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How long does it take to see a flat Brownian path on the average?

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Abstract: Let W_t be a standard Brownian motion and define $R(t, 1) = \max_{t-1 \leqslant s \leqslant t} W_s - \min_{t-1 \leqslant s \leqslant t} W_s$ for $t \leqslant 1$. Given $\varepsilon > 0$, let $\tau(\varepsilon) = \min\{t \geqslant 1\colon\ R(t, 1) \leqslant \varepsilon\}$. We prove that $\lim_{\varepsilon \to 0} \varepsilon^2 \log E(\tau(\varepsilon)) = \frac{1}{2}\pi^2$. We also give the lim inf behavior of $R(t, 1)$ and $\inf_{1 \leqslant s \leqslant t} R(s, 1)$.

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1. Introduction

In this paper we investigate certain patterns for Brownian motion and specifically patterns related to the *flatness* of Brownian motion.

Let W_t, $t \geqslant 0$, denote a standard Brownian motion. For $0 \leqslant h \leqslant t$, let

$$R(t, h) = \max_{t-h \leqslant s \leqslant t} W_s - \min_{t-h \leqslant s \leqslant t} W_s, \qquad R(t) = R(t, t).$$

$R(t, h)$ denotes the *range* of the Brownian motion over the time interval $[t-h, t]$ and is the width of the smallest rectangle of length h with sides parallel to the coordinate axis that covers the graph (s, W_s) for $s \in [t-h, t]$.

Given $\varepsilon > 0$, let

$$\tau(\varepsilon) = \min\{t \geqslant 1\colon\ R(t, 1) \leqslant \varepsilon\}.$$

Concerning the expected value of $\tau(\varepsilon)$, we have:

Theorem 1.1.

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log E(\tau(\varepsilon)) = \tfrac{1}{2}\pi^2.$$

Questions concerning the expected time to see patterns are not new. Chapter XIII of the classical book of Feller (1967) is devoted to the study of patterns connected with repeated trials. In Sections 7 and

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8, the theory is applied to certain patterns arising from the outcomes of the Bernoulli process. For example, the expected number of trials needed to see a success run of length r; the expected number of trials needed to see either a success run of length r or a failure run of length ρ; the probability of seeing a run of r successes prior to a run of ρ failures are calculated. The methods employed are quite general and by no means limited to runs: any finite pattern composed of the symbols S (success) and F (failure) may be substituted in place of success or failure runs (e.g. SFS). Much of this theory is developed through an analysis of a fundamental convolution equation relating the probabilities of the occurrence of the pattern at time n and the first occurrence of the pattern at time n. Gardner (1974) gives a nice discussion of a related mathematical game.

An elegant alternative approach to some of this theory is given by S. Li (1980). Let Z, Z_i, $i \geqslant 1$, be a sequence of i.i.d. discrete-valued random variables and let Σ denote the set of possible outcomes of Z. Let A_i, $1 \leqslant i \leqslant n$, be finite sequences over Σ. The number of trials needed to see the pattern A_i in a run will be denoted by N_{A_i}. $N = \min_{1 \leqslant i \leqslant n} N_{A_i}$ is the number of trials needed to see *any* of the patterns sequentially. Using martingale techniques, a system of $n+1$ linear equations is generated from which $E(N)$ and $P(N = N_{A_i})$ can be computed. Guibas and Odlyzko (1981) give a combinatorial treatment of this and related problems. Some of this theory has found application in the study of genetics (e.g. Shukla and Srivastava, 1985). More recently, Chrysaphinou and Papastavridis (1990) consider pattern problems associated with the outcomes of a stationary Markov chain. Móri (1991) considers the expected waiting time until *each* of some given patterns has occurred.

This discrete-time theory can be utilized in the study of $E(\tau(\varepsilon))$, but only with moderate success. Consider the analogue of $\tau(\varepsilon)$ for simple random walk. Let X_i, $i \geqslant 1$, be an i.i.d. sequence of random variables with $P(X_1 = \pm 1) = \frac{1}{2}$. Let $S_n = X_1 + \cdots + X_n$ $(S_0 = 0)$. For $\varepsilon > 0$ and a positive integer m, let

$$N(\varepsilon, m) = \min\left\{n \geqslant m\colon \max_{n-m \leqslant j \leqslant n} S_j - \min_{n-m \leqslant j \leqslant n} S_j \leqslant \varepsilon\sqrt{m}\right\}, \quad \varepsilon\sqrt{m} \geqslant 1.$$

If m is even and $\varepsilon\sqrt{m} = 1$, then $N(\varepsilon, m) = \min(N_{\boldsymbol{A}}, N_{\boldsymbol{B}})$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are the following patterns of length m:

$$\boldsymbol{A}\colon SFSF\ldots SF \quad \text{and} \quad \boldsymbol{B}\colon FSFS\ldots FS.$$

Using the method of S. Li (for example), we obtain

$$E(N(\varepsilon, m)) = 2^m - 1 = \exp((\log 2)/\varepsilon^2) - 1.$$

Up to a multiplicative constant, the exponent agrees with that given in Theorem 1.1. The comparison with Theorem 1.1 is made more precise as $\varepsilon\sqrt{m} \to \infty$. However, in this case, the calculation of $E(N(\varepsilon, m))$ (by any means) becomes formidable. While retaining some of the flavor of the discrete-time methods, the proof of Theorem 1.1 requires a different approach.

We also consider the related problem: how small can $R(t, 1)$ be? Put another way, let $\varepsilon(t)$ decrease monotonically as $t \to \infty$. How quickly can $\varepsilon(t)$ decrease and still permit $R(t, 1) \leqslant \varepsilon(t)$ with probability one for arbitrarily large t? Likewise, how slowly can $\varepsilon(t)$ decrease and yet ensure that eventually $R(t, 1) \geqslant \varepsilon(t)$ with probability one? Theorem 1.1 suggests that the function $\pi/\sqrt{2 \log t}$ is critical. In fact we have the precise statement:

Theorem 1.2.

$$\liminf_{t \to \infty} \sqrt{\log t}\, R(t, 1) = \pi/\sqrt{2} \quad a.s., \tag{1.1}$$

$$\liminf_{t \to \infty} \sqrt{\log t} \inf_{1 \leqslant s \leqslant t} R(s, 1) = \pi/\sqrt{2} \quad a.s. \tag{1.2}$$

Given $\delta > 0$, (1.1) implies that with probability one, for all $T > 0$, there exists a $t \geqslant T$ such that

$$R(t, 1) \leqslant (1+\delta)\pi/\sqrt{2 \log t}.$$

On the other hand, for almost all ω, there exists a $T(\omega)$ such that for all $t \geqslant T(\omega)$,

$$R(t, 1) \geqslant (1-\delta)\pi/\sqrt{2 \log t}. \tag{1.3}$$

(1.2) strengthens (1.3) in that the same conclusion holds with $\inf_{1 \leqslant s \leqslant t} R(s, 1)$ in place of $R(t, 1)$. Taylor (1972) investigates other properties of $R(t)$ in connection with Brownian path variation.

Theorem 1.1 and Theorem 1.2 rely on the *small-ball* estimate for $R(1)$:

$$P(R(1) \leqslant \varepsilon) \sim (8/\varepsilon^2) \exp(-\pi^2/(2\varepsilon^2)) \quad \text{as } \varepsilon \to 0. \tag{1.4}$$

This is obtained by Feller (1951) by observing the connection between the joint distribution function of the maximum and minimum of W_t for $0 \leqslant t \leqslant 1$ and a certain distribution function arising in the Kolmogorov–Smirnov theorem on empirical distributions. Combining the rescaling and stationary increments properties of Brownian motion, we observe that $R(t, h)$ is distributed as $\sqrt{h}\, R(1)$, hence (1.4) can be adapted easily to estimates for $R(t, h)$.

In Section 2 we give the proof of Theorem 1.1. In Section 3 we give the proof of Theorem 1.2. We conclude this paper with some brief remarks.

2. Proof of Theorem 1.1

Given $\varepsilon > 0$, let

$$\theta = \theta(\varepsilon) = \min\{k \geqslant 1, k \in \mathbb{Z}: R(k, 1) \leqslant \varepsilon\}.$$

Then $\tau(\varepsilon) \geqslant \theta(\varepsilon)$. Next we give a lower bound for $\tau(\varepsilon)$. Let $N \geqslant 2$ be an integer. For each integer $i \in [0, N-1]$, and each integer $k \geqslant 0$, let

$$i_k = k + i/N, \qquad h = 1 - 1/N,$$

$$T_i = T_i(\varepsilon) = \min\{i_k: R(i_k, h) \leqslant \varepsilon\}$$

and

$$T = T(\varepsilon) = \min\{0 \leqslant i \leqslant N-1: T_i(\varepsilon)\}.$$

It is clear that $T(\varepsilon) \leqslant \tau(\varepsilon)$. Hence we have

$$E(T(\varepsilon)) \leqslant E(\tau(\varepsilon)) \leqslant E(\theta(\varepsilon)).$$

First we show that

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log E(\tau(\varepsilon)) \leqslant \lim_{\varepsilon \to 0} \varepsilon^2 \log E(\theta(\varepsilon)) = \tfrac{1}{2}\pi^2 \tag{2.1}$$

Let $p = P(R(1) \leqslant \varepsilon)$ and $q = 1 - p$. Since the increments of W_t are independent, $P(\theta > m) = q^m$ for $m \geqslant 0$. Thus $E(\theta) = 1/p$ and (2.1) follows from (1.4).

Next we show that

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log E(\tau(\varepsilon)) \geqslant \tfrac{1}{2}\pi^2. \tag{2.2}$$

For convenience, we introduce the notation $\langle x\rangle = x \pmod N$. First we calculate $E(T_i(\varepsilon))$. Let $P = P(\sqrt{h}\cdot R(1)\leqslant \varepsilon)$ and $Q = 1-P$. For $0\leqslant i\leqslant N-2$,

$$\{T_i(\varepsilon)=i_k\}=\{R(i_1,h)>\varepsilon,\dots,R(i_{k-1},h)>\varepsilon,\ R(i_k,h)\leqslant\varepsilon\}.$$

By the independence of the increments of W_t, $P(T_i(\varepsilon)=i_k)=P\cdot Q^{k-1}$. Hence

$$E(T_i(\varepsilon))=\sum_{k=1}^{\infty}\left(k+\frac{i}{N}\right)P\cdot Q^{k-1}=\frac{1}{P}+\frac{i}{N},\quad 0\leqslant i\leqslant N-2.$$

Similarly, for $i=N-1$,

$$E(T_{N-1}(\varepsilon))=\frac{1}{P}+\frac{-1}{N}.$$

Consequently,

$$E(T_i(\varepsilon))=\frac{1}{P}+\frac{\langle i+1\rangle-1}{N},\quad 0\leqslant i\leqslant N-1. \tag{2.3}$$

For each $i\in\{0,\dots,N-1\}$, write

$$E(T_i(\varepsilon))=E(T)+E(T_i-T)=E(T)+\sum_{\substack{0\leqslant j\leqslant N-1\\ j\neq i}}E\big((T_i-T)I(T=T_j)\big). \tag{2.4}$$

If $i>j$, then

$$E\big((T_i-T)I(T=T_j)\big)=\sum_{l=0}^{\infty}\sum_{k=l}^{\infty}(i_k-j_l)\cdot P(T=T_j=j_l,\ T_i=i_k)=A+B$$

where

$$A=\sum_{l=0}^{\infty}(i_l-j_l)\cdot P(T=T_j=j_l,\ T_i=i_l),$$

$$B=\sum_{l=0}^{\infty}\sum_{k=l+1}^{\infty}(i_k-j_l)\cdot P(T=T_j=j_l,\ T_i=i_k).$$

Since $i_l-j_l=(i-j)/N$,

$$A\leqslant\frac{1-j}{N}\cdot P(T=T_j). \tag{2.5}$$

To estimate B we note that

$$\begin{aligned}&P(T=T_j=j_l,\ T_i=i_k)\\ &\quad\leqslant P\big(T=T_j=j_l,\ R(i_{l+1},h)>\varepsilon,\dots,R(i_{k-1},h)>\varepsilon,\ R(i_k,h)\leqslant\varepsilon\big)\\ &\quad\leqslant P(T=T_j=j_l)\cdot P\cdot Q^{k-l-1}.\end{aligned}$$

Thus for $i>j$,

$$\begin{aligned}B&\leqslant\sum_{l=0}^{\infty}P(T=T_j=j_l)\sum_{k=l+1}^{\infty}\left(k-l+\frac{1-j}{N}\right)P\cdot Q^{k-l-1}\\ &=P(T=T_j)\cdot\left(\frac{1}{P}+\frac{1-j}{N}\right).\end{aligned}$$

Combining this with (2.5), we obtain for $i > j$,

$$E\big((T_i - T)I(T = T_j)\big) \leqslant P(T = T_j) \cdot \left(\frac{1}{P} + 2 \cdot \frac{i-j}{N}\right).$$

The case $i < j$ can be handled similarly and in general we have

$$E\big((T_i - T)I(T = T_j)\big) \leqslant P(T = T_j) \cdot \left(\frac{1}{P} + 2 \cdot \frac{\langle i-j\rangle}{N}\right). \tag{2.6}$$

Now plugging (2.3) and (2.6) into (2.4) we obtain

$$\frac{1}{P} + \frac{\langle i+1\rangle - 1}{N} \leqslant E(T) + \frac{1}{P} \sum_{\substack{0 \leqslant j \leqslant N-1 \\ j \neq i}} P(T = T_j) + \frac{2}{N} \sum_{j=0}^{N-1} \langle i-j\rangle \cdot P(T = T_j).$$

But $\Sigma_{j \neq i} P(T = T_j) = 1 - P(T = T_i)$ so that

$$\frac{1}{P} \cdot P(T = T_i) + \frac{\langle i+1\rangle - 1}{N} \leqslant E(T) + \frac{2}{N} \sum_{j=0}^{N-1} \langle i-j\rangle \cdot P(T = T_j).$$

By adding these equations together and observing that

$$\frac{1}{N} \sum_{i=0}^{N-1} (\langle i+1\rangle - 1) = \tfrac{1}{2}(N-3) \quad \text{and} \quad \frac{2}{N} \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} \langle i+1\rangle \cdot P(T = T_j) = N - 1,$$

we obtain the inequality

$$1/P\big(\sqrt{h}\, R(1) \leqslant \varepsilon\big) \leqslant N E(T) + \tfrac{1}{2}(N+1).$$

Hence

$$\begin{aligned} \liminf_{\varepsilon \to 0} \varepsilon^2 \log E(\tau(\varepsilon)) &\geqslant \liminf_{\varepsilon \to 0} \varepsilon^2 \log E(T) \\ &\geqslant -\liminf_{\varepsilon \to 0} \varepsilon^2 \log P\big(\sqrt{h}\, R(1) \leqslant \varepsilon\big) = h \cdot \tfrac{1}{2}\pi^2. \end{aligned}$$

Letting h tend to 1, we obtain (2.2) which finishes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We only need to show

$$\liminf_{t \to \infty} \sqrt{\log t}\, R(t, 1) \leqslant \pi/\sqrt{2} \quad \text{a.s.} \tag{3.1}$$

and

$$\liminf_{t \to \infty} \sqrt{\log t} \inf_{1 \leqslant s \leqslant t} R(s, 1) \geqslant \pi/\sqrt{2} \quad \text{a.s.} \tag{3.2}$$

First we show that (3.1) holds. Choose $\lambda > \gamma > 1$ and for $k = 1, 2, \ldots$, let

$$A_k = \big\{\sqrt{\log k}\, R(k, 1) \leqslant \sqrt{\lambda} \cdot \pi/\sqrt{2}\big\}.$$

Since $R(k, 1)$ is distributed as $R(1)$, we have by (1.4), for k sufficiently large,

$$\lambda \cdot (\log k)^{-1} \cdot \log P(A_k) \geqslant -\gamma.$$

Thus $\sum_{k=1}^{\infty} P(A_k) = \infty$ and $P(A_k, \text{ i.o.}) = 1$, by the Borel–Cantelli lemma, which in turn implies (3.1).

Now we turn to the proof of (3.2). Fix an integer $N \geqslant 2$ and let $h = 1 - 1/N$. Let $t_k = k/N$, $k \geqslant 0$, $k \in \mathbb{Z}$. Note that for $s \geqslant 1$, $t_k \in [s - 1/N, s]$ for some k. Moreover, $t_k - h \geqslant s - 1$. Hence $R(t_k, h) \leqslant R(s, 1)$. Consequently, for $t \geqslant 1$,

$$\inf_{N-1 \leqslant j \leqslant [Nt]} R(t_j, h) \leqslant \inf_{1 \leqslant s \leqslant t} R(s, 1). \tag{3.3}$$

We will show that

$$\liminf_{k \to \infty} \sqrt{\log t_k} \inf_{N-1 \leqslant j \leqslant k} R(t_j, h) \geqslant \sqrt{h} \cdot \pi/\sqrt{2} \quad \text{a.s.} \tag{3.4}$$

Combining (3.4) with (3.3) and observing $\lim_{t \to \infty} \log(t_{[Nt]})/\log t = 1$, we obtain

$$\liminf_{t \to \infty} \sqrt{\log t} \inf_{1 \leqslant s \leqslant t} R(s, 1) = \sqrt{h} \cdot \pi/\sqrt{2} \quad \text{a.s.}$$

Letting h tend to 1 yields (3.2) and we are left to show (3.4).

Let λ be chosen so that $h > \lambda > 0$. Choose $1 > \gamma > 0$ so that $\gamma h > \lambda$. Let

$$B_k = \left\{ \sqrt{\log t_k} \cdot R(t_j, h) \leqslant \sqrt{\lambda} \cdot \pi/\sqrt{2} \text{ for some } N-1 \leqslant j \leqslant k \right\}.$$

Then for k sufficiently large,

$$\begin{aligned} P(B_k) &\leqslant k \cdot P\left(\sqrt{\log t_k} \cdot R(h) \leqslant \sqrt{\lambda} \cdot \pi/\sqrt{2}\right) \\ &= k \cdot P\left(\sqrt{\log t_k} \cdot R(1) \leqslant \sqrt{\lambda/h} \cdot \pi/\sqrt{2}\right) \\ &\leqslant N^{\gamma h/\lambda} \cdot k^{-(\gamma h/\lambda - 1)}, \end{aligned}$$

where the last inequality follows from (1.4). Since $\gamma h/\lambda > 1$, we have $\sum_{k=1}^{\infty} P(B_{2^k}) < \infty$ and $P(B_{2^k}, \text{ i.o.}) = 0$. This is true for all $\lambda < h$, which implies

$$\liminf_{k \to \infty} \sqrt{\log t_{2^k}} \inf_{N-1 \leqslant j \leqslant 2^k} R(t_j, h) \geqslant \sqrt{h} \cdot \pi/\sqrt{2} \quad \text{a.s.} \tag{3.5}$$

We are left to fill in the gaps. Let

$$D_k = \left\{ \sqrt{\log t_{2^k}} \cdot R(t_j, h) \leqslant \sqrt{\lambda} \cdot \pi/\sqrt{2} \text{ for some } 2^k < j \leqslant 2^{k+1} \right\}.$$

Arguing as above,

$$P(D_k) \leqslant (2^{k+1} - 2^k) \cdot P\left(\sqrt{\log t_{2^k}} \cdot R(h, h) \leqslant \sqrt{\lambda} \cdot \pi/\sqrt{2}\right) \leqslant N^{\gamma h/\lambda} \cdot 2^{-(\gamma h/\lambda - 1)k}.$$

Hence, we have $\sum_{k=1}^{\infty} P(D_k) < \infty$ and $P(D_k, \text{ i.o.}) = 0$ as well. Thus,

$$\liminf_{k \to \infty} \sqrt{\log t_{2^k}} \inf_{2^k < j \leqslant 2^{k+1}} R(t_j, h) \geqslant \sqrt{h} \cdot \pi/\sqrt{2} \quad \text{a.s.} \tag{3.6}$$

For $2^k < i \leqslant 2^{k+1}$ we have $\log t_i \geqslant \log t_{2^k}$ and

$$\inf_{N-1 \leqslant j \leqslant i} R(t_j, h) \geqslant \min\left[\inf_{N-1 \leqslant j \leqslant 2^k} R(t_j, h), \inf_{2^k < j \leqslant 2^{k+1}} R(t_j, h) \right].$$

Consequently, (3.4) follows from (3.5) and (3.6). □

4. Remarks

In essence, Theorem 1.1 tells us that

$$\log E(\tau(\varepsilon)) \sim -\log P(R(1) \leqslant \varepsilon) \quad \text{as } \varepsilon \to 0,$$

and an easy observation yields $P(R(1) \leqslant \varepsilon)E(\tau(\varepsilon)) \leqslant 1$. However, one cannot hope that

$$E(\tau(\varepsilon)) \sim 1/P(R(1) \leqslant \varepsilon) \quad \text{as } \varepsilon \to 0,$$

as indicated by the following proposition.

Proposition 4.1. $\lim_{\varepsilon \to 0} P(R(1) \leqslant \varepsilon)E(\tau(\varepsilon)) = 0$.

Proof. Let $t \geqslant 0$, then

$$P(R(t+1, 1) \leqslant \varepsilon) = \int_{[0,t]} P(R(t+1, 1) \leqslant \varepsilon \mid \tau(\varepsilon) = s) P(\tau(\varepsilon) \in \mathrm{d}s) + P(R(t+1, 1) \leqslant \varepsilon, \tau(\varepsilon) > t).$$

However $P(R(t+1, 1) \leqslant \varepsilon) = P(R(1) \leqslant \varepsilon)$ and for $s \in [0, t]$,

$$P(R(t+1, 1) \leqslant \varepsilon \mid \tau(\varepsilon) = s) = P(R(1) \leqslant \varepsilon),$$

by the independence of the increments of Brownian motion. Consequently,

$$P(R(1) \leqslant \varepsilon) P(\tau(\varepsilon) > t) = P(R(t+1, 1) \leqslant \varepsilon, \tau(\varepsilon) > t).$$

Integrating with respect to Lebesgue measure, we obtain

$$P(R(1) \leqslant \varepsilon) E(\tau(\varepsilon)) = \int_0^\infty P(R(t+1, 1) \leqslant \varepsilon, \tau(\varepsilon) > t) \, \mathrm{d}t. \tag{4.1}$$

Choose $1 > \delta > 0$. Then

$$\begin{aligned} &P(R(t+1, 1) \leqslant \varepsilon, \tau(\varepsilon) > t) \\ &\quad \leqslant P(R(t+1, 1) \leqslant \varepsilon, t < \tau(\varepsilon) \leqslant t+1-\delta) + P(t+1-\delta < \tau(\varepsilon) \leqslant t+1). \end{aligned}$$

An easy calculation reveals

$$\int_0^\infty P(t+1-\delta < \tau(\varepsilon) \leqslant t+1) \, \mathrm{d}t = \int_{1-\delta}^1 P(\tau(\varepsilon) > t) \, \mathrm{d}t \leqslant \delta. \tag{4.2}$$

Moreover,

$$\begin{aligned} &P(R(t+1, 1) \leqslant \varepsilon, t < \tau(\varepsilon) \leqslant t+1-\delta) \\ &\quad \leqslant P(R(t+1, \delta) \leqslant \varepsilon, t < \tau(\varepsilon) \leqslant t+1-\delta) \\ &\quad = P(R(\delta) \leqslant \varepsilon) \cdot P(t < \tau(\varepsilon) \leqslant t+1-\delta). \end{aligned}$$

Integrating this expression, we obtain

$$\begin{aligned} &\int_0^\infty P(R(t+1, 1) \leqslant \varepsilon, t < \tau(\varepsilon) \leqslant t+1-\delta) \, \mathrm{d}t \\ &\quad \leqslant P(R(\delta) \leqslant \varepsilon) \cdot \int_0^\infty P(t < \tau(\varepsilon) \leqslant t+1-\delta) \, \mathrm{d}t \\ &\quad \leqslant P(R(\delta) \leqslant \varepsilon). \end{aligned} \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), it follows that

$$\limsup_{\varepsilon \to 0} P(R(1) \leqslant \varepsilon) E(\tau(\varepsilon)) \leqslant \delta.$$

Letting δ tend to 0 finishes the proof. □

Theorem 1.2 can be compared to a result of Csörgő and Révész (1981). For $0 \leqslant h \leqslant t$, let

$$V(t, h) = \sup_{0 \leqslant s \leqslant h} |W(t-h+s) - W(t-h)|.$$

$V(t, h)$ and $R(t, h)$ are measures of the variation of an increment of Brownian motion. $V(t, h)$ differs from $R(t, h)$ in that it measures the largest variation of the Brownian motion from $W(t-h)$ over the interval $[t-h, t]$ rather than the difference between the maximum and the minimum over that same interval. As an application of their Theorem 1.7.1, Csörgő and Révész obtain

$$\liminf_{t \to \infty} \inf_{8 \log t/\pi^2 \leqslant s \leqslant t} V(s, 8 \log t/\pi^2) = 1 \quad \text{a.s.} \tag{4.4}$$

Setting aside the differences between $R(t, h)$ and $V(t, h)$, (4.4) addresses the possibility of seeing increasingly long increments of Brownian motion of consistent width while Theorem 1.2 addresses the possibility of seeing consistently long but increasingly flat increments of Brownian motion. The magnitudes of the scalings ($\log t$ versus $1/\sqrt{\log t}$) are consistent with the self-similarity of Brownian motion.

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References

Chrysaphinou, O. and S. Papastavridis (1990), The occurrence of sequence patterns in repeated dependent experiments, *Theory Probab. Appl.* **35**, 145–152.

Csörgő, M. and P. Révész (1981), *Strong Approximations in Probability and Statistics* (Academic Press, New York).

Feller, W. (1951), The asymptotic distribution of the range of sums of independent random variables, *Ann. Math. Statist.* **22**, 427–432.

Feller, W. (1967), *An Introduction to Probability Theory and Its Applications, Vol I* (Wiley, New York, 3rd ed.).

Gardner, M. (1974), Mathematical games, *Sci. Amer.* **10**, 120–125.

Guibas, L.J. and A.M. Odlyzko (1981), String overlaps, pattern matching, and nontransitive games, *J. Comb. Theory. Ser. A* **30**, 183–208.

Li, S.R. (1980), A martingale approach to the study of occurrence of sequence patterns in repeated experiments, *Ann. Probab.* **8**, 1171–1176.

Móri, T.F. (1991), On the waiting time till each of some given patterns occurs as a run, *Probab. Theory Rel. Fields* **87**, 313–323.

Shukla, R. and R.C. Srivastava (1985), The statistical analysis of direct repeats in nucleic acid sequences *J. Appl. Probab.* **22**, 15–24.

Taylor, S.J. (1972), Exact asymptotic estimates of Brownian path variation, *Duke J.* **39**, 219–241.