{1/p} tR\left(1 - \frac{N(sx_0)}{tR}\right)^{1/p} \tag{7.4}$$

where $\beta > 0$ is such that $\delta_B(\epsilon) \geq \beta\epsilon^p$ and $\delta_B(\epsilon)$ is the modulus of convexity for B.

Proof. Take $y \in A_s$ and let $z_1 = y + sx_0$, $z_2 = sx_0$. Then

$$\|(z_1 + z_2)/2 - tx_0\| \geq N(sx_0) \tag{7.5}$$

by definition of $N(\cdot)$ and that $y \in M_0$ (since (5.4) holds). Let $Q_1 = (z_1 - tx_0)/tR$, $Q_2 = (z_2 - tx_0)/tR$. Then $\|Q_1\| \leq 1$, $\|Q_2\| \leq 1$ since $y \in A_s$ and $\|Q_1 - Q_2\| = \|z_1 - z_2\|/tR = \|y\|/tR$. Thus by definition of $\delta_B(\epsilon)$

$$\begin{aligned}\beta(\|y\|/tR)^p \leq \delta_B(\|y\|/tR) &\leq 1 - \|(Q_1 + Q_2)/2\| \\ &= 1 - \|(z_1 + z_2)/2 - tx_0\|\,/(tR) \\ &\leq 1 - N(sx_0)/tR\end{aligned}$$

by (7.5). Hence the lemma is proved.

If G is a Gaussian vector with $\mu = \mathcal{L}(G)$ we write $G = G_1 + G_2$ where $G_1 = f(G)x_0/f(x_0)$ and $G_2 = G - G_1$. Then $G_2 \in M_0$, and by Lemma 7.1

$$\begin{aligned}&\mu(tD \cap \{x : 0 \leq f(x) \leq t^{1/2}\}) \qquad (7.6)\\ &= P(\|G_1 + G_2 - tx_0\| \leq tR, 0 \leq f(G) \leq t^{1/2}) \\ &\leq P(0 \leq f(G) \leq t^{1/2}, \|G_2\| \leq \left(\frac{1}{\beta}\right)^{1/p} tR(1 - N(G_1)/tR)^{1/p}) \\ &\leq I_t + J_t\end{aligned}$$

where

$$I_t = P(0 \le f(G) \le t^{1/2}, \|G - f(G)g/\|g\|_\mu^2\| \le \frac{3}{2}tR\left(\frac{1}{\beta}\right)^{1/p}(1 - N(G_1)/tR)^{1/p}),$$

$$J_t = P(0 \le f(G) \le t^{1/2}, \|f(G)g/\|g\|_\mu^2 - G_1\| > \tfrac{1}{2}tR\left(\tfrac{1}{\beta}\right)^{1/p}(1 - N(G_1)/tR)^{1/p}),$$

and $g = Sf \neq 0$. Since $G_1 = f(G)x_0/f(x_0)$,

$$\begin{aligned} N(G_1) &= d(G_1, M_t) \\ &= |t - f(G)/f(x_0)|R, \end{aligned}$$

and hence

$$1 - N(G_1)/tR = |f(G)|/(tf(x_0))$$

when $0 \le f(G) \le t^{1/2}$. Thus, since $f(G)$ and $G - f(G)g/\|g\|_\mu^2$ are independent, we have from Lemma 3.3 that as $t \to \infty$

$$\tfrac{1}{2} - I_t \approx t^{-(p-1)}.$$

Combining (7.1), (7.6), and that J_t converges to zero exponentially fast, we now have as $t \to \infty$ that

$$\begin{aligned} \tfrac{1}{2} - \mu(tD) &\ge \tfrac{1}{2} - I_t - o(t^{-p}) \\ &\approx t^{-(p-1)}. \end{aligned}$$

Hence Theorem 5 holds.

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