ON THE LOWER TAIL OF GAUSSIAN MEASURES ON l_p

Wenbo V. Li[1]

1. Introduction. Throughout this paper $(\xi_k)_{k=1}^{\infty}$ denotes a sequence of independent, centered, Gaussian random variables with variance one. We shall study the behavior of

$$P\Big((\sum_{k\geq 1} a_k|\xi_k|^p)^{1/p} \leq \varepsilon\Big) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{for} \quad p > 0 \tag{1.1}$$

where $(a_k)_{k=1}^{\infty}$ is a given sequence of positive numbers and $\sum_{k\geq 1} a_k < +\infty$.

In section 2 we give a lower bound of (1.1) for $p \geq 2$ and a upper bound of (1.1) for $p > 0$ when $\varepsilon > 0$ is small. In particular for $a_k = k^{-\alpha}$ and $\alpha > 1$, we obtain for $p > 2$ and $\varepsilon > 0$ small

$$-C_{\alpha,p} \cdot \varepsilon^{-p/(\alpha-1)} \leq \log P\Big((\sum_{k\geq 1} a_k|\xi_k|^p)^{1/p} \leq \varepsilon\Big) \leq -D_{\alpha,p} \cdot \varepsilon^{-p/(\alpha-1)} \tag{1.2}$$

where $C_{\alpha,p}$ and $D_{\alpha,p}$ are positive constants. This extends the results (Theorem 4.1-4.4) of Hoffmann-Jørgensen, Shepp and Dudley[4] for $p = 2$ and can be used to determine the nature rate of escape for an independent coordinate l_p-valued Brownian motion for $p > 2$ (see Cox[2] and Erickson[3]). As a consequence of (1.2), we give a positive answer to a conjecture in Erickson[3].

In section 3, as an application of the result given in section 2 for the lower bound of (1.1), we give a lower bound for $P(\sup_{0\leq t\leq 1} |X(t)| \leq \varepsilon)$ under certain conditions where $X(t) = \sum_{k\geq 1} \lambda_k \phi_k(t)\xi_k$, $0 \leq t \leq 1$. Note that $\{X(t) : 0 \leq t \leq 1\}$ is a mean zero Gaussian process but not necessarily a stationary process.

2. Upper and Lower Bound for l_p-norm. The following claim about the volume of the unit ball under l_p^n-norm is probably known, though we could not locate a reference (The claim is known and the author would like to thank the referee for providing a reference: Saint-Raymond[7]). Lemma 1 is a key step for the proof of our results.

Lemma 1.

$$V(n,p) = \int \cdots \int_{\sum_{i=1}^n |x_i|^p \leq 1} 1 \; dx_1 \ldots dx_n = 2^n \Gamma(\frac{1}{p}+1)^n \cdot \Gamma(\frac{n}{p}+1)^{-1}.$$

Proof. Let

$$y_k = (1 - \sum_{i=1}^{k} x_i^p)^{1/p}, \qquad k = 0, 1, 2, \ldots, n$$

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and

$$I_{m,n} = \int_0^1 \int_0^{y_1} \dots \int_0^{y_{n-2}} \int_0^{y_{n-1}} y_n^m \, dx_n \, dx_{n-1} \dots dx_1.$$

Note that

$$\int_0^{y_{n-1}} y_n^m \, dx_n = \int_0^{y_{n-1}} (y_{n-1}^p - x_n^p)^{m/p} dx_n = y_{n-1}^{m+1} \int_0^1 (1 - x^p)^{m/p} \, dx = y_{n-1}^{m+1} \, C_m$$

where

$$C_m = \int_0^1 (1 - x^p)^{m/p} \, dx \; = \Gamma(\frac{1}{p} + 1)\Gamma(\frac{m}{p} + 1)\Gamma(\frac{m+1}{p} + 1)^{-1}.$$

We have $I_{m,n} = C_m I_{m+1,n-1}$ and $I_{m,1} = C_m$. Hence

$$2^{-n} V(n,p) = I_{0,n} = C_0 I_{1,n-1} = \dots = (\prod_{m=0}^{n-2} C_m) I_{n-1,1} = \Gamma(\frac{1}{p} + 1)^n \Gamma(\frac{n}{p} + 1)^{-1}.$$

Lemma 2. *For x large enough, we have*

$$\sqrt{\pi} x^{x+\frac{1}{2}} e^{-x} \le \Gamma(1+x) \le 2\sqrt{\pi} x^{x+\frac{1}{2}} e^{-x}.$$

Proof. It is easy to see by Stirling's formula

$$\Gamma(1+x) = \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} \big(1 + \theta(x)\big), \quad |\theta(x)| \le e^{\frac{1}{12x}} - 1.$$

Theorem 1. *If ε is small and $p > 2$, we have*

$$P\big((\sum_{k\ge 1} a_k |\xi_k|^p)^{1/p} \le \varepsilon\big) \ge \Big(1 - \delta^{-1} E|\xi_1|^p \sum_{k \ge n+1} a_k\Big) (2\pi^{-1})^{n/2} \Big(\prod_{k=1}^n a_k\Big)^{-1/p} \cdot \tag{2.1}$$

$$\exp\Big(-\frac{1}{2}\big(\sum_{k=1}^n a_k^{-2/(p-2)}\big)^{(p-2)/p} \cdot \big(\varepsilon^p - \delta\big)^{2/p}\Big) \cdot (\varepsilon^p - \delta)^{n/p} \cdot \Gamma\big(\frac{1}{p} + 1\big)^n \cdot \Gamma\big(\frac{n}{p} + 1\big)^{-1}.$$

for all positive integer n and all $\delta \in (0, \varepsilon^p)$.

Proof. Note that for any positive integer n and $\delta \in (0, \varepsilon^p)$

$$P\big((\sum_{k\ge 1} a_k |\xi_k|^p)^{1/p} \le \varepsilon\big) \ge P\big(\sum_{k=1}^n a_k |\xi_k|^p \le \varepsilon^p - \delta\big) \cdot P\big(\sum_{k\ge n+1} a_k |\xi_k|^p \le \delta\big). \tag{2.2}$$

By putting in the Gaussian density and making the change of variables, we have for the first term on the right side of (2.2)

(2.3)

$$\begin{aligned} &P\big(\sum_{k=1}^n a_k |\xi_k|^p \le \varepsilon^p - \delta\big) \\ =&(2\pi)^{-n/2} \Big(\prod_{k=1}^n a_k\Big)^{-1/p} \int \cdots \int_{\sum_{k=1}^n |x_k|^p \le \varepsilon^p - \delta} \exp\big(-\frac{1}{2} \sum_{k=1}^n a_k^{-2/p} x_k^2\big) \, dx_1 \dots dx_n. \end{aligned}$$

Using Hölder's inequality, we obtain for the exponent in the integrand of (2.3)

$$\sum_{k=1}^{n} a_k^{-2/p} x_k^2 \le \Big(\sum_{k=1}^{n} a_k^{-2/(p-2)}\Big)^{(p-2)/p} \cdot \Big(\sum_{k=1}^{n} |x_k|^p\Big)^{2/p} \le \Big(\sum_{k=1}^{n} a_k^{-2/(p-2)}\Big)^{(p-2)/p} \cdot \big(\varepsilon^p - \delta\big)^{2/p}. \tag{2.4}$$

Now putting (2.3), (2.4) and Lemma 1 together yields

$$\begin{aligned} &P\Big(\sum_{k=1}^{n} a_k |\xi_k|^p \le \varepsilon^p - \delta\Big) \\ \ge &(2\pi^{-1})^{n/2} \Big(\prod_{k=1}^{n} a_k\Big)^{-1/p} \exp\Big(-\frac{1}{2}\Big(\sum_{k=1}^{n} a_k^{-2/(p-2)}\Big)^{(p-2)/p} \cdot \big(\varepsilon^p - \delta\big)^{2/p}\Big) \cdot \\ &(\varepsilon^p - \delta)^{n/p} \cdot \Gamma\Big(\frac{1}{p}+1\Big)^n \cdot \Gamma\Big(\frac{n}{p}+1\Big)^{-1}. \end{aligned} \tag{2.5}$$

By using Chebyshev's inequality we have for the second term on the right side of (2.2)

$$\begin{aligned} P\Big(\sum_{k\ge n+1} a_k |\xi_k|^p \le \delta\Big) &= 1 - P\Big(\sum_{k\ge n+1} a_k |\xi_k|^p > \delta\Big) \\ &\ge 1 - \delta^{-1} E\Big(\sum_{k\ge n+1} a_k |\xi_k|^p\Big) \\ &= 1 - \delta^{-1} E|\xi_1|^p \sum_{k\ge n+1} a_k. \end{aligned} \tag{2.6}$$

Combining (2.2), (2.5) and (2.6), we obtain (2.1) and finish the proof.

Remark. Our proof here (also the next theorem) is along the same lines as the proof for the case $p = 2$ in [4]. The main difference is that we benefit a lot from Lemma 1 and use Hölder's inequality to take care of the rest. In application of (2.1) one should try to maximize the right-hand side in n and δ for fixed ε. Many examples are given in [4] for the case $p = 2$. Similarly, lengthy estimates for particular $(a_k)_{k=1}^{\infty}$ also work in our setting. However, here we are only going to evaluate one of the most important cases (also see the remark after Theorem 2).

Example. If $p > 2$, $a_k = k^{-\alpha}$ and $\alpha > 1$, then we have for $\varepsilon > 0$ small

$$\log P\Big(\sum_{k\ge 1} k^{-\alpha} |\xi_k|^p \le \varepsilon^p\Big) \ge -C_{\alpha,p} \cdot \varepsilon^{-p/(\alpha-1)}. \tag{2.7}$$

In this case we have

$$\sum_{k=n+1}^{\infty} a_k = \sum_{k=n+1}^{\infty} k^{-\alpha} \le \sum_{k=n+1}^{\infty} \int_{k-1}^{k} x^{-\alpha}\, dx = \frac{1}{\alpha-1} n^{-(\alpha-1)}.$$

So there exists a constant $M_{\alpha,p}$ such that if $n \geq M_{\alpha,p}\delta^{-1/(\alpha-1)}$, then

$$1 - \delta^{-1}E|\xi_1|^p \sum_{k\geq n+1} a_k \geq e^{-1}.$$

Now for n large, we have by Lemma 2

$$\big(\prod_{k=1}^{n} a_k\big)^{-1/p} = (n!)^{\alpha/p} \geq n^{\alpha(n+1/2)/p} e^{-\alpha n/p};$$

$$\Gamma\big(\frac{n}{p}+1\big)^{-1} \geq (2\sqrt{\pi})^{-1}\big(\frac{n}{p}\big)^{n/p+1/2} e^{-n/p};$$

$$\big(\sum_{k=1}^{n} a_k^{-2/(p-2)}\big)^{(p-2)/p} = \big(\sum_{k=1}^{n} k^{2\alpha/(p-2)}\big)^{(p-2)/p} \leq n^{(2\alpha-2+p)/p}.$$

Thus by Theorem 1 for $n \geq M_{\alpha,p}\delta^{-1/(\alpha-1)}$ and ε small enough (hence δ small enough and n large enough), we obtain

$$\begin{aligned}
&\log P\big((\sum_{k\geq 1} a_k|\xi_k|^p)^{1/p} \leq \varepsilon\big)\\
\geq& -1 + \frac{n}{2}\log(2\pi^{-1}) + \frac{\alpha}{p}(n+\frac{1}{2})\log n - \frac{\alpha}{p}n - \frac{1}{2}n^{(2\alpha-2+p)/p}(\varepsilon^p-\delta)^{2/p}\\
&+ \frac{n}{p}\log(\varepsilon^p-\delta) + n\log\Gamma\big(\frac{1}{p}+1\big) - \log(2\sqrt{\pi}) - \big(\frac{n}{p}+\frac{1}{2}\big)\log\frac{n}{p} + \frac{n}{p}\\
\geq& \Big(\frac{1}{2}\log(2\pi^{-1}) - \frac{\alpha-1}{p} + \frac{1}{p}\log\big(n^{\alpha-1}(\varepsilon^p-\delta)\big)\Big)n - \frac{1}{2}n^{(2\alpha-2+p)/p}(\varepsilon^p-\delta)^{2/p}.
\end{aligned}$$

Now choose $\delta = \varepsilon^p/2$ and $n = [K_{\alpha,p}\varepsilon^{-p/(\alpha-1)}]$ where $K_{\alpha,p}$ is a constant such that

$$K_{\alpha,p}\varepsilon^{-p/(\alpha-1)} - 1 \geq M_{\alpha,p}\delta^{-1/(\alpha-1)};$$

$$2^{-1}\log(2\pi^{-1}) - (\alpha-1)/p + p^{-1}\log\big(n^{\alpha-1}(\varepsilon^p-\delta)\big) \geq 0$$

for all ε small. Then we have

$$\begin{aligned}
\log P\big(\sum_{k\geq 1} k^{-\alpha}|\xi_k|^p \leq \varepsilon^p\big) &\geq -2^{-1}\big(K_{\alpha,p}\varepsilon^{-p/(\alpha-1)}\big)^{(2\alpha-2+p)/p}\cdot 2^{-2/p}\cdot\varepsilon^2\\
&= -C_{\alpha,p}\varepsilon^{-p/(\alpha-1)}.
\end{aligned}$$

Theorem 2. *For any positive integer n, we have*

$$P\big((\sum_{k\geq 1} a_k|\xi_k|^p)^{1/p} \leq \varepsilon\big) \leq (2\pi^{-1})^{n/2}\Big(\prod_{k=1}^{n} a_k\Big)^{-1/p}\cdot\Gamma\big(\frac{1}{p}+1\big)^n\cdot\Gamma\big(\frac{n}{p}+1\big)^{-1}\cdot\varepsilon^n. \tag{2.8}$$

Proof. Observe that for any positive integer n,

$$
\begin{aligned}
&P\Big(\big(\sum_{k\geq 1} a_k|\xi_k|^p\big)^{1/p} \leq \varepsilon\Big) \leq P\Big(\sum_{k=1}^{n} a_k|\xi_k|^p \leq \varepsilon^p\Big)\\
=&(2\pi)^{-n/2}\Big(\prod_{k=1}^{n} a_k\Big)^{-1/p} \int\cdots\int_{\sum_{k=1}^{n}|x_k|^p\leq\varepsilon^p} \exp\big(-\frac{1}{2}\sum_{k=1}^{n} a_k^{-2/p}x_k^2\big)\,dx_1\ldots dx_n\\
\leq&(2\pi)^{-n/2}\Big(\prod_{k=1}^{n} a_k\Big)^{-1/p} \int\cdots\int_{\sum_{k=1}^{n}|x_k|^p\leq\varepsilon^p} 1\,dx_1\ldots dx_n.
\end{aligned}
$$

Hence the theorem is proved by Lemma 1.

Remark. In application of (2.8) one should try to minimize the right-hand side in n for fixed ε. Also the remark after Theorem 1 is valid here.

Example. If $p > 0$, $a_k = k^{-\alpha}$ and $\alpha > 1$, then we have

$$
\log P\big(\sum_{k\geq 1} k^{-\alpha}|\xi_k|^p \leq \varepsilon^p\big) \leq -D_{\alpha,p}\cdot\varepsilon^{-p/(\alpha-1)}. \tag{2.9}
$$

In this case we have for n large,

$$
\begin{aligned}
&\big(\prod_{k=1}^{n} a_k\big)^{-1/p} = (n!)^{\alpha/p} \leq (2\sqrt{\pi})^{\alpha/p} n^{\alpha(n+1/2)/p} e^{-\alpha n/p};\\
&\Gamma\big(\frac{n}{p}+1\big)^{-1} \leq \pi^{-1/2}\big(\frac{n}{p}\big)^{-(n/p+1/2)}\cdot e^{n/p}.
\end{aligned}
$$

Thus by Theorem 2 for n large, we have

$$
\begin{aligned}
\log P\big(\sum_{k\geq 1} k^{-\alpha}|\xi_k|^p \leq \varepsilon^p\big) &\leq +\frac{n}{2}\log(2\pi^{-1}) + \frac{\alpha}{p}\log(2\sqrt{\pi}) + \frac{\alpha}{p}(n+\frac{1}{2})\log n - \frac{\alpha}{p}n\\
&\quad + n\log\varepsilon + n\log\Gamma\big(\frac{1}{p}+1\big) - \frac{1}{2}\log\pi - \big(\frac{n}{p}+\frac{1}{2}\big)\log\frac{n}{p} + \frac{n}{p}\\
&=\Big(\frac{\alpha}{p}\log(2\sqrt{\pi}) - \frac{1}{2}\log\pi + \frac{\alpha}{2p}\log n - \frac{1}{2}\log\frac{n}{p} - \frac{n}{2}\log(2\pi^{-1}) - \frac{\alpha-1}{p}n\Big)\\
&\quad + \Big(\frac{\alpha}{p}n\log n + n\log\varepsilon + n\log\Gamma\big(\frac{1}{p}+1\big) - \frac{n}{p}\log n + \frac{\log p}{p}n\Big)\\
&\leq\Big(\log\Gamma\big(\frac{1}{p}+1\big) + \frac{\log p}{p} + \log(\varepsilon n^{(\alpha-1)/p})\Big)n.
\end{aligned}
$$

Choose $n = [\delta_{\alpha,p}\varepsilon^{-p/(\alpha-1)}]$ where $\delta_{\alpha,p} > 0$ is a constant such that

$$
\log\Gamma\big(\frac{1}{p}+1\big) + \frac{\log p}{p} + \frac{\alpha-1}{p}\log\delta_{\alpha,p} \leq -1,
$$

then we obtain

$$
\log P\big(\sum_{k\geq 1} k^{-\alpha}|\xi_k|^p \leq \varepsilon^p\big) \leq -[\delta_{\alpha,p}\varepsilon^{-p/(\alpha-1)}] \leq -D_{\alpha,p}\varepsilon^{-p/(\alpha-1)}.
$$

Remark. Combining (2.7) and (2.9) as given in (2.1), we see that for the canonical example $a_k = k^{-\alpha}$, $\alpha > 1$, our upper and lower bound estimates are sharp at the logarithmic level (up to a constant) for ε small.

Remark. Cox[2] showed that genuinely infinite dimensional Brownian motions on l_p sequence spaces have natural rates of escape, provided the coordinates are independent. But finding the rate functions depends heavily on the sharp asymptotic estimates for (1.1). Our estimates given here are an attempt to serve this need. In particular, our (1.2) settles the conjecture 3 in Erickson[3]. Namely, the natural rate of escape for the process $Z_\beta = \sum_{k\geq 1} k^{-\beta} B_k(t) e_k, t \geq 0$ with respect to the l_p ($p \geq 2$) norm is given by

$$\gamma_{p,\beta}(t) = t^{1/2}/(\log\log t)^{(\beta-1)/p} \quad \text{i.e.} \quad \varliminf_{t\to\infty} \gamma_{p,\beta}(t)^{-1}\|Z_\beta\|_p = C(p,\beta) \quad a.s. \tag{2.10}$$

for $0 < C(p,\beta) < \infty$, provided $\alpha = \beta p > 1$. Here $\{B_k(t), t \geq 0\}_{k\geq 1}$ is a sequence of mutually independent one dimensional standard Brownian motions all defined on the same probability space and e_k is kth unit coordinate vector. The proof of (2.10) is routine (see Erickson[3]) if we have (1.2). So we omit it here.

At the end of Erickson's paper[3], it was added that the above mentioned conjecture had been solved by Cox[1]. Unfortunately, the lower bound arguments about (1.1) in Cox[1] contains a flaw.

3. Bounds for sup-norm. Let $X(t) = \sum_{k\geq 1} \lambda_k \phi_k(t) \xi_k$, $0 \leq t \leq 1$, $\sum_{k\geq 1} \lambda_k < \infty$ and $\lambda_k > 0$. Here $\{\phi_k(t) : 0 \leq t \leq 1\}$ is a sequence of functions satisfying the condition

$$\sup_{k\geq 1} \sup_{0\leq t\leq 1} |\phi_k(t)| \leq M < +\infty. \tag{3.1}$$

By the way we define $X(t)$, it is clear that $\sup_{0\leq t\leq 1} |X(t)| < \infty$ a.s. and $X(t)$ is a mean zero Gaussian process but not necessarily a stationary process. Our next result gives a lower bound of the lower tail of $X(t)$ under the sup-norm. This bound can be evaluated by using Theorem 1.

Theorem 3. *If there exist $x \in (0,1)$ independent of p such that*

$$Q_p = \Big(\sum_{k\geq 1} \lambda_k^{xp/(p-1)}\Big)^{(p-1)/p} \leq Q < \infty, \tag{3.2}$$

then for any $m > 0$, we have

$$P\big(\sup_{0\leq t\leq 1} |X(t)| \leq \varepsilon\big) \geq P\Big(\sum_{k\geq 1} \lambda_k^{(1-x)m} |\xi_k|^m \leq (Q^{-1}M^{-1}\varepsilon)^m\Big).$$

Proof. Using Hölder's inequality for $q = p/(p-1)$ and $q' = p$, we have

$$\begin{aligned}
|X(t)|^p &\leq \Big(\sum_{k\geq 1} \lambda_k |\phi_k(t)\xi_k|\Big)^p \leq \Big(\sum_{k\geq 1} \lambda_k^{xq}\Big)^{p/q} \cdot \Big(\sum_{k\geq 1} \lambda_k^{(1-x)q'} |\phi_k(t)\xi_k|^{q'}\Big)^{p/q'} \\
&= \Big(\sum_{k\geq 1} \lambda_k^{xp/(p-1)}\Big)^{p-1} \cdot \Big(\sum_{k\geq 1} \lambda_k^{(1-x)p} |\phi_k(t)\xi_k|^p\Big) \leq \sum_{k\geq 1} \big(QM\lambda_k^{1-x} |\phi_k(t)\xi_k|\big)^p
\end{aligned}$$

where the last inequality holds by (3.1) and (3.2). Hence

$$\begin{aligned}
P\big(\sup_{0\le t\le 1}|X(t)|\le\varepsilon\big) &= P\Big(\lim_{p\to\infty}\Big(\int_0^1 |X(t)|^p\,dt\Big)^{1/p}\le\varepsilon\Big)\\
&= \lim_{p\to\infty} P\Big(\int_0^1 |X(t)|^p\,dt\ \le\varepsilon^p\Big)\\
&\ge \lim_{p\to\infty} P\Big(\sum_{k\ge 1}\big(QM\lambda_k^{1-x}|\xi_k|\big)^p\le\varepsilon^p\Big)\\
&= \lim_{p\to\infty} P\Big(\sum_{k\ge 1}\big(QM\lambda_k^{1-x}|\xi_k|\varepsilon^{-1}M\big)^p\le 1\Big)\\
&\ge \lim_{p\to\infty} P\Big(\sum_{k\ge 1}\big(QM\lambda_k^{1-x}|\xi_k|\varepsilon^{-1}\big)^m\le 1\Big)\\
&= P\Big(\sum_{k\ge 1}\big(\lambda_k^{(1-x)m}|\xi_k|^m\big)\le (Q^{-1}M^{-1}\varepsilon)^m\Big)
\end{aligned}$$

which finishes our proof.

Remark. If $\{\phi_k(t)\}_{k\ge1}$ are some orthonormal basis in $L^2[0,1]$, then we have the following upper bound:

$$P\big(\sup_{0\le t\le 1}|X(t)|\le\varepsilon\big)\le P\Big(\int_0^1 X^2(t)\,dt\le\varepsilon^2\Big)=P\Big(\sum_{k\ge1}\lambda_k^2\xi_k^2\le\varepsilon^2\Big).$$

The behavior of $P(\sum_{k\ge1}\lambda_k^2\xi_k^2\le\varepsilon^2)$ as $\varepsilon\to 0$ can be found in Li[6] and the reference there.

Finally we carry out the following simple example.

Example. If $\lambda_k=k^{-\alpha}$, $\alpha>1$ and (3.1) holds, then for any $\delta>0$ small, we have

$$\log P\big(\sup_{0\le t\le 1}|X(t)|\le\varepsilon\big)\ge -C\Big(\frac{1}{\varepsilon}\Big)^{1/(\alpha-1-\delta)}$$

where $C=C(\delta)$ is a positive constant.

Let $x=(1+\delta/2)\alpha^{-1}<1$, then we have

$$Q_p=\Big(\sum_{k\ge1}k^{-\alpha xp/(p-1)}\Big)^{(p-1)/p}\le\sum_{k\ge1}k^{-\alpha xp/(p-1)}\le\sum_{k\ge1}k^{-\alpha x}=\sum_{k\ge1}k^{-(1+\delta/2)}=C_\delta.$$

Thus applying Theorem 3 with $m=2\delta^{-1}$ gives us

$$\begin{aligned}
&\log P\big(\sup_{0\le t\le 1}|X(t)|\le\varepsilon\big)\\
&\ge \log P\Big(\sum_{k\ge1}k^{-(2(\alpha-1)/\delta-1)}|\xi_k|^{2/\delta}\le (C_\delta^{-1}M^{-1}\varepsilon)^{2/\delta}\Big)\\
&\ge -C_{\alpha,\delta}(C_\delta^{-1}M^{-1}\varepsilon)^{1/(\alpha-1-\delta)}\\
&= -C\Big(\frac{1}{\varepsilon}\Big)^{1/(\alpha-1-\delta)}
\end{aligned}$$

where the last inequality is by using (2.7) for $p = 2\delta^{-1}$.

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References

1. Cox, D.D. (1980). Normalized Brownian motion on Banach spaces. Ph.D. thesis, University of Washington, Seattle.

2. Cox, D.D. (1982). On the existence of natural rate of escape functions for infinite dimensional Brownian motions. *Ann. Prob.* **10**, 623-638.

3. Erickson, K.B. (1980). Rates of escape of infinite dimensional Brownian motion. *Ann. Prob.* **8**, 325-338.

4. Hoffmann-Jørgensen, J., Shepp, L.A. and Dudley, R.M. (1979). On the lower tail of Gaussian seminorms. *Ann. Prob.* **7**, 319-342.

5. Kuelbs, J. (1978). Rates of growth for Banach space valued independent increments processes. *Proc. of the 2nd Oberwolfach conference on probability on Banach spaces.* Lecture Notes in Math. **709**, Springer, Berlin, pp. 151-169.

6. Li, W.V. (1992). Comparison results for the lower tail of Gaussian seminorms. *J. Theor. Probab.* **5**, 1-31.

7. Saint-Raymond, J. (1981). Sur le volume des corps convexes symetriques. *Seminaire d'Initation a l'Analyse.* Math. Univ. Pierre et Marie Curie, **46**, Univ. Paris VI, Paris, Exp. No 11.

Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706 USA