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On the future infima of some transient processes

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Summary. Let $(X(t), t \in S)$ be a real-valued stochastic process with $\mathbb{P}(X(0)=0)=1$ and $\mathbb{P}(\lim_{t\to\infty} X(t)=\infty)=1$. In this paper we are interested in the reluctance of such a process to tend to infinity. This entails determining the rate of escape of the associated process $(\inf_{s\geq t} X(s), t\in S)$, the so-called *future infima process*.

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1 Introduction

Let $(X(t), t\in S)$ be a real-valued stochastic process with $\mathbb{P}(X(0)=0)=1$ and $\mathbb{P}(\lim_{t\to\infty} X(t)=\infty)=1$, where S may denote either $[0, \infty)$ or the nonnegative integers. In this paper, we are interested in the reluctance of such a process to tend to infinity. As a measure of this reluctance, we introduce the *future infima process* associated with $(X(t), t\in S)$: for $t\in S$, let

$$I(t) \stackrel{\text{df}}{=} \inf_{s\geq t} X(s),$$

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we note that $(I(t), t \in S)$ is an increasing process which tends to infinity as $t \to \infty$. Primarily we are interested in comparing the rates of escape of $X(t)$ and $I(t)$; consequently, we study the lim sup behavior of $X(t)$, $I(t)$ and $X(t)-I(t)$ as $t \to \infty$.

The following terminology will be useful. Let $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing. We say that $\varphi \in \mathcal{U}(X)$ (the upper-class with respect to the process $(X(t), t \in S)$) if almost surely $X(t) \leqq \varphi(t)$ for all t sufficiently large. We say that $\varphi \in \mathcal{L}(X)$ (the lower class with respect to the process $(X(t), t \in S)$) if almost surely $X(t) > \varphi(t)$ infinitely often as $t \to \infty$. We will denote the upper and lower classes with respect to the process $(I(t), t \in S)$ by $\mathcal{U}(I)$ and $\mathcal{L}(I)$, respectively. Whenever possible, we will compare $\mathcal{U}(X)$ and $\mathcal{U}(I)$ as well as $\mathcal{L}(X)$ and $\mathcal{L}(I)$. Since $I(t) \leqq X(t)$, we always have $\mathcal{U}(X) \subset \mathcal{U}(I)$ and $\mathcal{L}(I) \subset \mathcal{L}(X)$: we will show that the containments can be proper.

Throughout, let $L(x) = L_1(x) \overset{\text{df}}{=} \ln(x \vee e)$ and, for $k \geqq 2$, let $L_k(x) \overset{\text{df}}{=} L(L_{k-1}(x))$.

In Sect. 2, we consider $\mathbb{R}^1$-valued random walk with positive drift. Let $\xi, \xi_1, \xi_2, \ldots$ be independent and identically distributed random variables. We assume that $\mathbb{P}(\xi < 0) > 0$, $\mathbb{E}(\xi) > 0$ and $\mu(t) \overset{\text{df}}{=} \mathbb{E}(e^{t\xi})$, the moment generating function of ξ, is defined in a neighborhood of the origin. Let $X_0 \overset{\text{df}}{=} 0$ and, for $n \geqq 1$, let $X_n \overset{\text{df}}{=} \xi_1 + \ldots + \xi_n$ and $I_n \overset{\text{df}}{=} \inf_{j \geqq n} X_j$. In our first theorem, we demonstrate that

$$\limsup_{n \to \infty} \frac{X_n - I_n}{L(n)} = \frac{1}{|r|} \quad \text{a.s.,} \tag{1.1}$$

where r is the unique negative solution to $\mu(t) = 1$.

In Sect. 3, we develop related results for Brownian motion with drift. Let $(B(t), t \geqq 0)$, denote a standard, one-dimensional Brownian motion and let $f: \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing with $f(0) = 0$. Let

$$X_f(t) \overset{\text{df}}{=} B(t) + f(t) \quad \text{and} \quad I_f(t) \overset{\text{df}}{=} \inf_{s \geqq t} X_f(t),$$

for all $t \geqq 0$. For $f(t) = mtL^p(t)$ with $m > 0$ and $p \geqq 0$, we demonstrate that

$$\limsup_{t \to \infty} \frac{X_f(t) - I_f(t)}{L^{1-p}(t)} = \frac{1}{2m} \quad \text{a.s.} \tag{1.2}$$

In the case $p = 0$, this is the Brownian analogue to (1.1).

The relative proximity of $I_f(t)$ to $X_f(t)$ implies that $(I_f(t), t \geqq 0)$, suitably normalized, inherits all of the classical limit theorems from $(X_f(t), t \geqq 0)$. In fact we can say more: let $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing and let $\varphi(t)$

$=f(t)+\sqrt{t}\psi(t)$. By Kolmogorov's integral test for Brownian motion (see Itô and McKean (1965)), one obtains

$$\varphi\in\mathscr{U}(X_f) \qquad \text{if and only if } \int_1^\infty t^{-1}\psi(t)\exp(-\psi^2(t)/2)\,dt<\infty.$$

However, with probability one $X_f(t)-cL^{1-p}(t)\leqq I_f(t)\leqq X_f(t)$ for t sufficiently large and $c>(2m)^{-1}$. As a consequence,

$$\varphi\in\mathscr{U}(I_f) \qquad \text{if and only if } \int_1^\infty t^{-1}\psi(t)\exp(-\psi^2(t)/2)\,dt<\infty.$$

Using the integral test of Feller (see Feller (1970) or Bai (1989)), a similar conclusion can be drawn for the random walk examples of Sect. 1.

In Sect. 4, we consider a transient Bessel process, i.e., one-dimensional positive diffusion, $(X(t), t\geqq 0)$, determined by the local generator

$$\mathscr{L}f(x)\overset{\text{df}}{=}\frac{1}{2}\left(f''(x)+\frac{d-1}{x}f'(x)\right),$$

where $d>2$. For integer $d\geqq 3$, the radial part of a d-dimensional Brownian motion is such a process. Let $I(t)\overset{\text{df}}{=}\inf_{s\geqq t} X(s)$. Then

$$\text{(1.3)} \qquad \limsup_{t\to\infty}\frac{X(t)-I(t)}{\sqrt{2t\,L_2(t)}}=\limsup_{t\to\infty}\frac{I(t)}{\sqrt{2t\,L_2(t)}}=1 \quad \text{a.s.}$$

One consequence of this development is that $(1-\varepsilon)\sqrt{2t\ln\ln t}\in\mathscr{L}(I)$ for every $\varepsilon>0$. Moreover, in contrast with the one-dimensional results of Sects. 1 and 2, this suggests that $(X(t), t\in S)$ recovers from large excursions from the origin. In the same theorem, we give sufficient conditions (in terms of an integral test) for inclusion in $\mathscr{U}(I)$: in particular, this shows that $\mathscr{U}(X)$ is a proper subset of $\mathscr{U}(I)$. When $d=3$, the results of Sect. 4 can be obtained from a theorem of J.W. Pitman (see Remark 4.1.1).

Finally, in Sect. 5, we discuss analogous results for high-dimensional $\mathbb{Z}^d$-valued random walk ($d\geqq 3$). Let $\xi_1, \xi_2, \ldots$ be independent identically distributed $\mathbb{Z}^d$-valued bounded random variables with zero mean vector and identity covariance matrix. Let $(S_n, n\geqq 0)$ be the associated random walk, which we assume is strongly aperiodic. For $0\leqq t\leqq 1$, let $j_n(t)$ denote the element of $C([0,1])$ obtained by linearly interpolating the points

$$\{(kn^{-1}, n^{-1/2}\inf_{j\geqq k}|S_j|);\ 0\leqq k\leqq n\}.$$

We demonstrate that $j_n(\cdot)$ converges weakly in $C([0,1])$ to the stochastic process

$$(\inf_{s \geqq t} X_s;\ 0 \leqq t \leqq 1),$$

where $(X(t), t \geqq 0)$ is the radial part of a standard d-dimensional Brownian motion. It should be noted that this result cannot be obtained directly from Donsker's invariance principle. In addition, this section includes laws of the iterated logarithm which are analogous to the Brownian motion results of Sect. 4.

2 A transient walk on $\mathbb{R}^1$

Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed $\mathbb{R}^1$-valued random variables with $\mathbb{P}(\xi < 0) > 0$ and $\mathbb{E}(\xi) > 0$. Let $X_0 \stackrel{\text{df}}{=} 0$ and, for all $n \geqq 1$, let $X_n \stackrel{\text{df}}{=} \sum_{i=1}^{n} \xi_i$. Of course this defines a transient random walk on $\mathbb{R}^1$. We define the associated future infima process; that is, for $n \geqq 0$ let

$$I_n \stackrel{\text{df}}{=} \inf_{j \geqq n} X_j.$$

For $t \in \mathbb{R}$, let $\mu(t) \stackrel{\text{df}}{=} \mathbb{E}(e^{t\xi})$, the moment generating function of ξ. We will assume that $\mu(\cdot)$ is defined in a neighborhood of the origin. We observe that $\mu'(0) = \mathbb{E}(\xi) > 0$. Since $\mathbb{P}(\xi \leqq \delta) > 0$ for some $\delta < 0$, $\mu(t) \geqq e^{t\delta}\,\mathbb{P}(\xi \leqq \delta)$ for $t \leqq 0$. Moreover, since $t \mapsto \mu(t)$ is convex, it follows that there exists a unique negative solution to the equation $\mu(t) = 1$, which we will denote by r, i.e., $r < 0$ and $\mu(r) = 1$.

The main result of this section is:

Theorem 2.1 *With probability one,*

$$\limsup_{n \to \infty} \frac{X_n - I_n}{L(n)} = \frac{1}{|r|}.$$

Naturally, the proof of Theorem 2.1 is composed of upper and lower bound arguments. The upper bound will follow from a gambler's ruin calculation for $(X_n, n \geqq 1)$, which is the content of the next lemma.

Lemma 2.2 *For all $n \geqq 0$ and $b > 0$, $\mathbb{P}(X_n - I_n \geqq b) \leqq e^{rb}$.*

Proof. First we make a preliminary calculation. Let $\beta<0$ and $\alpha>0$. Define stopping times

$$\tau_\beta \stackrel{\text{df}}{=} \min\{k: X_k \leqq \beta\} \quad \text{and} \quad \tau_\alpha \stackrel{\text{df}}{=} \min\{k: X_k \geqq \alpha\}.$$

Let $\tau=\tau_\alpha \wedge \tau_\beta$.

Since $\mu(r)=1$, $(e^{rX_n}, n\geqq 0)$ is a mean one, positive martingale. By Doob's optional sampling theorem we obtain

$$\begin{aligned} 1 &= \mathbb{E}(e^{rX_\tau}|\tau_\alpha<\tau_\beta)\,\mathbb{P}(\tau_\alpha<\tau_\beta)+\mathbb{E}(e^{rX_\tau}|\tau_\beta<\tau_\alpha)\,\mathbb{P}(\tau_\beta<\tau_\alpha) \\ &\geqq \mathbb{E}(e^{rX_\tau}|\tau_\beta<\tau_\alpha)\,\mathbb{P}(\tau_\beta<\tau_\alpha). \end{aligned}$$

Since $\{\tau_\beta<\tau_\alpha\}$ implies $rX_\tau \geqq r\beta$, it follows that $\mathbb{E}(e^{rX_\tau}|\tau=\tau_\beta)\geqq e^{r\beta}$. By sending α to infinity, we obtain

$$\mathbb{P}(\tau_\beta<\infty)\leqq e^{-r\beta}. \tag{2.1}$$

To finish the proof, observe that

$$X_n - I_n = -\inf_{j\geqq n}(X_j - X_n) \stackrel{\text{D}}{=} -I_0.$$

Consequently, by an application of (2.1), we obtain

$$\mathbb{P}(X_n - I_n \geqq b) = \mathbb{P}(I_0 \leqq -b) = \mathbb{P}(\tau_{-b}<\infty) \leqq e^{rb},$$

which proves the lemma. □

We will need some additional terminology to prove the lower bound in Theorem 2.1. Let $Y, Y_1, Y_2, \ldots$ be i.i.d. random variables with $\mathbb{E}(Y)=0$ and $M(t)\stackrel{\text{df}}{=}\mathbb{E}(e^{tY})$ defined in a neighborhood of the origin. Let $S_n \stackrel{\text{df}}{=} Y_1+\ldots+Y_n$ for all $n\geqq 1$. Let $\rho(x)$ denote the Chernoff function associated with $-Y$, i.e.,

$$\begin{aligned} \rho(x) &\stackrel{\text{df}}{=} \inf_t e^{-tx}\mathbb{E}(e^{t(-Y)}) \\ &= \inf_t e^{tx} M(t). \end{aligned}$$

For $c>0$, let

$$a(c) \stackrel{\text{df}}{=} \sup\{x: \rho(x)\geqq e^{-1/c}\} \quad \text{and} \quad \varphi(n) \stackrel{\text{df}}{=} \lfloor cL(n) \rfloor \tag{2.2}$$

(where $\lfloor x \rfloor$ denotes the integer part of x). Then

$$\liminf_{n\to\infty} \frac{S_{n+\varphi(n)} - S_n}{\varphi(n)} = -a(c) \quad \text{a.s.} \tag{2.3}$$

This is a modification of the celebrated Erdős-Rényi law of large numbers

$$\lim_{n\to\infty} \min_{0\le k\le n-\varphi(n)} \frac{S_{n+\varphi(n)}-S_n}{\varphi(n)} = -a(c) \quad \text{a.s.}$$

(see, for example, Theorem 2.4.3 of Csörgő and Révész (1981) and its proof).

Proof of Theorem 2.1 To prove the upper bound, let

$$F_k = \{X_k - I_k \geqq -(1+\varepsilon)\, L(k)/r\},$$

for $\varepsilon>0$ and $k\geqq 1$. Then, by Lemma 2.2, $\mathbb{P}(F_k)\leqq k^{-(1+\varepsilon)}$ and $\sum_k \mathbb{P}(F_k)<\infty$. By the Borel-Cantelli lemma $\mathbb{P}(F_k, \text{ i.o.})=0$. It follows that

$$\limsup_{n\to\infty} \frac{X_n - I_n}{L(n)} \leqq -\frac{(1+\varepsilon)}{r} \quad \text{a.s.}$$

We obtain the desired upper bound upon letting $\varepsilon\to 0$.

Let $m \stackrel{\text{df}}{=} \mathbb{E}(\xi)$ and let $Y_i \stackrel{\text{df}}{=} \xi_i - m$ for all $i\geqq 1$. Let $S_n \stackrel{\text{df}}{=} Y_1+\ldots+Y_n$ for all $n\geqq 1$. Let $c>0$ and let $a(c)$ and $\varphi(n)$ be as in (2.2)

Observe that

$$I_n \leqq X_{n+\varphi(n)} = S_{n+\varphi(n)} + m(n+\varphi(n)).$$

Consequently

$$X_n - I_n \geqq -(S_{n+\varphi(n)} - S_n) - m\,\varphi(n).$$

Since $\varphi(n)\sim cL(n)$, by (2.3) we obtain

$$\limsup_{n\to\infty} \frac{X_n-I_n}{L(n)} \geqq -c \liminf_{n\to\infty} \frac{S_{n+\varphi(n)}-S_n}{\varphi(n)} - cm$$
$$= c(a(c)-m) \quad \text{a.s.}$$

For the appropriate choice of c, we will recover the desired constant.

Let $\gamma \stackrel{\text{df}}{=} \mu'(r)$ and let $\lambda(t) \stackrel{\text{df}}{=} \log(e^{-\gamma t}\mu(t)) = -\gamma t + \log \mu(t)$. Then $\lambda(\cdot)$ is convex and $\lambda'(r)=0$. Consequently, $\lambda(t)\geqq \lambda(r) = -\gamma r$ for all t, which is to say $\inf_t e^{-\gamma t}\mu(t) = e^{-\gamma r}$. As a consequence

$$\rho(m-\gamma) = \inf_t e^{(m-\gamma)t}\,\mathbb{E}(e^{tY})$$
$$= \inf_t e^{-\gamma t}\mu(t)$$
$$= e^{-\gamma r}.$$

Let $c=1/(\gamma r)$. Then it follows that $\rho(m-\gamma)=e^{-\gamma r}=e^{-1/c}$, which says that $a((\gamma r)^{-1})\geqq m-\gamma$. Hence $c(a(c)-m)\geqq 1/|r|$. We conclude that

$$\limsup_{n\to\infty}\frac{X_n-I_n}{L(n)}\geqq\frac{1}{|r|}\qquad\text{a.s.,}$$

which is the desired lower bound. □

3 One-dimensional Brownian motion with positive drift

Throughout let $(B(t), t\geqq 0)$ denote a standard, one-dimensional Brownian motion. Let $f\colon \mathbb{R}^+\to\mathbb{R}^+$ be a nondecreasing function with $f(0)=0$ and let $X_f(t)=B(t)+f(t)$. We define the future infima process associated with X_f: for $t\geqq 0$, let

$$I_f(t)=\inf_{s\geqq t} X_f(t)$$

The main result of this section follows:

Theorem 3.1 *If $f(t)=mtL^p(t)$ for $m>0$ and $p\geqq 0$, then*

$$\limsup_{t\to\infty}\frac{X_f(t)-I_f(t)}{L^{1-p}(t)}=\frac{1}{2m}\qquad\text{a.s.}$$

Observe that when $p=0$ we obtain the Brownian motion analogue to Theorem 2.1. Setting $p=1$ in Theorem 3.1, we obtain: for $f(t)=mtL(t)$,

$$\limsup_{t\to\infty}(X_f(t)-I_f(t))=\frac{1}{2m}\qquad\text{a.s.}$$

In this case, no renormalization is necessary.

When $p>1$, $X_f(t)-I_f(t)\to 0$ with probability one as $t\to\infty$. Theorem 3.1 gives the rate with which this difference tends to zero:

$$\limsup_{t\to\infty} L^{p-1}(t)\,(X_f(t)-I_f(t))=\frac{1}{2m}\qquad\text{a.s.}$$

As with Theorem 2.1, the proof of Theorem 3.1 is composed of upper and lower bound arguments. The upper bound will follow quite naturally from a gambler's ruin calculation, which is the content of our next lemma.

Lemma 3.2 *Let $b>0$. If $f'>0$ and $f''\geqq 0$ eventually, then, for t sufficiently large,*

$$\mathbb{P}(X_f(t)-I_f(t)\geqq a)\leqq e^{-2f'(t)a}.$$

Proof. First we consider the case $f(t)=mt$, where $m>0$; the general case will be obtained from this.

Observe that

$$X_f(t)-I_f(t)=-\inf_{s\geqq t}(B(s)-B(t)+m(s-t)),$$

which indicates that $X_f(t)-I_f(t)$ has the same distribution as $-I_f(0)$. By Itô's formula, $(\exp(-2mX_f(t)), t\geqq 0)$ is a mean one positive martingale. Consequently, by a gambler's ruin calculation,

(3.1) $$\mathbb{P}(I_f(0)\leqq -b)=P(X_f(\cdot)\text{ ever hits }-b)=e^{-2mb},$$

which is what we wished to show in the case $f(t)=mt$.

In general, observe that the eventual convexity and monotonicity of f implies

$$f(s)-f(t)\geqq f'(t)(s-t) \quad \text{with} \quad f'(t)>0$$

for all $s\geqq t$ sufficiently large. Thus

$$X_f(t)-I_f(t)=-\inf_{s\geqq t}(B(s)-B(t)+f(s)-f(t))$$
$$\leqq -\inf_{s\geqq t}(B(s)-B(t)+f'(t)(s-t)).$$

This last random variable is distributed as $-I_g(0)$, where $g(s)=f'(t)s$. Consequently, by (3.1)

$$\mathbb{P}(X_f(t)-I_f(t)\geqq b)\leqq \mathbb{P}(I_g(0)\leqq -b)=e^{-2f'(t)b},$$

which verifies the lemma in question. □

To obtain the lower bound in Theorem 3.1, we will use a theorem of Hanson and Russo on the increments of the Wiener process (see Theorem 2.2 of Hanson and Russo (1983)), which we will briefly describe.

Let a_t be measurable with $0\leqq a_t\leqq t$ for all $t>0$. Let

$$b_t=(2a_t(\log(t/a_t)+\log_2(t)))^{-1/2}.$$

Let $\mathscr{L}$ denote the set of limit points of $b_t(B(t)-B(t-a_t))$. If $a_t t^\alpha\to\infty$ as $t\to\infty$ for all $\alpha>0$, then, by the theorem of Hanson and Russo, $\mathbb{P}(\mathscr{L}=[-1,1])=1$. In particular,

(3.2) $$\liminf_{t\to\infty} b_t(B(t)-B(t-a_t))=-1 \quad \text{a.s.}$$

Proof of Theorem 3.1 It is easily checked that $f(t)=mtL^p(t)$ with $m>0$ and $p\geqq 0$, satisfies the hypothesis of Lemma 3.3. In fact, for t sufficiently large,

$$f'(t)=mL^p(t)+mpL^{p-1}(t).$$

Let $t_1 \stackrel{\text{df}}{=} 1$ and, for $k\geqq 2$, let

$$t_k \stackrel{\text{df}}{=} t_{k-1}+\frac{1}{L^{p+1}(k)}.$$

Consequently, $t_k \sim k/L^{p+1}(k)$ and $f'(t_k)\sim mL^p(k)$ as $k\to\infty$. As a consequence, by the mean value theorem,

$$f(t_{k+1})-f(t_k)=o(1) \quad \text{as} \quad k\to\infty. \tag{3.3}$$

For $k\geqq 1$, let

$$D_k = \sup_{t_k\leqq t\leqq t_{k+1}} (B(t)-B(t_k)).$$

We will show that D_k is small as $k\to\infty$.

Let $\varepsilon>0$ be given. Then, by the reflection principle, the Markov property and Brownian scaling we obtain

$$\begin{aligned}\mathbb{P}(D_k>\varepsilon)&=2\,\mathbb{P}(B(t_{k+1}-t_k)>\varepsilon)\\ &=2\,\mathbb{P}(B(1)>\varepsilon/\sqrt{t_k-t_{k+1}})\\ &\leqq 2e^{-\varepsilon^2 L^{p+1}(k)/2}\end{aligned}$$

where we have used the well-known estimate: for x sufficiently large, $\mathbb{P}(B(1)>x)\leqq e^{-x^2/2}$.

We will need to consider two cases: if $p=0$, then

$$P(D_k>\varepsilon)\leqq\frac{2}{k^{\varepsilon^2/2}}.$$

Consequently, for $\varepsilon>\sqrt{2}$, we obtain $\sum_{k=1}^{\infty}\mathbb{P}(D_k>\varepsilon)<\infty$. By the Borel-Cantelli lemma, it follows that

$$\limsup_{k\to\infty} D_k\leqq\sqrt{2} \quad \text{a.s.} \tag{3.4}$$

If, however, $p>0$, then $\sum_{k=1}^{\infty}\mathbb{P}(D_k>\varepsilon)<\infty$ for all $\varepsilon>0$; consequently, by the Borel-Cantelli lemma,

$$\lim_{k\to\infty} D_k=0 \quad \text{a.s.} \tag{3.5}$$

Let $q>1$. By Lemma 3.3

$$\mathbb{P}(X_f(t_k)-I_f(t_k)>qL^{1-p}(t_k)/2m)\leqq e^{-qL(k)-qp}$$
$$=e^{-qp}\frac{1}{k^q},$$

which is summable by our choice of q. From the Borel-Cantelli lemma, we obtain

$$\limsup_{k\to\infty}\frac{X_f(t_k)-I_f(t_k)}{L^{1-p}(t_k)}\leqq\frac{1}{2m}\quad \text{a.s.} \tag{3.6}$$

Finally, for $t_k\leqq t\leqq t_{k+1}$ we have

$$\begin{aligned}X_f(t)-I_f(t)&\leqq X_f(t)-I_f(t_k)\\&=(B(t)-B(t_k))+(f(t)-f(t_k))+(X_f(t_k)-I_f(t_k))\\&\leqq D_k+(f(t_{k+1})-f(t_k))+(X_f(t_k)-I_f(t_k))\end{aligned}$$

In conjunction with (3.3) through (3.6), this demonstrates that

$$\limsup_{t\to\infty}\frac{X_f(t)-I_f(t)}{L^{1-p}(t)}\leqq\frac{1}{2m}\quad \text{a.s.}$$

which gives the upper bound in question.

To obtain the lower bound, let

$$a_t\overset{\text{df}}{=}\frac{L^{1-2p}(t)}{2m^2}.$$

Since a_t is nonnegative, we obtain

$$I_f(t)\leqq X_f(t+a_t)=B(t+a_t)+f(t+a_t).$$

As a consequence,

$$X_f(t)-I_f(t)\geqq-(B(t+a_t)-B(t))+f(t)-f(t+a_t).$$

By the mean value theorem, we obtain

$$f(t)-f(t+a_t)\sim-\frac{1}{2m}L^{1-p}(t)\quad\text{as}\quad t\to\infty.$$

By (3.2)

$$\liminf_{t\to\infty}\frac{B(t+a_t)-B(t)}{L^{1-p}(t)}=\frac{-1}{m}\quad \text{a.s.,}$$

(where we have written the increment forward in time). Consequently, we see that

$$\limsup_{t\to\infty} \frac{X_f(t)-I_f(t)}{L^{1-p}(t)} \geqq \frac{1}{2m} \quad \text{a.s.,}$$

which is the lower bound in question. □

4 High-dimensional Bessel processes

Throughout let $(X(t); t \geqq 0)$ be a Bessel process of index $d>2$, i.e., a one-dimensional diffusion on $[0, \infty)$ with local generator

$$\mathscr{L} f(x) = \frac{1}{2} f''(x) + \frac{d-1}{2x} f'(x)$$

for all $f \in C^2([0, \infty))$ and $d>2$ (see Revuz and Yor (1991), p. 411). For integer $d \geqq 3$, $(X(t), t \geqq 0)$ can be realized as the radial part of a d-dimensional Brownian motion.

As with the previous examples, we consider the future infima process associated with $(X(t), t \geqq 0)$, i.e.,

$$I(t) \stackrel{\text{df}}{=} \inf_{s \geqq t} X(s).$$

It is worth mentioning that $I(t)$ inherits scaling from $X(t)$, i.e., for any $c>0$, $(I(t); t \geqq 0)$ and $(c^{-1/2} I(ct); t \geqq 0)$ have the same finite dimensional distributions. Concerning the process $I(t)$ we have:

Theorem 4.1 *With probability one:*

(1)
$$\limsup_{t\to\infty} \frac{I(t)}{\sqrt{2t\, L_2(t)}} = 1$$

(2)
$$\limsup_{t\to\infty} \frac{X(t)-I(t)}{\sqrt{2t\, L_2(t)}} = 1.$$

(3) *Let $\varphi(t) = \sqrt{t}\,\psi(t)$ be nondecreasing in $t>0$ and assume that $\psi(t) \to \infty$ as $t \to \infty$. Then*

$$\int_1^\infty \psi(t)^{d-2}\, t^{-1} \exp(-\psi^2(t)/2)\, dt < \infty \quad \textit{implies} \quad \varphi(t) \in \mathscr{U}(I).$$

Theorem 4.1(3) should be compared with the Kolmogorov and the Dvoretsky-Erdős integral tests (see, for example, Itô and McKean (1965), pp. 161 ff.). We shall state the latter below for convenience:

Theorem A *(Kolmogorov-Dvoretsky-Erdős) Let $\psi: \mathbb{R}^+ \mapsto \mathbb{R}^+$ be nondecreasing and unbounded as $t \to \infty$. For all $d \geqq 1$,*

$$\mathbb{P}(X(t) > \sqrt{t}\,\psi(t) \text{ i.o.}) = \begin{cases} 0 \\ 1 \end{cases}$$

if and only if

$$\int_1^\infty \psi(t)^d\, t^{-1} \exp(-\psi^2(t)/2)\, dt \begin{cases} < \infty \\ = \infty \end{cases}.$$

Remark 4.1.1 The proof of Theorem 4.1(3) is inspired by the first half of Kolmogorov's test (see Itô and McKean (1965), p. 34). In it we exploit the analogy between first hitting times of spheres by $X(t)$ and $\sup_{s \leqq t} X(s)$, and last exit times from spheres by $X(t)$ and the process $\inf_{s \geqq t} X(s)$. When $d=3$ much more is true. Indeed, by the celebrated theorem of J.W. Pitman (Theorem 3.5, Chap. VI of Revuz and Yor (1991)), the process $((X(t), I(t)), t \geqq 0)$ has the same finite dimensional distributions as $((2 \sup_{s \leqq t} X(s) - X(t), \sup_{s \leqq t} X(s)), t \geqq 0)$, where $(X(t), t \geqq 0)$ is a 1-dimensional Brownian motion. Indeed if $d=3$, then $X(t) - I(t)$ is a Bessel process of index one and Theorem 4.1(2) follows easily from the law of the iterated logarithm. Moreover, Theorem A implies the necessity of Theorem 4.1(3) in three dimensions. We do not have a proof for this necessity when $d \neq 3$.

From Theorem A and Theorem 4.1(3) one can easily show that for $d > 2$,

$$X(t) > \sqrt{t} \cdot \sqrt{2 L_2 t + (d+2) L_3 t + 2 L_4 t}, \qquad \text{i.o.}$$

while for all $\varepsilon > 0$

$$I(t) < \sqrt{t} \cdot \sqrt{2 L_2 t + d L_3 t + (2+\varepsilon) L_4 t}, \qquad \text{eventually}$$

with probability one.

Lim inf results for $(I(t))$ are not interesting; a real variable argument shows the following:

Lemma B *Let $\varphi: \mathbb{R}^1_+ \mapsto \mathbb{R}^1_+$ be a function satisfying: $\varphi(t) \downarrow 0$ and $\sqrt{t}\,\varphi(t) \uparrow \infty$ as $t \to \infty$. Then for any continuous function $x: \mathbb{R}^1_+ \mapsto \mathbb{R}^1_+$ for which $\lim_{t \to \infty} x(t) = \infty$,*

$$x(t) \leqq \sqrt{t}\,\varphi(t), \qquad \text{i.o.}$$

if and only if

$$\inf_{s \geqq t} x(s) \leqq \sqrt{t}\,\varphi(t) \quad \text{i.o.}$$

Hence we need only consult the lim inf results for $X(t)$, (e.g., see Motoo (1959), p. 27). A simple consequence of this development is that

$$\liminf_{t \to \infty} \frac{I(t)}{\sqrt{2t\,L_2\,t}} = 0, \qquad \text{a.s.}$$

In conjunction with Theorem 4.1, this demonstrates that the random sequence

$$\{(2t\,L_2\,t)^{-1/2}\,I(t),\ t \geqq 0\}$$

converges and clusters in [0, 1].

The main ingredient in the proof of Theorem 4.1 is the solution to the gambler's ruin problem for Bessel processes (see for example Itô and McKean (1965) or Revuz and Yor (1991)),

$$\mathbb{P}(I_1 \geqq x \mid X_1 = y) = \mathbb{P}(X(s) \geqq x \text{ for all } s \geqq 1 \mid X_1 = y) = 1 - (x/y)^{d-2}.$$

For $d > 0$, let

$$\Gamma_d \overset{\text{df}}{=} \frac{2}{\Gamma(d/2)\,2^{d/2}}.$$

It is well known that the density of X_1 is given by

$$\text{(4.1)} \qquad \frac{\mathbb{P}(X_1 \in dx)}{dx} = \Gamma_d\,x^{d-1} \exp(-x^2/2)\,1_{[0,\infty)}(x).$$

To derive the integral test (Theorem 4.1(3)), we will need the distribution of the last exit time from a sphere of radius a by a transient Bessel process. Let $a > 0$ and let

$$\sigma_a = \sup\{t > 0 : X(t) \leqq a\}.$$

Lemma 4.2 *Let* $d > 2$.
(1) *The density of* I_1 *is given by*

$$\frac{\mathbb{P}(I_1 \in dx)}{dx} = \Gamma_{d-2}\,x^{d-3} \exp(-x^2/2)\,1_{[0,\infty)}(x).$$

(2) *The density of* σ_a *is given by*

$$\frac{\mathbb{P}(\sigma_a \in dt)}{dt} = \frac{\Gamma_{d-2}}{2t}(a/\sqrt{t})^{d-2} \exp(-a^2/2t)\,1_{[0,\infty)}(t).$$

Remark 4.2.1 Observe that I_1 in dimension $d \geqq 3$ has the same distribution as X_1 in dimension $d-2$. A similar but deeper phenomenon has been observed in Ciesielski and Taylor (1962).

Proof. By (4.1),

$$\mathbb{P}(I_1 \geqq x) = \Gamma_d \int_x^\infty \mathbb{P}(I_1 \geqq x \mid X_1 = y)\, y^{d-1} e^{-y^2/2}\, dy.$$

Thus,

$$\mathbb{P}(I_1 \geqq x) = \Gamma_d \int_x^\infty (y^{d-2} - x^{d-2})\, y e^{-y^2/2}\, dy$$
$$= (d-2)\,\Gamma_d \int_x^\infty y^{d-3} e^{-y^2/2}\, dy,$$

by an integration by parts. Since $(d-2)\,\Gamma_d = \Gamma_{d-2}$, the density of I_1 is as stated.

To verify (2), observe that $\{\sigma_a \leqq t\} = \{I(t) \geqq a\}$, by Lemma 4.3 and scaling we have

$$\mathbb{P}(\sigma_a \leqq t) = \mathbb{P}(I_1 \geqq a t^{-1/2}) = \Gamma_{d-2} \int_{at^{-1/2}}^\infty y^{d-3} e^{-y^2/2}\, dy.$$

We obtain the result upon differentiating with respect to t. □

Proof of Theorem 4.1 Throughout, let $F(t) = (2t L_2(t))^{-1/2}$. Recall (see Revuz and Yor (1991)) that

$$\limsup_{t \to \infty} F(t)\, X(t) = 1, \qquad \text{a.s.} \tag{4.2}$$

Fix $\varepsilon \in (0, 1/2)$ and define stopping times as follows:

$$U(n) \overset{\text{df}}{=} \inf\{s \geqq n : F(s)\, X(s) \geqq (1-\varepsilon)\}. \tag{4.3}$$

By (4.2) and (4.3), $U(n) \geqq n$ and is finite, almost surely. Moreover, $\lim_n U(n) = \infty$ with probability one.

Let us prove (1) first. By applying the strong Markov property at the stopping time, $U(n)$:

$$\begin{aligned}
\mathbb{P}(F(U(n))\, I_{U(n)} \geqq (1-2\varepsilon)) &= \mathbb{P}(I_{U(n)} \geqq (1-2\varepsilon)\, X_{U(n)}/(1-\varepsilon)) \\
&= \int \mathbb{P}(I_{U(n)} \geqq (1-2\varepsilon)\, u/(1-\varepsilon) \mid X_{U(n)} = u)\, \mathbb{P}(X_{U(n)} \in du) \\
&= \int \mathbb{P}(I_0 \geqq (1-2\varepsilon)\, u/(1-\varepsilon) \mid X_0 = u)\, \mathbb{P}(X_{U(n)} \in du) \\
&= \int \mathbb{P}(I_0 \geqq (1-2\varepsilon)\, u/1-\varepsilon) \mid X_0 = u)\, \mathbb{P}(X_{U(n)} \in du) \\
&= 1 - \left(\frac{1-2\varepsilon}{1-\varepsilon}\right)^{d-2} \\
&\overset{\text{df}}{=} \kappa_0(\varepsilon).
\end{aligned} \tag{4.4}$$

Since $U(n) \geqq n$, by definition,

$$\mathbb{P}(F(t)\,I(t) \geqq (1-2\varepsilon) \text{ some } t \geqq n) \geqq \mathbb{P}(F(U(n))\,I_{U(n)} \geqq (1-2\varepsilon)) = \kappa_0(\varepsilon).$$

Therefore for all $\varepsilon \in (0, 1/2)$,

$$\mathbb{P}(\limsup_{t\to\infty} F(t)\,I(t) \geqq 1-2\varepsilon) = \lim_{n\to\infty} \mathbb{P}(F(t)\,I(t) \geqq (1-2\varepsilon) \text{ some } t \geqq n)$$
$$= \kappa_0(\varepsilon) > 0.$$

By Shiga and Watanabe (1973), $(X(t))$ is the same process as $(tX(1/t))$. Therefore, by the Blumenthal $0-1$ law (see Revuz and Yor (1991)), the tail σ-field of $(X(t))$ is trivial. Hence the above development shows that for all $\varepsilon \in (0, 1/2)$,

$$\limsup_{t\to\infty} F(t)\,I(t) \geqq 1-2\varepsilon,$$

almost surely. Letting $\varepsilon \to 0$ along a countable sequence we arrive at the lower bound. The upper bound is a consequence of (4.2) and the fact that $I(t) \leqq X(t)$. This proves part (1). The proof of part (2) is similar. Indeed,

$$\begin{aligned}(4.5)\quad \mathbb{P}(F(U(n))(X_{I(n)} - I_{U(n)}) \geqq (1-2\varepsilon)) &= \mathbb{P}(I_{U(n)} \leqq 2\varepsilon\, X_{U(n)}/(1-\varepsilon)) \\ &= \int \mathbb{P}(I_0 \leqq 2\varepsilon u \mid X_0 = u)\,\mathbb{P}(X_{U(n)} \in du) \\ &= \left(\frac{2\varepsilon}{1-\varepsilon}\right)^{d-2} \\ &\overset{\text{df}}{=} \kappa_1(\varepsilon).\end{aligned}$$

Hence $U(n) \geqq n$ implies that,

$$\begin{aligned}&\mathbb{P}(\limsup_{t\to\infty} F(t)\,(X(t) - I(t)) \geqq 1-2\varepsilon) \\ &\quad = \lim_{n\to\infty} \mathbb{P}(F(t)\,(X(t) - I(t)) \geqq (1-2\varepsilon) \text{ some } t \geqq n) \\ &\quad \geqq \mathbb{P}(F(U(n))\,(X_{U(n)} - I(U(n))) \geqq (1-2\varepsilon)) \\ &\quad = \kappa_1(\varepsilon) > 0.\end{aligned}$$

Another application of the triviality of the tail σ-field of $(X(t))$ establishes (2).

Finally, to prove (3), assume that φ satisfies the hypotheses of the theorem. Let $0 < a < b < \infty$ and let

$$E \overset{\text{df}}{=} \{I(t) \geqq \varphi(t) \text{ for some } a \leqq t \leqq b\}.$$

We will estimate $\mathbb{P}(E)$.

Let $a=t_0<t_1<\dots<t_m=b$ be a partition of the interval $[a, b]$. For each integer $1\leqq k\leqq m$, let

$$E_k \stackrel{\mathrm{df}}{=} \{I(t)\geqq\varphi(t) \text{ for some } t_{k-1}<t\leqq t_k \text{ but } I(t)<\varphi(t) \text{ for all } a\leqq t\leqq t_{k-1}\}.$$

Then $E=\{\sigma_{\varphi(a)}\leqq a\}\cup\left(\bigcup_{k=1}^{m} E_k\right)$ and

$$\mathbb{P}(E)=\mathbb{P}(\sigma_{\varphi(a)}\leqq a)+\sum_{k=1}^{m}\mathbb{P}(E_k).$$

However, since φ is nondecreasing,

$$\begin{aligned}\mathbb{P}(E_k)&\leqq\mathbb{P}(I_{t_{k-1}}<\varphi(t_{k-1}), I_{t_k}\geqq\varphi(t_{k-1}))\\&=\mathbb{P}(t_{k-1}<\sigma_{\varphi(t_{k-1})}\leqq t_k)\\&=\frac{\Gamma_{d-2}}{2}\int_{t_{k-1}}^{t_k}\left(\frac{\varphi(t_{k-1})}{\sqrt{u}}\right)^{d-2}\exp(-\varphi^2(t_{k-1})/2u)\frac{du}{u},\end{aligned}$$

where this last equality is from Lemma 4.5. We also have

$$\mathbb{P}(\sigma_{\varphi(a)}\leqq a)=\frac{\Gamma_{d-2}}{2}\int_0^{a\varphi^{-2}(a)} u^{-d/2}e^{-1/2u}\,du.$$

As the mesh size of the partition tends to zero, we obtain

$$\begin{aligned}\mathbb{P}(E)\leqq&\frac{\Gamma_{d-2}}{2}\int_0^{a\varphi^{-2}(a)} u^{-d/2}e^{-1/2u}\,du\\&+\frac{\Gamma_{d-2}}{2}\int_a^b\left(\frac{\varphi(u)}{\sqrt{u}}\right)^{d-2}e^{-\varphi^2(u)/2u}\frac{du}{u}.\end{aligned}$$

Since $a\varphi^{-2}(a)=\psi^{-1}(a)\to 0$ as $a\to\infty$, the first integral tends to zero as $a\to\infty$. Since

$$\int_1^\infty(\varphi(u)/\sqrt{u})^{d-2}u^{-1}e^{-\varphi^2(u)/2u}\,du<\infty,$$

$$\lim_{a\to\infty}\lim_{b\to\infty}\int_a^b(\varphi(u)/\sqrt{u})^{d-2}e^{-\varphi^2(u)/2u}\,du/u=0.$$

Consequently,

$$\lim_{a\to\infty}\lim_{b\to\infty} \mathbb{P}(I(t)\geqq\varphi(t) \text{ for some } a\leqq t\leqq b)=0,$$

and $\mathbb{P}(I(t)\geqq\varphi(t)$ i.o.$)=0$, which demonstrates (b). □

5 High-dimensional random walk results

Let $\xi_1, \xi_2, \ldots$ be independent, identically distributed random vectors in $\mathbb{Z}^d (d\geqq 3)$ with zero mean vector and identity covariance matrix. For simplicity assume that $\mathbb{P}(|\xi_1|\leqq M)=1$, for some finite constant M. Let $S_n=\sum_{i=1}^{n}\xi_i$ be the associated random walk. Suppose that (S_n) is strongly aperiodic (see Spitzer (1964)). Define the resolvent density or Green function of the walk,

$$g(x)\stackrel{\text{df}}{=}\sum_{i=1}^{\infty}\mathbb{P}(S_i=-x) \qquad x\in\mathbb{Z}^d.$$

We define the future infima process associated with (S_n) (and its truncated version), i.e., for $1\leqq n\leqq N$,

$$J_n\stackrel{\text{df}}{=}\inf_{j\geqq n}|S_j|, \qquad J_n^N\stackrel{\text{df}}{=}\min_{n\leqq j\leqq N}|S_j|. \tag{5.1}$$

Throughout, let

$$\gamma_d\stackrel{\text{df}}{=}\frac{1}{2\pi^{d/2}}\Gamma(d/2-1). \tag{5.2}$$

Define $C([0,1])$ to be the space of real continuous functions on $[0,1]$ endowed with uniform topology. Let j_n be the extension of $\{n^{-1/2}J_k;\ 1\leqq k\leqq n\}$ to $C([0,1])$ by linear interpolation, i.e., for all $0\leqq t\leqq 1$,

$$j_n(t)\stackrel{\text{df}}{=}(nt-[nt])(n^{-1/2}J_{1+[nt]}-n^{-1/2}J_{[nt]})+n^{-1/2}J_{[nt]}.$$

What follows is a central limit theorem for the process $\{j_n, n\geqq 1\}$ which does not appear to be an immediate consequence of Donsker's invariance principle. (For this and more, see Billingsley (1968).)

Theorem 5.1 *For all* $d \geqq 3$, $j_n \Rightarrow I$, *where* I *is the future infima process of the radial part of a Brownian motion (as defined in Sect. 4) and* $\Rightarrow$ *denotes weak convergence.*

Remark 5.1.1 The proof of this theorem requires some standard potential theory for random walk. Theorem 5.1 generalizes easily to random walk with positive definite covariance matrix Q: in this case one replaces the standard Brownian motion with a Brownian motion having covariance matrix Q.

With Theorem 5.1 in mind, it is natural to ask whether the laws of the iterated logarithm of Sect. 2 hold in the context of random walk. This is the content of our next theorem.

Theorem 5.3 *With probability one,*

$$\limsup_{n\to\infty} \frac{J_n}{\sqrt{2n\,L_2(n)}} = 1 \quad \textit{and,}$$

$$\limsup_{n\to\infty} \frac{\max_{m\leqq n}|S_m| - J_n}{\sqrt{2n\,L_2(n)}} = \limsup_{n\to\infty} \frac{|S_n| - J_n}{\sqrt{2n\,L_2(n)}} = 1.$$

Remark 5.3.1 This result has already been observed by Erdős and Taylor for simple, symmetric random walk (see Theorem 9 of Erdős and Taylor (1960)).

The proof of Theorem 5.3 is the same as that of its Brownian analogues (Theorem 4.1) and, as such, will not be presented here. The essential difference is this: the gambler's ruin problem cannot be solved explicitly in this setting. Instead one can use the estimates in Lemma 5.5 and Lemma 5.6.

Fundamental to the proof of Theorem 5.1 are estimates for certain gambler's ruin probabilities; the following sequence of lemmas is directed towards obtaining these estimates. The next lemma can be found in Spitzer (1964), pp. 77–79.

Lemma 5.4 *The following hold for the random walks of this section.*

(i) $\lim_n \sup_x |(2\pi n)^{d/2}\,\mathbb{P}(S_n = x) - \exp(-|x|^2/2n)| = 0.$

(ii) $\lim_n \sup_x |x|^2 n^{-1} |(2\pi n)^{d/2}\,\mathbb{P}(S_n = x) - \exp(-|x|^2/2n)| = 0.$

Lemma 5.5 *Under the assumptions of Lemma* 5.4, $|x|^{d-2} g(x) \sim \gamma_d$, *as* $|x| \to \infty$.

Proof. Section 26 of Spitzer (1964) contains a proof of this for the case $d=3$. For the most part, we shall mimic this proof for the general case $d \geqq 3$: the only exception may be that the last part of our proof is slightly cleaner.

Define for $s>0$ and $x\in\mathbb{R}^d$, $p_s(x)=(2\pi s)^{-d/2}\exp(-|x|^2/2s)$. Then, for any $K>0$,

$$(5.3)\qquad |g(x)-\sum_{n=1}^{\infty} p_n(x)| \leqq \sum_{n=1}^{[K|x|^2]} (\mathbb{P}(S_n=x)+p_n(x)) + \sum_{n=1+[K|x|^2]}^{\infty} |\mathbb{P}(S_n=x)-p_n(x)| \overset{\text{df}}{=} T_1^{5.3}+T_2^{5.3}.$$

As in Spitzer, we estimate the two terms separately. By Lemma 5.4(i), $\sup_x \mathbb{P}(S_n=x) \leqq c_{3.1}\, n^{-d/2}$ and, trivially, $\sup_x p_n(x) \leqq c_{5.3}\, n^{-d/2}$. Therefore

$$(5.4)\qquad T_1^{5.3} \leqq c_{5.4} \sum_{n=1}^{[K|x|^2]} n^{-d/2} \leqq c_{5.5}\, K^{-(d-2)/2} |x|^{2-d}.$$

Fix $\varepsilon>0$ arbitrarily small. By Lemma 5.4(ii), for all n large enough,

$$\sup_x |x|^2 |\mathbb{P}(S_n=x)-p_n(x)| \leqq \varepsilon n^{-(d-2)/2}.$$

Therefore, for all $|x|$ large enough,

$$(5.5)\qquad T_2^{5.3} \leqq \varepsilon |x|^{-2} \sum_{n=[K|x|^2]+1}^{\infty} n^{-(d-2)/2} \leqq c_{5.6}\, \varepsilon |x|^{2-d} K^{-(d-4)/2}.$$

Therefore,

$$\limsup_{|x|\to\infty} |x|^{2-d} |g(x)-\sum_n p_n(x)| \leqq c_{5.5}\, K^{-(d-2)/2}.$$

Since K is arbitrary, it suffices to show that as $|x|\to\infty$,

$$(5.6)\qquad |x|^{2-d} \sum_n p_n(x) \sim \gamma_d.$$

Write

$$\sum_n p_n(x) = \sum_n \int_{n-1}^{n} p_n(x)\, ds.$$

Since $\{p_s(x);\, s\geqq 1\}$ is decreasing for each $x\in\mathbb{Z}^d$,

$$\int_1^{\infty} p_s(x)\, ds \leqq \sum_n p_n(x) \leqq \int_0^{\infty} p_s(x)\, ds.$$

However, elementary calculations reveal that for all $a>0$, as $|x|\to\infty$,

$$\int_a^{\infty} p_s(x)\,ds=|x|^{2-d}\int_{a|x|^{2-d}} (2\pi s)^{-d/2}\exp(-1/2s)\,ds\sim\gamma_d|x|^{2-d},$$

which verifies (5.6) and, hence, the lemma. □

Define first hitting times,

$$\tau_k=\inf\{j\colon k-M<|S_j|<k+M\}. \tag{5.7}$$

Next we obtain gambler's ruin estimates for τ_k.

Lemma 5.6 *Fix $\eta>1$. Then under the assumptions of Lemma 5.4, as $k\to\infty$,*

$$\mathbb{P}(\tau_k=\infty\,|\,S_0=a)\sim\frac{|a|^{d-2}-k^{d-2}}{|a|^{d-2}},$$

uniformly over all $a\in\mathbb{Z}^d$ with $|a|\geqq(k+2M)\eta$.

Proof. Note that $g(S_n)$ is a martingale. Moreover, it is bounded by Lemma 5.5. Therefore by Doob's optional stopping theorem,

$$\begin{aligned}g(a)&=\mathbb{E}(g(S_{\tau_k\wedge\tau_l})\,|\,S_0=a)\\&=\mathbb{E}(g(S_{\tau_l})\,1_{\{\tau_l<\tau_k\}}\,|\,S_0=a)+\mathbb{E}(g(S_{\tau_k})\,1_{\{\tau_k<\tau_l\}}\,|\,S_0=a).\end{aligned}$$

Since $g(S_n)$ is a bounded martingale, we see that $g(S_{\tau_l})\to 0$ almost surely and in $L_1(\mathbb{P})$ as $\ell\to\infty$. Fix $\varepsilon>0$ small. If k is sufficiently large, then uniformly over all $|a|>k+2M$,

$$\begin{aligned}&(1-\varepsilon)\,\gamma_d|k+M|^{2-d}\,\mathbb{P}(\tau_k<\infty\,|\,S_0=a)\\&\quad\leqq g(a)\leqq(1+\varepsilon)\,\gamma_d|k-M|^{2-d}\,\mathbb{P}(\tau_k<\infty\,|\,S_0=a).\end{aligned}$$

Likewise, for all k large enough, Lemma 5.5 guarantees that uniformly over all $|a|>k+2M$,

$$(1-\varepsilon)\,\gamma_d|a|^{2-d}\leqq g(a)\leqq(1+\varepsilon)\,\gamma_d|a|^{2-d}.$$

Therefore,

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)\frac{|k-M|^{d-2}}{|a|^{d-2}}\leqq\mathbb{P}(\tau_k<\infty\,|\,S_0=a)\leqq\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\frac{|k+M|^{d-2}}{|a|^{d-2}}.$$

Hence uniformly over all $|a| \geqq (k+2M)\eta$,

$$\frac{|a|^{d-2}}{|a|^{d-2}-k^{d-2}} \mathbb{P}(\tau_k=\infty \mid S_0=a) \leqq \frac{|a|^{d-2}-|k-M|^{d-2}}{|a|^{d-2}-k^{d-2}} + 2\varepsilon \frac{|k-M|^{d-2}}{|a|^{d-2}-k^{d-2}}$$
$$\leqq \frac{|a|^{d-2}-|k-M|^{d-2}}{|a|^{d-2}-k^{d-2}} + \frac{2\varepsilon}{\eta^{d-2}-1}$$
$$\to 1+\frac{2\varepsilon}{\eta^{d-2}-1}, \quad \text{as} \quad k\to\infty.$$

Since $\varepsilon>0$ is arbitrary, we get the upper bound. The lower bound is similar. □

Finally, we state the following elementary lemma without proof.

Lemma 5.7 *Suppose Z and Z_n are a sequence of $C([0,1])$-valued random variables that are monotonically increasing. If the finite dimensional distributions of Z_n converge to those of Z, then $Z_n \Rightarrow Z$.*

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1 Fix $m\geqq 1$ and $0\leqq t_1 \leqq \dots \leqq t_m \leqq 1$. By Lemma 5.7 it is sufficient to show that

$$(j_n(t_1), \dots, j_n(t_m)) \xrightarrow{\mathrm{D}} (I_{t_1}, \dots, I_{t_m}).$$

Let $0\leqq \lambda_1 \leqq \lambda_2 \leqq \dots \leqq \lambda_m$. Since $j_n(t) \leqq j_{n+1}(t)$ for all $0\leqq t\leqq 1$, the proof would be complete upon showing that, as $n\to\infty$,

$$\mathbb{P}(j_n(t_i) \geqq \lambda_i \text{ for all } i\leqq m) \to \mathbb{P}(I_{t_i} \geqq \lambda_i \text{ for all } i\leqq m).$$

However,

$$\mathbb{P}(J_{[nt_i]} \geqq \sqrt{n}\,\lambda_i \text{ for all } i\leqq m) \leqq \mathbb{P}(j_n(t_i) \geqq \lambda_i \text{ for all } i\leqq m)$$
$$\leqq \mathbb{P}(J_{1+[nt_i]} \geqq \sqrt{n}\,\lambda_i \text{ for all } i\leqq m).$$

We will show that, as $n\to\infty$,

$$\text{(5.8)} \qquad \mathbb{P}(J_{[nt_i]} \geqq \sqrt{n}\,\lambda_i \text{ for all } i\leqq m) \to \mathbb{P}(I_{t_i} \geqq \lambda_i \text{ for all } i\leqq m).$$

The proof of (5.8) can be used to show that, as $n\to\infty$,

$$\mathbb{P}(J_{1+[nt_i]} \geqq \sqrt{n}\,\lambda_i \text{ for all } i\leqq m) \to \mathbb{P}(I_{t_i} \geqq \lambda_i \text{ for all } i\leqq m),$$

which would finish the proof.

By the Markov property, the left expression in (5.8) can be written as

$$(5.9)\quad \mathbb{P}(J_{[nt_i]}^{[nt_i+1]} \geqq \sqrt{n}\lambda_i \text{ for all } i \leqq m-1,\ J_{[nt_m]} \geqq \sqrt{n}\lambda_m)$$
$$= \sum_{\substack{y\in\mathbb{Z}^d \\ |y|\geqq \sqrt{n}\lambda_m}} \mathbb{P}(J_{[nt_i]}^{[nt_i+1]} \geqq \sqrt{n}\lambda_m \text{ for all } i \leqq m-1 \mid S_{[nt_m]}=y)$$
$$\cdot\, \mathbb{P}(J_{[nt_m]} \geqq \sqrt{n}\lambda_m \mid S_{[nt_m]}=y)\, \mathbb{P}(S_{[nt_m]}=y).$$

Fix an arbitrary $\eta>1$. By another application of the Markov property, for all $y\in\mathbb{Z}^d$ with $|y|\geqq(\lambda_m\sqrt{n}+2M)\eta$,

$$(5.10)\quad \mathbb{P}(\tau_{\sqrt{n}\lambda_m+M}=\infty \mid S_0=y) \leqq \mathbb{P}(J_{[nt_m]} \geqq \sqrt{n}\lambda_m \mid S_{[nt_m]}=y)$$
$$\leqq \mathbb{P}(\tau_{\sqrt{n}\lambda_m-M}=\infty \mid S_0=y).$$

By Lemma 5.6, for all $\varepsilon>0$ there exists a positive integer, n_0, such that for all $n\geqq n_0$ and for all $y\in\mathbb{Z}^d$ with $|y|\geqq(\lambda_m\sqrt{n}+2M)\eta$,

$$(5.11)\quad \mathbb{P}(\tau_{\sqrt{n}\lambda_m-M}=\infty \mid S_0=y) \leqq (1+\varepsilon)\frac{|y|^{d-2}-|\sqrt{n}\lambda_m|^{d-2}}{|y|^{d-2}},$$

$$(5.12)\quad \mathbb{P}(\tau_{\sqrt{n}\lambda_m+M}=\infty \mid S_0=y) \geqq (1-\varepsilon)\frac{|y|^{d-2}-|\sqrt{n}\lambda_m|^{d-2}}{|y|^{d-2}}.$$

By picking n_0 even larger, we can guarantee that for all $n\geqq n_0$

$$(5.11\text{c})\quad \#\{y\in\mathbb{Z}^d\colon \sqrt{n}\lambda_m \leqq |y| < (\sqrt{n}\lambda_m+2M)\eta\} \leqq c_{5.7}\, n^{d/2}(\eta-1)^d$$

for some absolute constant $c_{5.7}=c_{5.7}(d,\lambda_m)$. Define

$$H_n \overset{\text{df}}{=} \{\omega\colon J_{[nt_i]}^{[nt_i+1]}(\omega) \geqq \sqrt{n}\lambda_i \text{ for all } i\leqq m-1\}$$

$$K_n \overset{\text{df}}{=} \{\omega\colon S_{[nt_m]}(\omega) \geqq \sqrt{n}\lambda_m\}.$$

With this notation in mind, (5.9), (5.10) and (5.11c) imply that for all $n\geqq n_0$

$$(5.14)\quad \mathbb{P}(J_{[nt_i]} \geqq \sqrt{n}\lambda_i \text{ for all } i\leqq m)$$
$$\leqq (1+\varepsilon) \sum_{\substack{y\in\mathbb{Z}^d \\ |y|\geqq(\sqrt{n}\lambda_m+2M)\eta}} (1-|y/\sqrt{n}|^{2-d}\lambda_m^{d-2})$$
$$\cdot\, \mathbb{P}(H_n \mid S_{[nt_m]}=y)\, \mathbb{P}(S_{[nt_m]}=y)$$
$$+(1+\varepsilon)\lambda_m^{d-2} \sum_{\substack{y\in\mathbb{Z}^d \\ \sqrt{n}\lambda_m\leqq|y|<(\sqrt{n}\lambda_m+2M)\eta}} \mathbb{P}(S_{[nt_m]}=y)$$
$$\leqq (1+\varepsilon)\,\mathbb{E}((1-\lambda_m^{d-2}|S_{[nt_m]}/\sqrt{n}|^{2-d})\,1_{H_n\cap K_n})$$
$$+c_{5.7}(1+\varepsilon)\lambda_m^{d-2} \sum_{\substack{y\in\mathbb{Z}^d \\ \sqrt{n}\lambda_m\leqq|y|<(\sqrt{n}\lambda_m+2M)\eta}} n^{-d/2}$$
$$\leqq (1+\varepsilon)\,\mathbb{E}((1-\lambda_m^{d-2}|S_{[nt_m]}/\sqrt{n}|^{2-d})\,1_{H_n\cap K_n})$$
$$+(1+\varepsilon)\,c_{5.8}\,\lambda_m^{d-2}(\eta-1)^d.$$

Here $c_{5.8} = c_{5.8}(d, M)$ is a universal constant. Therefore, since $\eta > 1$ and $\varepsilon > 0$ are arbitrary,

$$\limsup_{n\to\infty} \mathbb{P}(J_{[nt_i]} \geqq \sqrt{n}\lambda_i \text{ for all } i \leqq m)$$
$$\leqq \limsup_{n\to\infty} \mathbb{E}((1 - \lambda_m^{d-2} |S_{[nt_m]}/\sqrt{n}|^{2-d})\, 1_{H_n \cap K_n}).$$

By approximating

$$1_{H_n \cap K_n} \equiv 1_{\{J_{[nt_i]}^{[nt_{i+1}]} \geqq \sqrt{n}\lambda_i \text{ for all } i \leqq m-1\}}\, 1_{\{S_{[nt_m]} \geqq \sqrt{n}\lambda_m\}}$$

by continuous functionals of $\{n^{-1/2} S_{[nt_i]};\ i \leqq m\}$ in the usual way, Donsker's invariance principle can be used to show that

$$\text{(5.15)} \qquad \limsup_{n\to\infty} \mathbb{P}(J_{[nt_i]} \geqq \sqrt{n}\lambda_i \text{ for all } i \leqq m)$$
$$\leqq \mathbb{E}((1 - |X_{t_m}/\lambda_m|^{2-d})\, 1_{\{I_{t_i}^{t_{i+1}} \geqq \lambda_i \text{ for all } i \leqq m-1\}}\, 1_{\{|X_{t_m}| \geqq \lambda_m\}})$$
$$= \mathbb{E}(1_{\{I_{t_i}^{t_{i+1}} \geqq \lambda_i \text{ for all } i \leqq m-1\}}\, 1_{\{|X_{t_m}| \geqq \lambda_m\}}\, \mathbb{P}(I_{t_m} \geqq \lambda_m | X_{t_m}))$$
$$= \mathbb{P}(I_{t_i}^{t_{i+1}} \geqq \lambda_i \text{ for all } i \leqq m-1,\ I_{t_m} \geqq \lambda_m)$$
$$= \mathbb{P}(I_{t_i} \geqq \lambda_i \text{ for all } i \leqq m),$$

where (5.15) follows from the Markov property of (X_t) and the gambler's ruin problem for Bessel processes. Likewise one gets a lower bound on the lim inf by using (5.12). This proves (5.8) and, hence, Theorem 3.1. □

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