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Gaussian Samples Approach "Smooth Points" Slowest

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Let K be the unit ball of the Hilbert space which generates a Gaussian measure μ on the real separable Banach space B. Some recent results establish that suitable normalized μ-Gaussian samples approach the smoothest points on the "boundary of K" slowest. As a partial explanation of this phenomenon we show that these points contain that portion of the boundary closest to points outside K. We also examine what happens if K is replaced by an arbitrary compact convex C in B, and attempt to characterize the set on the "boundary of C" which is closest to points outside C. Another result shows how this phenomenon characterizes B as a reflexive Banach space, and we also include some examples of interest.

1. INTRODUCTION

Let μ be a centered Gaussian measure on a real separable Banach space B with norm $\|\cdot\|$, and dual space B^*. If K is the unit ball of the Hilbert space H_μ which generates μ, and $X, X_1, X_2, \ldots$ are i.i.d. Gaussian random vectors with law μ, then it is well known that with probability one the sequence $\{X_n/(2\,\mathrm{Ln})^{1/2}\}$ converges to and clusters throughout K in the B-norm. Furthermore, in [KLT] we have shown, aside from some minor technical assumptions on $\psi(\varepsilon) = -\log P(\|X\| \leqslant \varepsilon)$, that the points in the set

$$E = \{f \in B\colon \|f\|_\mu = 1, f = Sh, h \in B^*\} \tag{1.1}$$

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are approached slowest by the sequence $\{X_n/(2\,\mathrm{Ln})^{1/2}\}$. Here $f = Sh$ is given by the Bochner integral

$$Sh = \int_B xh(x)\,d\mu(x) \qquad (h \in B^*) \tag{1.2}$$

and $\|\cdot\|_\mu$ is the Hilbert norm on H_μ with $\|f\|_\mu^2 = \langle f, f\rangle_\mu = \int_B h^2(x)\,d\mu(x)$ when $f = Sh$, $h \in B^*$. Hence E is a subset of the H_μ boundary of the unit ball K, and if dim $H_\mu < \infty$, then it is the entire boundary so the result is to be expected. However, if dim $H_\mu = \infty$, then E is only H_μ dense in the H_μ-boundary of K, but so is the set $F = \{f\colon \|f\|_\mu = 1, f \notin SB^*\}$. Furthermore, if dim $H_\mu = \infty$, then the points of E are in some sense the smoothest points on the boundary of K since $f = Sh$ implies the μ-inner product $\langle x, f\rangle_\mu = h(x)$, and hence $\langle x, f\rangle_\mu$ extends to all of B via the continuity of h; see, for example, Lemma 2.1 in [K] for this and further details on the construction of K and the relationship between H_μ and B. Thus one would perhaps expect these points to be approximated the fastest among the points on the boundary of K. When dim $H_\mu = \infty$ this is not the case, and one possible explanation is that the typical sample element simply does not match the "smooth" elements in E very well. For example, think of the irregular sample paths of Brownian motion, and recall from [KLT] that for μ, Wiener measure, we have shown that E consists precisely of those functions f on the surface of K such that $f'(s)$ has a version of bounded variation on $[0, 1]$. Beyond the analytic considerations in Propositions 1 and 2 of [KLT], we have not found a way to demonstrate this possibility, but there is another more geometric explanation, and this is what we turn to now.

Our other explanation is that when dim $H_\mu = \infty$, the H_μ-boundary of K has layers, and the points of E are those which are furthest from zero. That K has layers of some sort can be seen from the comparison results in [KLT], and the next theorem explains why we say E is the furthest from zero. In particular, Theorem 1 says that, looking from the outside of K, the points in E are closest (in the B-norm), and since $0 \in K$ we interpret this as E being in the layer furthest from zero.

THEOREM 1. *If $y \in \bar{H}_\mu$, $y \notin K$, then there exists a unique point p in K such that*

$$\|y - p\| = \inf_{x \in K} \|y - x\| \tag{1.3}$$

and $p \in E$; i.e., $\|p\|_\mu = 1$ and $p = Sh$ for some $h \in B^*$.

Remark. It is easy to construct examples when Theorem 1 fails if $y \notin \bar{H}_\mu$. However, the support of μ is $\bar{H}_\mu$ and hence with probability one the sequence $\{X_n/(2\,\mathrm{Ln})^{1/2}\}$ is in $\bar{H}_\mu$. Thus from the point of view of this problem $\bar{H}_\mu$ is the set of importance.

Remark. Whether the interpretation of Theorem 1 given above is the entire reason for E being approached slowest is unclear to us, but the geometric content of this result is interesting in its own right. For example, if $\dim H_\mu = \infty$, $y \in H_\mu$, and $y \notin (SB^* \cap K)$, then Theorem 1 says the point in K closest to y is not the point $y/\|y\|_\mu$, but some point on the boundary of K in SB^*. Furthermore, the dense set of points on the boundary of K which are not in SB^* are never the closest point for any point outside K. If B is reflexive, then the results in Section 4 show that E, as defined in (1.1), is precisely the set of points in K which are closest to points in $\bar{H}_\mu \cap K^c$. In fact, there also is a converse result, which states that whenever E is exactly the set of points in K closest to points in $\bar{H}_\mu \cap K^c$, then $\bar{H}_\mu$ is reflexive. Hence if $\bar{H}_\mu = B$, then B is reflexive.

2. Proof of Theorem 1

If $\dim H_\mu < \infty$, then $H_\mu = \bar{H}_\mu = SB^*$ and E is the entire boundary of K, so there is nothing to prove. Hence assume $\dim \bar{H}_\mu = \infty$ for the remainder of the proof.

Take $y \in \bar{H}_\mu$, $y \notin K$. Then it is well known that K is compact in B, and since the function $f(x) = \|x - y\|$ is positive and continuous on K there exists a point $p \in K$ such that (1.3) holds. The next lemma identifies p in a useful way and shows p is unique. To state this lemma we define

$$I(x) = \begin{cases} \|x\|_\mu^2 & \text{if } x \in H_\mu \\ +\infty & \text{if } x \notin H_\mu. \end{cases} \tag{2.1}$$

Lemma 1. *If* $y \in \bar{H}_\mu$, $y \notin K$,

$$r = \inf_{x \in K} \|y - x\|,$$

and $V = \{x\colon \|x - y\| < r\}$, *then there exists a unique point p on the boundary of V such that*

$$I(p) = \inf_{x \in V} I(x) = \inf_{x \in \bar{V}} I(x) = 1. \tag{2.2}$$

Furthermore, p is the unique point in K closest to y, i.e., $\|y - p\| = r$.

Proof. First we show that there is a unique point on the boundary of V such that (2.2) holds. Since $V \cap \bar{H}_\mu \neq \varnothing$ and V is open, we have $V \cap H_\mu \neq \emptyset$. Thus $\inf_{x \in V} I(x) < \infty$, so we choose $\{x_j\} \subseteq V$ such that

$$I(x_j) \leqslant \inf_{x \in V} I(x) + 1/j. \tag{2.3}$$

Then, by convexity of V, $(x_i + x_j)/2 \in V$, and hence

$$I((x_i + x_j)/2) \geqslant \inf_{x \in V} I(x). \tag{2.4}$$

Since

$$I(x_i - x_j) = 2I(x_i) + 2I(x_j) - I(x_i + x_j), \tag{2.5}$$

we have from (2.3), (2.4), and (2.5) that $\{x_j\}$ is Cauchy in H_μ. Thus $\{x_j\}$ converges, say to p, with $p \in H_\mu \cap \bar{V}$ since H_μ-convergence implies B-convergence. Hence, $I(p) = \inf_{x \in V} I(x)$ and p must be unique or $\{x_j\}$ would need to converge to two points. Since V is open it is trivial p cannot be in V, and hence $p \in \partial V$. Hence it remains to show that $I(p) = \inf_{x \in \bar{V}} I(x)$. To see this let $a \in (\partial V) \cap H_\mu$, $d \in V \cap H_\mu$ and let $L(a, d) = \{ta + (1-t)\, d\colon 0 < t \leqslant 1\}$. Then $L(a, d) \subseteq V \cap H_\mu$ and since $I(x)$ is convex on H_μ we have $\inf_{x \in L(a,d)} I(x) \leqslant \min(I(a), I(d))$. Thus $\inf_{a \in \partial V} I(a) \geqslant \inf_{x \in V} I(x)$, which implies $\inf_{x \in \bar{V}} I(x) \geqslant \inf_{x \in V} I(x)$. Hence (2.2) is proved since $\inf_{x \in \bar{V}} I(x) \leqslant 1$ and

$$\inf_{x \in V} I(x) \geqslant 1. \tag{2.6}$$

To see $\inf_{x \in \bar{V}} I(x) \leqslant 1$ follows easily since by the compactness of K there exists a point $b \in K \cap \bar{V}$ such that

$$\|y - b\| = r.$$

Thus $\inf_{x \in \bar{V}} I(x) \leqslant I(b) = 1$. That (2.6) holds follows since $V \cap K = \emptyset$.

Finally, to check that p is the unique point in K closest to y, suppose there exists $p_1 \neq p$ in K such that $\|y - p_1\| = r$. Then p_1 is on the boundary of V and $I(p_1) \leqslant 1$ (it is in K), and hence this contradicts the uniqueness of p, satisfying (2.2). Thus the lemma is proved.

To complete the proof of Theorem 1 we now apply Lemma 1 and the Hahn–Banach separation theorem in the form presented on page 64 of [S].

Hence take $y \in \bar{H}_\mu$, $y \notin K$, and let r be as in Lemma 1. If $A = \bar{V}$, then $\text{Int}\, A = V$ and $K \cap (\text{Int}\, A) = \emptyset$, so by [S] there exists $h \in B^*$, $h \neq 0$ on $\bar{H}_\mu$ such that

$$\inf_{x \in A} h(x) \geqslant \sup_{x \in K} h(x). \tag{2.7}$$

Since $h \neq 0$ on $\bar{H}_\mu$ we have $\sup_{x \in K} h(x) > 0$. Hence there exists $\alpha > 0$ such that if $h_0 = \alpha h$ we have $\int_B h_0^2(x)\, d\mu(x) = 1$. Thus $\|Sh_0\|_\mu = 1$ and by Cauchy–Schwarz,

$$h_0(p) = \langle Sh_0, p \rangle_\mu \leqslant \|Sh_0\|_\mu \, \|p\|_\mu = 1. \tag{2.8}$$

On the other hand, $\|Sh_0\|_\mu = 1$ implies $\sup_{x \in K} h_0(x) = 1$ and hence (2.7) applied to $h_0 = \alpha h$ with $\alpha > 0$ implies

$$\inf_{x \in A} h_0(x) \geqslant 1. \tag{2.9}$$

In particular, since $p \in \partial V \subseteq A$ from the proof of Lemma 1, we thus have

$$h_0(p) \geqslant 1. \tag{2.10}$$

Thus (2.8) and (2.10) imply $h_0(p) = 1$.

Now let $g = Sh_0$. Then $\|g\|_\mu = 1$ and $h_0(g) = 1$ as well. Thus $g \in K$ and we claim $g = p$ to complete the proof of the theorem.

If $g \neq p$, then we have two points in K with $\|g\|_\mu = \|p\|_\mu = 1$ and $h_0(g) = h_0(p) = 1$. Take $x_0 = \alpha(g + p)$ such that $\|x_0\|_\mu^2 = 1$. That is,

$$\begin{aligned} \|x_0\|_\mu^2 &= \alpha^2(\|g\|_\mu^2 + 2\langle g, p \rangle_\mu + \|p\|_\mu^2) \\ &= 2\alpha^2(1 + \langle g, p \rangle_\mu). \end{aligned} \tag{2.11}$$

Now $h_0(g) = h_0(p) = 1$ and $p \neq g$ implies $\langle g, p \rangle_\mu < \|g\|_\mu \, \|p\|_\mu = 1$, so (2.11) implies

$$\|x_0\|_\mu^2 < 4\alpha^2.$$

Thus $\|x_0\|_\mu^2 = 1$ implies $2\alpha > 1$. Thus $x_0 \in K$ and

$$h_0(x_0) = \alpha h_0(g + p) = 2\alpha > 1,$$

which contracts the fact that

$$\sup_{x \in K} h_0(x) = 1.$$

Thus $g = p$ and the theorem is proved.

3. A Relationship with $I(y, \delta)$

If $y \in \bar{H}_\mu$ and $\delta > 0$, then we define

$$I(y, \delta) = \inf_{\|x - y\| \leqslant \delta} I(x).$$

From [G], or as in the proof of Lemma 1 above, it is easy to check that for each $y \in \bar{H}_\mu$, $\delta > 0$, there is a unique point $y_\delta \in H_\mu$ such that

$$I(y, \delta) = \|y_\delta\|_\mu^2. \tag{3.1}$$

Hence the mapping $y \to y_\delta$ is well defined, and using the ideas in the proof of Theorem 1 it follows that y_δ is actually in SB^*, which is a proper subset of H_μ when $\dim H_\mu = \infty$.

4. Closest Points for General Compact, Convex, Symmetric Sets

Let $B = H$ be an infinite-dimensional Hilbert space, and assume X is an infinite-dimensional, centered, Gaussian vector with values in H. Then $X = \sum_{n \geq 1} \lambda_n g_n e_n$ where $\{e_n : n \geq 1\}$ is orthonormal in H, $\{g_n : n \geq 1\}$ is a sequence of independent $N(0, 1)$ random variables, and $\lambda_n > 0$ satisfies $\sum_{n \geq 1} \lambda_n^2 < \infty$. Then

$$K = \left\{ x = \sum_{n \geq 1} x_n e_n : \sum_{n \geq 1} x_n^2 / \lambda_n^2 \leq 1 \right\}, \tag{4.1}$$

and an easy computation shows that E, as given in (1.1), is the proper subset of the H_μ-boundary of K given by

$$E = \left\{ f = \sum_{n \geq 1} f_n e_n : \sum_{n \geq 1} f_n^2 / \lambda_n^2 = 1, \sum_{n \geq 1} f_n^2 / \lambda_n^4 < \infty \right\}. \tag{4.2}$$

In particular, Theorem 1 applies, and all points of K, closest to points outside K, are contained in E.

In what follows we see that when $B = H$, every point of E is the closest point of K to some point outside K. Hence the subset E of K is precisely the set of points which are closest to points outside K.

When these Hilbert space examples were shown to Professor Walter Rudin, he pointed out in [R] that if K is replaced by the Hilbert cube

$$C = \left\{ x = \sum_{n \geq 1} x_n e_n : |x_n| \leq \frac{1}{n} \right\} \tag{4.3}$$

and

$$\Sigma = \{ x \in C : |x_n| = 1/n \text{ for at least one } n \}, \tag{4.4}$$

then every point in Σ is the nearest point in C for somepoint outside C (see Rudin's observation below). Since Σ can be viewed as the "entire boundary of C," it seemed to be of interest to try to understand the differences in these examples and to characterize the set on the "boundary of C" which is closest to points outside C.

Hence let C be a compact, convex, symmetric set in a real separable Banach space B, and let $F = \bigcup_{n \geqslant 1} nC$. Then F is a Banach space in the norm

$$\lambda(x) = \inf\{t > 0 \colon x \in tC\}, \tag{4.5}$$

and B infinite-dimensional implies F is a proper subset of B. Also, for some $\alpha \in (0, \infty)$ we have $\|x\| \leqslant \alpha\lambda(x)$ for all $x \in B$, and we write $\bar{F}$ to denote the B-closure of F in B. Hence if C is as in (4.3), then the F-boundary of C is Σ, and for $C = K$ the F-boundary is $\{x = \sum_{n \geqslant 1} x_n e_n \colon \sum_{n \geqslant 1} x_n^2/\lambda_n^2 = 1\}$. Our next theorem is an easy result which shows how to identify that portion of the F-boundary of C which contains the points of C closest to points outside C.

For its statement, as well as subsequent results, we need some definitions. We say a Banach space B is strictly convex if every point on the boundary of the unit ball U of B is an extreme point; i.e., if $\|p\| = 1$, $p = tx + (1-t)y$ for $0 < t < 1$, and $x \neq y$, then either $x \notin U$ or $y \notin U$. If C is a compact, symmetric, convex subset of B we say C is strictly convex whenever C, as the unit ball in (F, λ), is strictly convex in the sense indicated previously. A point p is a strongly exposed point of C if there is an $f \in B^*$ such that $f(p) = 1$ and $f(x) < 1$ for $x \in C$, $x \neq p$, and $\sup\{\|x - y\| \colon x, y \in C \cap \{x \colon f(x) > 1 - \delta\}\} \to 0$ as $\delta \downarrow 0$.

THEOREM 2. *Let C be a compact, convex, symmetric subset of B, a real separable Banach space. Let $y \in \bar{F}$, $y \notin C$. Then there exists a point $p \in C$ such that*

$$\inf_{x \in C} \|y - x\| = \|y - p\|. \tag{4.6}$$

If $\partial_\lambda C$ denotes the F-boundary of C, $y \in \bar{F}$, and

$$E = \{q \in \partial_\lambda C \colon \exists f \in B^* \text{ such that } f(q) = 1, \sup_{x \in C} f(x) \leqslant 1\}, \tag{4.7}$$

then for any p satisfying (4.6) *we have $p \in E$. If C is strictly convex, then E consists of the strongly exposed points of C and p satisfying* (4.6) *is unique.*

Remark. It is easy to see from the proof of Theorem 1 that the sets E defined via (1.1) and (4.7) are the same in the setting of Theorem 1. Of course, the definition in (4.7) is general, and has nothing to do with Gaussian measures.

Proof of Theorem 2. Since C is compact and $d(x) = \|y - x\|$ is continuous on C, there exists a $p \in C$ such that $d(p) = \inf_{x \in C} \|y - x\|$. Since $y \in \bar{F}$, it follows that p must be in the F boundary of C. Of course, if $y \notin \bar{F}$, it may be that $p = 0$.

To see $p \in \partial_\lambda C$, the F-boundary of C, let

$$r = d(p) = \inf_{x \in C} \|y - x\|,$$

and set $V = \{q \in B: \|q - y\| < r\}$. Then V is open, $V \cap C = \emptyset$, so by the Hahn–Banach theorem in the form on page 64 of [S] we have $f \in B^*$ such that

$$\sup_{x \in C} f(x) = 1 \leqslant \inf_{x \in V} f(x).$$

Now $p \in C \cap \bar{V}$, and hence $f(p) = 1$. Thus $p \in E$ provided p is in the F-boundary of C. If $p \notin \partial_\lambda C$, then $\lambda(p) < 1$, and set

$$L = \{x: x = ty + (1-t)p, 0 \leqslant t \leqslant 1\}.$$

Let $t_0 = \sup\{t \geqslant 0: \lambda(ty + (1-t)p) \leqslant 1\}$. Then $\lambda(p) < 1$ implies $t_0 > 0$, so set $L_0 = \{x = ty + (1-t)p,\ 0 \leqslant t \leqslant t_0\}$. Then $L_0 \subseteq C$ and hence

$$d(p) = \|p - y\| = \inf_{x \in C} d(x) \leqslant \inf_{x \in L_0} d(x). \tag{4.8}$$

But

$$\begin{aligned} \inf_{x \in L_0} d(x) &= \inf_{0 \leqslant t \leqslant t_0} \|ty + (1-t)p - y\| \\ &= (1 - t_0)\|p - y\| < \|p - y\|, \end{aligned} \tag{4.9}$$

which is a contradiction to (4.8). Thus $p \in \partial_\lambda C$ as claimed, and the first part of the theorem is proved.

To prove the last assertion of the theorem it is easy to see that if p satisfies (4.6), and C is strictly convex as a subset of (F, λ), then p is unique. That is, if p is not unique, then there exists p_1, p_2 satisfying (4.6) for a given $y \in \bar{F} \cap C^c$. Hence p_1, p_2 are both in $C \cap \bar{V}$ where V is as above. By convexity, $\Gamma = \{x = tp_1 + (1-t)p_2:\ 0 \leqslant t \leqslant 1)$ is also in $C \cap \bar{V}$, and $\Gamma \subseteq \partial_\lambda C$ by the argument used previously. Hence $p_1 \neq p_2$ implies every point in the boundary $\partial_\lambda C$ is not an extreme point. This contradicts the strict convexity of C, so $p_1 = p_2$ and we have uniqueness. To see $p \in E$ is a strongly exposed point of C, suppose $f \in B^*$, $f(p) = 1$, and $\sup_{x \in C} f(x) = 1$. If $q \in C$, $q \neq p$, and $f(q) = 1$, then the line $\Gamma = \{x = tq + (1-t)p: 0 \leqslant t \leqslant 1\}$ $\subseteq C$ and $f(\gamma) = 1$ for all $\gamma \in \Gamma_0$. Then $\Gamma \subseteq \partial_\lambda C$ since $\sup_{x \in C} f(x) = 1$ and f is continuous on (F, λ) with $C =$ unit ball of (F, λ). Hence we again have a contradiction of the strict convexity of C, so $f(p) = 1$, and $f(x) < 1$ for all $x \in C$, $x \neq p$. Thus p is an exposed point of C and it remains to show

$$\lim_{\delta \downarrow 0} \sup\{\|x - y\|: x, y \in C \cap \{x: f(x) > 1 - \delta\}\} = 0. \tag{4.10}$$

To verify (4.10) assume there exist points $\{x_n\}$ such that $x_n \in C \cap \{x: f(x) > 1 - (1/n)\}$ and $\|x_n - p\| \geq \delta_0$ for some $\delta_0 > 0$ (recall $f(p) = 1$ and $p \in C$). Now C compact implies there is a subsequence $\{x_{n_k}\}$ such that

$$\lim_k x_{n_k} = q$$

in B, and $q \in C$. Thus $f(q) = \lim_k f(x_{n_k}) = 1$ and $\|q - p\| \geq \delta_0$. This contradicts $f(x) < 1$ for all $x \in C$, $x \neq p$. Hence $\delta_0 = 0$ and p is strongly exposed by f. Thus Theorem 2 is proved since E clearly contains the strongly exposed points of C.

Our next result establishes that E is precisely the set of points of $\bar{F}$ closest to points outside C whenever B is reflexive. We also include some converse results.

THEOREM 3. *Let B denote a real separable Banach space and assume C is a compact convex symmetric subset of B with (F, λ) the Banach space described above. Let $\partial_\lambda C$ denote the F-boundary of C, assume E is given by* (4.7), *and define*

$$\tilde{E} = \{p \in \partial_\lambda C : \exists v \in \bar{F} \cap C^c \text{ such that } \|v - p\| = d(v, C)\}, \tag{4.11}$$

where

$$d(v, C) = \inf_{x \in C} \{\|v - x\|\}. \tag{4.12}$$

Then the following hold:

(1) *If B is reflexive, then $E = \tilde{E}$. Furthermore, if B is strictly convex, $v \in \bar{F} \cap C^c$, and $\|v - p\| = d(v, C)$ for some $p \in \partial_\lambda C$, then p will be the unique point in C closest to v.*

(2) *If $E = \tilde{E}$ for all compact, convex, symmetric C in B, then B is reflexive.*

(3) *If $E = \tilde{E}$ for all $C = K$, K the unit ball of H_μ, and μ an arbitrary centered Gaussian measure on B, then B is reflexive.*

(4) *If $E = \tilde{E}$ for $C = K$, K the unit ball of H_μ, and μ a centered Gaussian measure on B such that $\overline{H_\mu} = B$, then B is reflexive.*

Remarks. (a) Since the unit ball K of H_μ is strictly convex, Theorem 2 and the results of [KLT] imply that normalized i.i.d. samples from a Gaussian measure on B of the form $\{X_n/(2\,Ln)^{1/2}: n > 1\}$ converge slowest to strongly exposed points of the boundary of K. If B is reflexive, Theorem 3, part 1, then shows that the strongly exposed points of K are precisely those points on the H_μ-boundary of K closest to points in $\overline{H_\mu} \cap K^c$. Furthermore, Theorem 3, part 4, says that if $E = \tilde{E}$ and $\overline{H_\mu} = B$, then B is reflexive.

(b) The proof of (4) in Theorem 3 shows $\overline{H_\mu}$ is reflexive when $\overline{H_\mu}$ is a proper subspace of B.

Proof of Theorem 3. To simplify notation we assume throughout that $\bar{F} = B$. If $\bar{F}$ is a proper closed subspace of B, simply replace B by $\bar{F}$, and the proof goes as we indicate since closed subspaces of reflexive spaces are reflexive. Hence we lose no generality in this assumption.

If $f \in B^*$, define $M_t(f) = \{x : f(x) = t\}$. If f is understood, we simply write M_t. Then the following lemma holds.

LEMMA 2. *Let B be reflexive, $f \in B^*$, and for $p \in M_1 = M_1(f)$, define*

$$d(p, M_0) = \inf_{m \in M_0} \|p - m\|.$$

Then there exists $m_0 \in M_0$ such that

$$d(p, M_0) = \|p - m_0\|, \tag{4.13}$$

and

$$\|p - m_0\| = 1/\|f\|_*, \tag{4.14}$$

where $\|f\|_* = \sup_{\|x\| \leqslant 1} |f(x)|$.

Proof of Lemma 2 Since B is reflexive, [C, p. 136] implies there is a point $m_0 \in M_0$ such that (4.13) holds. Since $p \in M_1$, $m_0 \in M_0$, we have

$$1 = f(p - m_0) \leqslant \|f\|_* \|p - m_0\|,$$

and hence

$$\|p - m_0\| \geqslant 1/\|f\|_*. \tag{4.15}$$

On the other hand, if $\|p - m_0\| = r$, then the open ball $W = \{x \in B: \|x\| < r\}$ does not intersect M_1; i.e.,

$$\begin{aligned} d(0, M_1) &= \inf \{\|x\| : x \in M_1\} \\ &= \inf \{\|p - m\| : m \in M_0\} \\ &= \|p - m_0\|. \end{aligned}$$

However, if $r > 1/\|f\|_*$, then $\sup_{x \in W} f(x) = r\|f\|_* > 1$, so convexity gives us a contradiction to $W \cap M_1 = \varnothing$. Thus $\|p - m_0\| = r \leqslant 1/\|f\|_*$, and combining this with (4.15) we get (4.14). Hence Lemma 2 is proved.

To complete the proof of Theorem 3, part 1, we now take $p \in E$ and $f \in B^*$ such that $f(p) = 1$ and $\sup_{x \in C} f(x) \leqslant 1$. Let

$$v = p + (p - m_0) \tag{4.16}$$

where m_0 is from Lemma 2 above. Then $v \notin C$ since $f(v)=2$, and we also have

$$\inf_{x \in M_1} \|v-x\| = \inf_{m \in M_0} \|v-(p+m)\|$$
$$= \inf_{m \in M_0} \|p-m_0-m\|$$
$$= d(p, M_0)$$
$$= \|p-m_0\|.$$

Thus if $V=\{x\colon \|x-v\|<\|p-m_0\|\}$, it follows that $V \cap M_1 = \varnothing$, and hence since $\sup_{x \in C} f(x) \leqslant 1$ we have $V \cap C = \varnothing$. Now $p \in \bar{V} \cap C$ since $\|p-v\|=\|p-m_0\|$ by (4.16), and $p \in C$ by assumption. Hence v is a point outside C and p is a point of C closest to v; i.e., $\inf_{x \in C} \|v-x\|=\|v-p\|$. Thus $E=\tilde{E}$.

To finish the proof we must show that if B is strictly convex, and v is an arbitrary point in $\bar{F} \cap C^c$ with $p \in C$ such that

$$\|v-p\| = \inf_{x \in C} \|v-x\|,$$

then p is uniqiue. From Theorem 2, $p \in E$, and as in the proof of Theorem 2, there exists $f \in B^*$ such that

$$f(p)=1, \quad \sup_{x \in C} f(x) \leqslant 1, \qquad f(v)>1.$$

To see p is unique assume $x \in C$ and $\|x-v\|=\|p-v\|=r$. Then

$$f(x)=f(v)+f(x-v)$$
$$=f(v)-\|f\|_* \|x-v\|$$
$$=f(v)-\|f\|_* r.$$

Now $v-p=(f(v)-1)(p-m_0)$ from the argument used in Lemma 2 where $p-m_0=\inf\{\|p-m\|\colon m \in M_0\}$. Thus from Lemma 2

$$\|v-p\|=(f(v)-1)\|p-m_0\|=(f(v)-1)/\|f\|_*,$$

and from the above we obtain

$$f(x) \geqslant 1.$$

Thus $x \in C \cap M_1$, so both x and p are in $C \cap M_1$ and in $\bar{V}$ as well. Thus $L(x,p)=\{y=tx+(1-t)p\colon 0 \leqslant t \leqslant 1\}$ is such that $L(x,p) \cap (C \cap M_1) = L(x,p)$ and $L(x,p) \subseteq \bar{V}$. However,

$$L(x, p) \cap V = \varnothing$$

since $L(x, p) \subseteq C$. If $x \neq p$, then every interior point of the line segment $L(x, p)$ is a boundary point of V which is not an extreme point.

Since V is isometric to $U/\|f\|_*$, where U is the unit ball of B, the strict convexity of B implies $x = p$. Hence p is the unique point of C closest to v, and Theorem 3, part 1, is proved.

To finish the proof of Theorem 3, it suffices to prove part (4), since parts (2) and (3) then follow immediately.

Let $X = \sum_{k \geqslant 1} g_k x_k$ where $\{g_k : k \geqslant 1\}$ are i.i.d. $N(0, 1)$ random variables, $x_k = y_k/k^2$, $k \geqslant 1$, and $\{y_k : k \geqslant 1\}$ is a dense subset of the unit ball of B. Then $\mu = \mathscr{L}(X)$ is a centered Gaussian measure, and defining H_μ as above, it is easy to check that $\overline{H_\mu} = B$.

Let $C = K$, the unit ball of H_μ. Then C is compact, convex, and symmetric in B, $F = \bigcup_{n \geqslant 1} nC = H_\mu$, and $\lambda = \|\cdot\|_\mu$.

Since C is strictly convex in F, it is by definition strictly convex in B, and hence by Theorem 2

$$E = \{p \in \partial_\lambda C : \exists f \in B^* \text{ such that } f(p) = 1 \text{ and } f(x) < 1 \ \forall x \in C,\ x \neq p\}.$$

If E is not reflexive, then there exists $g \in B^*$ such that g does not attain its supremum on the unit ball of B, see [J, p. 167]. Hence $g \neq 0$ on C (as F is dense in B), so take $p \in C$ such that $g(p) > g(x)$ for all $x \in C$, $x \neq p$. A unique point p exists as indicated since $C = K$ is strictly convex. Then $g(p) > 0$ as C is symmetric and we let $f = g(\cdot)/g(p)$. Then $p \in \partial_\lambda C$, $f(p) = 1$, and $f(x) < 1$ for all $x \in C$, $x \neq p$, so $p \in E$. Since $E = \tilde{E}$, there exists $v \notin C$ such that

$$\|p - v\| = d(v, C).$$

Hence let

$$V = \{x : \|x - v\| < \|p - v\|\}.$$

Then

$$V \cap C = \varnothing \qquad \text{and} \qquad C \cap \bar{V} = \{p\}$$

by Theorem 2 and the strict convexity of C. Hence by the Hahn–Banach separation theorem in the form presented on page 64 of [5], there exists $h \in B^*$ such that

$$h(p) = 1 \qquad \text{and} \qquad \sup_{x \in C} h(x) \leqslant \inf_{x \in V} h(x).$$

Thus both f and h are support functionals for C at p. Since both, when restricted to H_μ, are also continuous on H_μ in the norm $\|\cdot\|_\mu$, they are also support functionals for $C = K$ at p in the Hilbert space

$(H_\mu, \|\cdot\|_\mu)$. Hence $f=h$ on H_μ, as the unit ball of a Hilbert space is smooth [B, pp. 177–179], and since $\overline{H_\mu}=B$ we thus have $f=h$ on B. Hence $d(v, M_1(f)) = \|v-p\|$, and if $f(v)=r$ we have as in the argument of Lemma 2 that

$$\begin{aligned} d(v, M_1(f)) &= \inf_{m \in M_1(f)} \|v-m\| \\ &= \inf_{m \in M_0(f)} \|v-p-m\| \\ &= d(0, M_{r-1}(f)) \\ &= (r-1)/\|f\|_*. \end{aligned}$$

Thus $\|v-p\| = (r-1)/\|f\|_*$ and setting $w=(v-p)/\|v-p\|$ we have $\|w\|=1$ and

$$f(w) = \|f\|_*.$$

This is a contradiction to our choice of f, hence B must be reflexive, and Theorem 3 is proved.

Rudin's Observation for the Hilbert Cube. If

$$p = \sum_{m \geqslant 1} p_m e_m \in \Sigma,$$

as in (4.3) and (4.4), then $|p_m| = 1/m_0$ for some m_0. Define

$$v = p + (e_{m_0}/m_0)\operatorname{sgn}(p_{m_0}).$$

Then $v \notin C$ and $\|v-p\|_2 = |1/m_0| = \inf_{x \in C} \|x-v\|_2$ where $\|\cdot\|_2$ denotes the inner product norm of H, so we easily see that the entire F boundary of C denotes the points closest to points outside C.

5. A Final Example

Let $B=\ell^2$, $x=\sum_{n\geqslant 1} x_n e_n$, where $\{e_n: n \geqslant 1\}$ is the canonical basis for ℓ^2. Let

$$C = \left\{x \in \ell^2 : \sum_{n \geqslant 1} |x_n/\lambda_n|^r \leqslant 1\right\}, \tag{5.1}$$

where $1 \leqslant r < \infty$ and $\lambda_n > 0$ is such that $\lim_n \lambda_n = 0$. Then C is something we call an ℓ^r-ellipsoid. When $r=2$, C is of the form which describes the sets K related to Gaussian measures, but otherwise C is not such a set. If $1 \leqslant r \leqslant 2$, then C is compact, convex, and symmetric in ℓ^2 under the condition $\lim_n \lambda_n = 0$. If $r > 2$, we further require $\sum_{n \geqslant 1} \lambda_n^2 < \infty$, and then again we have C compact, convex, and symmetric in ℓ^2. Hence by applying Theorem 2 and Theorem 3, we have that E, as given by (4.7), is precisely the set of points on the boundary of C which is the closest to points outside C. What we want to show is that

$$E=\left\{p\in C: \sum_{n>1} |p_n/\lambda_n|^r=1 \text{ and } \sum_{n\geqslant 1} |p_n|^{2r/q}\lambda_n^{-2r}<\infty\right\}, \tag{5.2}$$

where q is the conjugate exponent of r; i.e., $1/q+1/r=1$.

To verify (5.2), take $p\in\partial_\lambda C$, $p\in E$. Then there exists $f\in\ell^2$, $f=\sum_{n\geqslant 1} f_n e_n$, $\sum_{n\geqslant 1} f_n^2<\infty$, such that

$$\begin{aligned} &\text{(a)}\quad f(p)=\sum_{n\geqslant 1} f_n p_n=1\\ &\text{(b)}\quad \sup_{x\in C} f(x)\leqslant 1. \end{aligned} \tag{5.3}$$

Here $\lambda(x)=\inf\{t>0: x\in tC\}$ is a norm on $F=\bigcup_{n\geqslant 1} nC$, and $\lambda(x)=(\sum_{n\geqslant 1} |p_n/\lambda_n|^r)^{1/r}$ on F. Hence f is also continuous on F.

Since $p=\sum_{n\geqslant 1} p_n e_n\in C$, we see that the definition of C implies $\sum_{n\geqslant 1} \pm p_n e_n\in C$ for *all* choises of ± 1's. Let $\varepsilon_n=(\operatorname{sgn} f_n)(\operatorname{sgn} p_n)$ where we interpret $\varepsilon_n=1$ if p_n or $f_n=0$. Let $\tilde p=\sum_{n\geqslant 1}\varepsilon_n p_n e_n$. Then $\tilde p\in C$ and by (5.3.b)

$$1\geqslant f(\tilde p)=\sum_{n\geqslant 1} |f_n p_n|. \tag{5.4}$$

Now (5.3.a) implies

$$f(p)=\sum_{n\geqslant 1} f_n p_n=1. \tag{5.5}$$

Hence $\sum_{n\geqslant 1} f_n p_n\geqslant\sum_{n\geqslant 1} |f_n p_n|$, so $f_n p_n\geqslant 0$ for all $n\geqslant 1$. Also, (5.3.b) and the above argument implies

$$\sup_{x\in C}\sum_{n\geqslant 1} |f_n x_n|\leqslant 1.$$

Thus

$$\sup_{x\in C}\sum_{n\geqslant 1} |\lambda_n f_n x_n/\lambda_n|\leqslant 1. \tag{5.6}$$

Then by combining (5.3), (5.6), and $p\in C$ we see

$$1=\sup_{x\in C}\sum_{n\geqslant 1} |\lambda_n f_n x_n/\lambda_n|=\left(\sum_{n\geqslant 1} |\lambda_n f_n|^q\right)^{1/q}, \tag{5.7}$$

where the second equality in (5.7) follows from the duality of ℓ^q and ℓ^r since $x \in C$ iff $\{x_n/\lambda_n : n \geq 1\}$ is in the unit ball of ℓ^r. (Hence f continuous on F implies $\sum_{n \geq 1} |\lambda_n f_n|^q < \infty$.) Therefore

$$\sum_{n \geq 1} |\lambda_n f_n|^q = 1, \tag{5.8}$$

and $p \in C$, (5.5), (5.7), and (5.8) thus imply

$$1 = \sum_{n \geq 1} f_n p_n = \Big(\sum_{n \geq 1} |\lambda_n f_n|^q\Big)^{1/q} \cdot \Big(\sum_{n \geq 1} (p_n/\lambda_n)^r\Big)^{1/r}. \tag{5.9}$$

Now equality holding in Holder's inequality implies

$$\alpha\, |\lambda_n f_n|^q = \beta\, |p_n/\lambda_n|^r \qquad (n \geq 1), \tag{5.10}$$

where $\alpha\beta \neq 0$. Since both sums are one, $\alpha = \beta$, and thus

$$|p_n| = |\lambda_n|^{1+1/r}\, |f_n|^{q/r} \qquad (n \geq 1). \tag{5.11}$$

or, equivalently,

$$|f_n| = |p_n|^{r/q}\, \lambda_n^{-(r/q+1)} = |p_n|^{r/q}\, \lambda_n^{-r} \qquad (n \geq 1) \tag{5.12}$$

Now f continuous on ℓ^2 implies $\sum_{n \geq 1} f_n^2 < \infty$, so (5.12) implies

$$\sum_{n \geq 1} p_n^{2r/q}\, \lambda_n^{-2r} < \infty. \tag{5.13}$$

Thus $p \in E$ implies p is in the right-hand side of (5.2).

Conversely, if p is in the right-hand side of (5.2), then $p \in \partial_\lambda C$ since $\sum_{n \geq 1} |p_n/\lambda_n|^r = 1$. Furthermore, setting

$$f_n = |p_n|^{r/q}\, \lambda_n^{-r}\, \operatorname{sgn}(p_n) \qquad (n \geq 1), \tag{5.14}$$

we have $f = \sum_{n \geq 1} f_n e_n$ satisfying $\sum_{n \geq 1} f_n^2 < \infty$. Hence f is continuous on ℓ^2 and

$$f(p) = \sum_{n \geq 1} f_n p_n = \sum_{n \geq 1} |p_n|^{(r/q)+1}\, \lambda_n^{-r} = \sum_{n \geq 1} |p_n/\lambda_n|^r = 1 \tag{5.15}$$

since r and q are conjugate. Now using (5.14) we see

$$\begin{aligned} \sup_{x \in C} f(x) &= \sup_{x \in C} \sum_{n \geq 1} f_n x_n \leq \sup_{x \in C} \Big(\sum_{n \geq 1} |\lambda_n f_n|^q\Big)^{1/q} \Big(\sum_{n \geq 1} |x_n/\lambda_n|^r\Big)^{1/r} \\ &\leq \Big(\sum_{n \geq 1} (\lambda_n |p_n|^{r/q}\, \lambda_n^{-r})^q\Big)^{1/q} = \sum_{n \geq 1} |p_n/\lambda_n|^r = 1 \end{aligned} \tag{5.16}$$

since $q(r-1) = r$. Thus the right-hand side of (5.2) is contained in E, and (5.2) holds.

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