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Limit theorems for the square integral of Brownian motion and its increments

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Strassen's functional law of the iterated logarithm can be used to prove limit results about Brownian motion, but the limiting constants are given implicitly in many cases. In this paper, we provide a probabilistic method that can give the limiting constants explicitly for the square integral of Brownian motion and its increments.

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Brownian motion * law of the iterated logarithm * increments of Brownian motion * Karhunen-Loève expansion

1. Introduction

Let $\{W(t), t \geq 0\}$ be a standard Brownian motion. Strassen's (1964) functional law of the iterated logarithm implies that for any $\theta, 0 \leq \theta < 1$,

$$\limsup_{T\to\infty} \frac{1}{T^2 \log\log T} \int_{\theta T}^{T} W^2(t)\, dt = 2\lambda(\theta) \quad \text{a.s.} \tag{1.1}$$

and for any $\alpha, 0 < \alpha < 1$,

$$\limsup_{T\to\infty} \frac{1}{T^2 \log\log T} \int_{0}^{T-\alpha T} |W(t+\alpha T) - W(t)|^2\, dt = 2\tau(\alpha) \quad \text{a.s.} \tag{1.2}$$

where

$$\lambda(\theta) = \sup_{f\in K} \int_{\theta}^{1} f^2(t)\, dt, \tag{1.3}$$

$$\tau(\alpha) = \sup_{f\in K} \int_{0}^{1-\alpha} |f(t+\alpha) - f(t)|^2\, dt \tag{1.4}$$

and

$$K = \left\{ f : f(0) = 0, f \text{ is absolutely continuous on } [0, 1] \text{ and } \int_0^1 (f'(t))^2\, dt \leq 1 \right\}.$$

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In order to see how the sample path behavior changes when θ and α vary, it is desirable to find an explicit expression for $\lambda(\theta)$ and $\tau(\alpha)$ (see the discussion below). By using (1.3) and the calculus of variations, Strassen (1964) proves that $\lambda(0)=4/\pi^2$. To find $\lambda(\theta)$ and $\tau(\alpha)$ explicitly by using (1.3) and (1.4) seems hard (at least the method of calculus of variations does not seem to work). In general, evaluating the sup of some functional over K can be very hard (see Csáki and Révész, 1979; Hanson and Russo, 1989).

In this paper we find $\lambda(\theta)$, $0\leq\theta\leq 1$, and $\tau(\alpha)$, $\frac{1}{2}\leq\alpha\leq 1$, explicitly by a probabilistic argument. We have the following results.

Theorem 1. *Let $a(T)$ and $b(T)$ be non-decreasing functions of T for which*

$$a(T)<b(T), \qquad \lim_{T\to\infty} b(T)=\infty \quad \text{and} \quad \lim_{T\to\infty} a(T)/b(T)=\theta, \quad 0\leq\theta\leq 1.$$

Then

$$\limsup_{T\to\infty} \frac{1}{b^2(T)\log\log b(T)} \int_{a(T)}^{b(T)} W^2(t)\,\mathrm{d}t = 2\lambda(\theta) \quad a.s. \tag{1.5}$$

where $\lambda(\theta)$ is the largest solution of the equation

$$\theta \sin\frac{1-\theta}{\sqrt{x}} = \sqrt{x}\cos\frac{1-\theta}{\sqrt{x}}. \tag{1.6}$$

From (2) of Lemma 4 below, we have $\lambda(1)=0$ which means the normalizer in (1.5) is too big when $\theta=1$. Our next result provides the right normalizer when $\theta=1$.

Theorem 2. *Let $a(T)$ and $b(T)$ be non-decreasing functions of T for which*

$$a(T)<b(T), \qquad \lim_{T\to\infty} b(T)=\infty,$$

$$\lim_{t\to\infty} a(T)/b(T)=1 \quad \text{and} \quad b(T)/(b(T)-a(T))$$

is monotonically non-decreasing. Then

$$\limsup_{T\to\infty} \frac{1}{b(T)\log\log b(T)} \cdot \left(\frac{1}{b(T)-a(T)} \int_{a(T)}^{b(T)} W^2(t)\,\mathrm{d}t\right) = 2 \quad a.s. \tag{1.7}$$

About (1.2), we have the following result.

Theorem 3. *If $\frac{1}{2}\leq\alpha<1$, then*

$$\limsup_{T\to\infty} \frac{1}{T^2\log\log T} \int_0^{T-\alpha T} |W(t+\alpha T)-W(t)|^2\,\mathrm{d}t$$

$$=(1-\alpha)^2\zeta\left(\frac{\alpha}{1-\alpha}\right) \quad a.s. \tag{1.8}$$

where $\zeta(a)$ *is the largest solution of the equation*

$$\tan\frac{1}{\sqrt{x}}=\frac{\sqrt{x}}{2a-1} \quad \textit{for } a\geqslant 1. \tag{1.9}$$

The proof of Theorem 1 and part of Theorem 3 are based on the upper tail estimates of $\int_\theta^1 W^2(t)\,\mathrm{d}t$ and $\int_0^{1-\alpha}|W(t+\alpha)-W(t)|^2\,\mathrm{d}t$, inequalities for the measure of a translated ball, and a martingale form of the Borel-Cantelli lemma. This approach is adopted from Csáki (1981) which gives an upper-lower class result for $\int_0^T W^2(t)\,\mathrm{d}t$ as $T\to\infty$. The proof of Theorem 2 and part of Theorem 3 are based on the following well known result of Csörgő and Révész (1979) for the increments of a Brownian motion. It can be seen that the way we prove Theorem 2 is not going to work for Theorem 1.

Theorem A (Csörgő and Révész, 1979). *Let* $a_T (T\geqslant 0)$ *be a monotonically non-decreasing function of* T *for which* $0<a_T\leqslant T$ *and* T/a_T *is monotonically non-decreasing. Then*

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|W(t+a_T)-W(t)|=1 \quad \text{a.s.} \tag{1.10}$$

and

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(t+s)-W(t)|=1 \quad \text{a.s.} \tag{1.11}$$

where

$$\beta_T=\left(2a_T\left(\log\frac{T}{a_T}+\log\log T\right)\right)^{-1/2}. \qquad \square$$

In order to see what Theorem 1 and Theorem 2 tell us, note the following result which is similar to Theorem 1 and Theorem 2. Let $a(T)<b(T)$, $b(T)$ be non-decreasing functions of T and $\lim_{T\to\infty}b(T)=\infty$, then

$$\limsup_{T\to\infty}\frac{1}{(b(T)\log\log b(T))^{1/2}}\sup_{a(T)\leqslant t\leqslant b(T)}|W(t)|=\sqrt{2} \quad \text{a.s.} \tag{1.12}$$

To see that (1.12) holds note that

$$|W(b(T))|\leqslant\sup_{a(T)\leqslant t\leqslant b(T)}|W(t)|\leqslant\sup_{0\leqslant t\leqslant b(T)}|W(t)|$$

and that the law of the interated logarithm implies both of the following:

$$\limsup_{t\to\infty}\frac{1}{(b(T)\log\log b(T))^{1/2}}\sup_{0\leqslant t\leqslant b(T)}|W(t)|=\sqrt{2} \quad \text{a.s.} \tag{1.13}$$

and

$$\limsup_{T\to\infty}\frac{|W(b(T))|}{(b(T)\log\log b(T))^{1/2}}=\sqrt{2} \quad \text{a.s.} \tag{1.14}$$

Comparing (1.12) with (1.5) and (1.7), we can see that for the sup-norm, the normalizer only depends on the end point of the block $[a(T), b(T)]$ (that is $b(T)$), but for the L_2-norm, the normalizer depend on the end point of the block $[a(T), b(T)]$ and the length of the block.

Comparing (1.12) and (1.14) with (1.5) and (1.7), we can see the following. If the block $[a(T), b(T)]$ is short (i.e. $a(T)/b(T)\to 1$ as $T\to\infty$), then the upper limit of the L_2-average

$$\left(\frac{1}{b(T)-a(T)}\int_{a(T)}^{b(T)} W^2(t)\,\mathrm{d}t\right)^{1/2} \tag{1.15}$$

is the same as $\sup_{a(T)\le t\le b(T)}|W(t)|$ which tells us when $\sup_{a(T)\le t\le b(T)}|W(t)|$ get big, it will stay big on short block $[a(T), b(T)]$ or in another words it does not have time on short block $[a(T), b(T)]$ to get small. But if the block $[a(T), b(T)]$ is long (i.e. $a(T)/b(T)\to\theta<1$ as $T\to\infty$), then from (1.5),

$$\limsup_{T\to\infty}\frac{1}{b(T)\log\log b(T)}\cdot\left(\frac{1}{b(T)-a(T)}\int_{a(T)}^{b(T)} W^2(t)\,\mathrm{d}t\right) = \frac{2\lambda(\theta)}{1-\theta}\quad \text{a.s.} \tag{1.16}$$

where $\lambda(\theta)$ is the largest zero of the equation (1.6). By (4) of Lemma 4, $\lambda(\theta)/(1-\theta)$ is strictly increasing to 1 on $[0, 1]$. Hence the upper limit of (1.15) is strictly increasing in θ and smaller than the upper limit of $\sup_{a(T)\le t\le b(T)}|W(t)|$. This tells us the longer the block $[a(T), b(T)]$ is, the smaller the L_2-average is, which is intuitively clear.

To illustrate more about what Theorem 1 and Theorem 2 tell us, we give here the following examples:

Example 1. For $x\ge 0$, let $b(T)=(x+1)T$ and $a(T)=xT$, then our Theorem 1 says

$$\limsup_{T\to\infty}\frac{1}{T^2\log\log T}\int_{xT}^{(x+1)T} W^2(t)\,\mathrm{d}t = 2(x+1)^2\lambda\left(\frac{x}{x+1}\right)\quad \text{a.s.} \tag{1.17}$$

From (4) of Lemma 4, $(x+1)^2\lambda(x/(x+1))$ is a strictly increasing function. Hence (1.17) tells us that although the length of interval is the same, the integral of $W(t)$ square on the interval is increasing when the interval moves further away from 0.

Example 2. Let $b(T)=T$, $a(T)=T-\log T$ first and then $b(T)=T+\log T$, $a(T)=T$. Then by Theorem 2, we have

$$\limsup_{T\to\infty}\frac{1}{T\log T\log\log T}\int_{T-\log T}^{T} W^2(t)\,\mathrm{d}t = \limsup_{T\to\infty}\frac{1}{T\log T\log\log T}\int_{T}^{T+\log T} W^2(t)\,\mathrm{d}t = 2\quad \text{a.s.} \tag{1.18}$$

Example 3. Let $b(T)=\log T$ and $a(T)=0$ in Theorem 1, then

$$\limsup_{T\to\infty} \frac{1}{(\log T)^2 \log\log\log T} \int_0^{\log T} W^2(t)\,\mathrm{d}t = \frac{8}{\pi^2} \quad \text{a.s.} \tag{1.19}$$

Comparing (1.18) with (1.19), we can see how different the behavior of $W(t)$ on $[0, \log T]$ and $[T-\log T, T]$ or $[T, T+\log T]$ are.

Turning to Theorem 3, we rewrite (1.8) as

$$\limsup_{T\to\infty} \frac{1}{T\log\log T}\left(\frac{1}{T-\alpha T}\int_0^{T-\alpha T} |W(t+\alpha T)-W(t)|^2\,\mathrm{d}t\right)$$

$$= (1-\alpha)\zeta\left(\frac{\alpha}{1-\alpha}\right) \quad \text{a.s.}$$

By (4) of Lemma 5, $(1-\alpha)\zeta(\alpha/(1-\alpha))$ is strictly increasing to 2 on $[\frac{1}{2}, 1]$. This tells us that on the L_2-average, the longer we look at the increment, the larger the L_2-average increment is.

2. Lemmas

Throughout this section, we assume that ξ_n are independent and normally distributed with mean zero and variance 1. The following lemma was first proved by Zolotarev (1961). A more general form was proved by Hwang (1980).

Lemma 1. *If $\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots > 0$ and $\sum_{n\geq 1}\lambda_n < \infty$, then*

$$P\left(\sum_{n\geq 1} \lambda_n \xi_n^2 > y\right) \sim K \cdot y^{-1/2} \exp\left(-\frac{y}{2\lambda_1}\right) \quad \textit{as } y\to\infty$$

where

$$K = (2\pi^{-1}\lambda_1)^{1/2} \prod_{n=2}^{\infty} (1-\lambda_n/\lambda_1)^{-1/2}. \qquad \Box$$

The following two lemmas are basic for the proof of our theorems.

Lemma 2. *For $0 \leq \theta < 1$,*

$$P\left(\int_\theta^1 W^2(t)\,\mathrm{d}t \geq y\right) \sim K(\theta)\cdot y^{-1/2}\exp\left(-\frac{y}{2\lambda(\theta)}\right) \quad \textit{as } y\to\infty$$

where $K(\theta)$ is a constant and $\lambda(\theta)$ is the largest solution of the equation (1.6).

Proof. Since $W(t)$ is a Gaussian processes with mean zero and covariance function $r(s,t) = EW(s)W(t) = \min(s,t)$ for $s, t \in [\theta, 1]$, we have in distribution

$$\int_\theta^1 W^2(t)\,\mathrm{d}t = \sum_{n\geq 1} \lambda_n \xi_n^2, \quad \lambda_n > 0,$$

by the Karhunen–Loève expansion (see Ash and Gardner, 1975; Kac and Siergert, 1947). Here λ_n are the eigenvalues of the equation

$$\lambda f(t)=\int_\theta^1 r(s,t)f(s)\,\mathrm{d}s, \quad \theta\leqslant t\leqslant 1.$$

From the above equation, we obtain one boundary condition $\lambda f(\theta)=\int_\theta^1 \theta f(s)\,\mathrm{d}s$ and

$$\lambda f(t)=\int_\theta^t sf(s)\,\mathrm{d}s+\int_t^1 tf(s)\,\mathrm{d}s, \quad \theta\leqslant t\leqslant 1. \tag{2.1}$$

Differentiating (2.1), we obtain another boundary condition $f'(1)=0$ and

$$\lambda f'(t)=\int_t^b f(s)\,\mathrm{d}s, \quad \theta\leqslant t\leqslant 1. \tag{2.2}$$

Differentiating (2.2) again, we obtain $\lambda f''(t)+f(t)=0$. Hence

$$f(t)=c_1\sin\frac{t}{\sqrt{\lambda}}+c_2\cos\frac{t}{\sqrt{\lambda}} \tag{2.3}$$

where c_1 and c_2 are constants. Substituting (2.3) into the two boundary conditions and simplifying them yields

$$\left(\sqrt{\lambda}\sin\frac{\theta}{\sqrt{\lambda}}-\theta\cos\frac{\theta}{\sqrt{\lambda}}+\theta\cos\frac{1}{\sqrt{\lambda}}\right)c_1 +\left(\sqrt{\lambda}\cos\frac{\theta}{\sqrt{\lambda}}+\theta\sin\frac{\theta}{\sqrt{\lambda}}-\theta\sin\frac{1}{\sqrt{\lambda}}\right)c_2=0 \tag{2.4}$$

and

$$\left(\cos\frac{1}{\sqrt{\lambda}}\right)c_1-\left(\sin\frac{1}{\sqrt{\lambda}}\right)c_2=0. \tag{2.5}$$

In order to find nontrivial constants c_1 and c_2 i.e. $c_1^2+c_2^2\neq 0$, the determinant of equations (2.4) and (2.5) has to be zero, that is

$$\theta\sin\frac{1-\theta}{\sqrt{\lambda}}=\sqrt{\lambda}\cos\frac{1-\theta}{\sqrt{\lambda}}. \tag{2.6}$$

Hence we have by Lemma 1, as $y\to\infty$,

$$P\left(\int_\theta^1 W^2(t)\,\mathrm{d}t\geqslant y\right)=P\left(\sum_{n\geqslant 1}\lambda_n\xi_n^2\geqslant y\right)\sim K(\theta)\cdot y^{-1/2}\exp\left(-\frac{y}{2\lambda(\theta)}\right)$$

where $K(\theta)$ is a constant and $\lambda(\theta)$ is the largest λ that satisfies the equation (2.6). □

Lemma 3. *If $a\geqslant 1$, then*

$$P\left(\int_0^1 |W(t+a)-W(t)|^2\,\mathrm{d}t>y\right) \sim K(a)\cdot y^{-1/2}\exp\left(-\frac{y}{\zeta(a)}\right) \quad \textit{as } y\to\infty \tag{2.7}$$

where $\zeta(a)$ is the largest solution of the equation (1.9).

Proof. Let $X(t)=W(t+a)-W(t), t\geqslant 0$ and $a\geqslant 1$. Then $\{X(t): 0\leqslant t\leqslant 1\}$ is a Gaussian processes with mean zero and covariance function

$$r(s,t)=EX(s)X(t)=\max(0, a-|s-t|) \quad \text{for } s,t\in[0,1].$$

To find the eigenvalues associated with the covariance function $r(s,t)$ of $X(t)$, we need to solve the integral equation

$$\lambda f(t)=\int_0^1 r(s,t)f(s)\,\mathrm{d}s, \quad 0\leqslant t\leqslant 1.$$

That is, for $a\geqslant 1$,

$$\lambda f(t)=\int_0^t (a-t+s)f(s)\,\mathrm{d}s+\int_t^1 (a+t-s)f(s)\,\mathrm{d}s, \quad 0\leqslant t\leqslant 1. \tag{2.8}$$

We may differentiate (2.8) with respect to t to obtain

$$\lambda f'(t)=-\int_0^t f(s)\,\mathrm{d}s+\int_t^1 f(s)\,\mathrm{d}s. \tag{2.9}$$

Differentiate again to obtain $\lambda f''(t)=-2f(t)$. Hence

$$f(t)=c_1\sin\sqrt{2\lambda^{-1}}\,t+c_2\cos\sqrt{2\lambda^{-1}}\,t. \tag{2.10}$$

Setting $t=0$ in (2.8) and (2.9), we obtain boundary conditions

$$\lambda f(0)=\int_0^1 (a-s)f(s)\,\mathrm{d}s \quad \text{and} \quad \lambda f'(0)=\int_0^1 f(s)\,\mathrm{d}s. \tag{2.11}$$

Substituting (2.10) into (2.11) and simplifying yields

$$\left(a+(1-a)\cos\sqrt{\frac{2}{\lambda}}+\sqrt{\frac{\lambda}{2}}\sin\sqrt{\frac{2}{\lambda}}\right)c_1$$
$$+\left((a-1)\sin\sqrt{\frac{2}{\lambda}}-\sqrt{\frac{\lambda}{2}}\left(1+\cos\sqrt{\frac{2}{\lambda}}\right)\right)c_2=0$$

and

$$(1+\cos\sqrt{2\lambda^{-1}})c_1+(\sin\sqrt{2\lambda^{-1}})\,c_2=0.$$

In order that there are non-zero choices for c_1 and c_2, the determinant of the above two equations has to be zero. We obtain after some simplification

$$\left((2a-1)\sin\frac{1}{\sqrt{2\lambda}}-\sqrt{2\lambda}\cos\frac{1}{\sqrt{2\lambda}}\right)\cos\frac{1}{\sqrt{2\lambda}}=0. \tag{2.12}$$

Let $\frac{1}{2}\zeta(a)$ be the largest λ that satisfies (2.12). Then clearly $\zeta(a)$ is the largest solution of the equation (1.9). Hence by Lemma 1 we obtain (2.7). □

Note that we are unable to find a similar expression like (2.7) for $0<a<1$ due to the complexity of finding the largest eigenvalue of (2.8) for $0<a<1$. This is why our Theorem 3 just holds for $\frac{1}{2}\leqslant \alpha<1$.

The following two lemmas state the basic properties of $\lambda(\theta)$ and $\zeta(a)$.

Lemma 4. *Let $\lambda(\theta)$ be defined as in Theorem 1, then*

$$(1)\qquad \frac{\mathrm{d}\lambda(\theta)}{\mathrm{d}\theta}=-\frac{2\theta^2\lambda(\theta)}{\lambda(\theta)+\theta^2(1-\theta)},\qquad 0\leqslant\theta\leqslant 1;$$

(2) *$\lambda(\theta)$ is a continuous, strictly decreasing function on $[0,1]$. In particular, $\lambda(0)=4\pi^{-2}\geqslant\lambda(\theta)\geqslant\lambda(1)=\lim_{\theta\to 1}\lambda(\theta)=0$ by (2.6) and $\lambda(\theta/b)\to\lambda(\theta)$ as $b\to 1^+$;*

(3) *$\frac{1}{2}(1-\theta^2)>\lambda(\theta)>\theta(1-\theta)$ on $(0,1)$;*

(4) *$(1-\theta)^{-1}\lambda(\theta)$ is a strictly increasing function on $[0,1]$ and $\lim_{\theta\to 1}(1-\theta)^{-1}\lambda(\theta)=1$. In particular, $(x+1)^2\lambda(x/(1+x))$ is a strictly increasing function on $[0,\infty)$.*

Proof. By implicit differentiation of (2.6), we have

$$\sec^2\frac{1-\theta}{\sqrt{\lambda(\theta)}}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1-\theta}{\sqrt{\lambda(\theta)}}\right)=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sqrt{\lambda(\theta)}}{\theta}\right). \tag{2.13}$$

Note that from (2.6),

$$\sec^2\frac{1-\theta}{\sqrt{\lambda(\theta)}}=1+\tan^2\frac{1-\theta}{\sqrt{\lambda(\theta)}}=1+\frac{\lambda(\theta)}{\theta^2}. \tag{2.14}$$

Substituting (2.14) into (2.13) and simplifying, we see (1) holds and hence (2) holds.

By using the inequality $\tan x>x$ on $0<x<\frac{1}{2}\pi$ and (2.6), we have

$$\sqrt{\lambda(\theta)}/\theta>(1-\theta)/\sqrt{\lambda(\theta)},$$

which is our lower bound in (3). Now turn to the upper bound in (3). If $0\leqslant\theta\leqslant\frac{1}{3}$, then $\lambda(\theta)\leqslant 4\pi^{-2}\leqslant\frac{1}{2}(1-\theta^2)$ by (2). If $\frac{1}{3}<\theta<1$, then

$$(1-\theta)/\sqrt{\lambda(\theta)}<(1-\theta)/\sqrt{\theta(1-\theta)}<\sqrt{2}.$$

Hence, using the inequality $\tan x<2x/(2-x^2)$ on $(0,\sqrt{2})$ and (2.6), we obtain

$$\frac{\sqrt{\lambda(\theta)}}{\theta}=\tan\frac{1-\theta}{\sqrt{\lambda(\theta)}}<2\left(\frac{1-\theta}{\sqrt{\lambda(\theta)}}\right)\Big/\left(2-\frac{(1-\theta)^2}{\lambda(\theta)}\right)$$

which gives the upper bound in (3) after simplification.

Using (1) and the lower bound in (3), we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\lambda(\theta)}{1-\theta}\right)=(1-\theta)^{-2}\left(\lambda(\theta)+(1-\theta)\frac{\mathrm{d}\lambda(\theta)}{\mathrm{d}\theta}\right)$$

$$=(1-\theta)^{-2}\lambda(\theta)\cdot\frac{\lambda(\theta)-\theta^2(1-\theta)}{\lambda(\theta)+\theta^2(1-\theta)}>0.$$

Hence (4) follows. □

Lemma 5. *Let $\zeta(a)$ be defined as in Theorem 3, then*

(1) $$\frac{\mathrm{d}\zeta(a)}{\mathrm{d}a}=\frac{4\zeta^2(a)}{(2a-1)^2+2a\zeta(a)},\quad a\geqslant 1;$$

(2) *$\zeta(a)$ is a continuous, strictly increasing function on $[1,\infty)$;*

(3) *$2a-\frac{1}{2}>\zeta(a)>2a-1$ on $[1,\infty)$;*

(4) *$\zeta(a)/a$ is a strictly increasing function on $[1,\infty)$ and $\lim_{a\to\infty}\zeta(a)/a=2$. In particular, $(1-\alpha)\zeta(\alpha/(1-\alpha))$ is a strictly increasing function on $[\frac{1}{2},1)$ and*

$$\lim_{\alpha\to 1}(1-\alpha)\zeta\left(\frac{\alpha}{1-\alpha}\right)=2.$$

Proof. It is similar to the proof of Lemma 4. □

The next lemma is a version of the Borel–Cantelli lemma. A proof can be found in Donskar and Varadhan (1977).

Lemma 6. *Let F_k be an increasing sequence of σ-fields and $A_k\in F_k$.*
If $\sum_{k\geqslant 1}P(A_k/F_{k-1})=\infty$ a.s., then $P(A_k\ \text{i.o.})=1$. □

Our next lemma is a particular case of Theorem 2.1 in Hoffmann-Jørgensen, Shepp and Dudley (1979) which is a well known fact about the measure of the translated ball.

Lemma 7. *For any $b>a\geqslant 0$, $s>0$ and $z\in\mathbb{R}$,*

$$P\left(\int_a^b (W(t)+z)^2\,\mathrm{d}t\geqslant s\right)\geqslant P\left(\int_a^b W^2(t)\,\mathrm{d}t\geqslant s\right). \qquad \square$$

Using Lemma 7 and the basic properties of Brownian motion, we have the following lemma which is intuitively clear.

Lemma 8. *For any $b>a>c\geqslant 0$ and $s>0$,*

$$P\left(\int_a^b W^2(t)\,\mathrm{d}t\geqslant s\,\middle|\,W(c)\right)\geqslant P\left(\int_{a-c}^{b-c} W^2(t)\,\mathrm{d}t\geqslant s\right)\quad \text{a.s.}$$

Proof. For any $x<y$, let $G=\{x<W(c)<y\}$, then G is $W(c)$ measurable. By using Lemma 7 and the fact that standard Brownian motion has independent and stationary

increments, we have

$$\int_G P\left(\int_a^b W^2(t)\,\mathrm{d}t \geqslant s \,\middle|\, W(c)\right) \mathrm{d}P$$

$$= P\left(\int_a^b W^2(t)\,\mathrm{d}t \geqslant s,\, x < W(c) < y\right)$$

$$= \int_x^y P\left(\int_a^b W^2(t)\,\mathrm{d}t \geqslant s \,\middle|\, W(c) = z\right) \mathrm{d}P(W(c) < z)$$

$$= \int_x^y P\left(\int_a^b (W(t) - W(c) + z)^2\,\mathrm{d}t \geqslant s \,\middle|\, W(c) = z\right) \mathrm{d}P(W(c) < z)$$

$$= \int_x^y P\left(\int_a^b (W(t) - W(c) + z)^2\,\mathrm{d}t \geqslant s\right) \mathrm{d}P(W(c) < z)$$

$$\geqslant \int_x^y P\left(\int_a^b (W(t) - W(c))^2\,\mathrm{d}t \geqslant s\right) \mathrm{d}P(W(c) < z)$$

$$= P\left(\int_{a-c}^{b-c} W^2(t)\,\mathrm{d}t \geqslant s\right) \cdot P(G)$$

where the forth equality is by the vector form of Corollary 4.38 in Breiman (1968). Hence by the monotone class theorem, the lemma is proved. □

Lemma 9. *For $b > a > 0$, $d > 0$ and $s > 0$, we have*

$$P\left(\int_0^b |W(t+d) - W(t)|^2\,\mathrm{d}t \geqslant s \,\middle|\, W(a)\right)$$

$$\geqslant P\left(\int_0^1 \left|W\left(t + \frac{d}{b-a}\right) - W(t)\right|^2 \mathrm{d}t \geqslant \frac{s}{(b-a)^2}\right) \quad \text{a.s.}$$

Proof. Note that $W(t)$ has independent and stationary increments and that as stochastic processes $W((b-a)t) = \sqrt{b-a}\; W(t)$. Hence we can proceed as follows.

$$P\left(\int_0^b |W(t+d) - W(t)|^2\,\mathrm{d}t \geqslant s \,\middle|\, W(a)\right)$$

$$\geqslant P\left(\int_a^b |W(t+d) - W(t)|^2\,\mathrm{d}t \geqslant s \,\middle|\, W(a)\right) \quad \text{a.s.}$$

$$= P\left(\int_a^b |W(t+d) - W(t)|^2\,\mathrm{d}t \geqslant s\right)$$

$$= P\left(\int_0^{b-a} |W(t+d) - W(t)|^2\,\mathrm{d}t \geqslant s\right)$$

$$= P\left(\int_0^1 \left|W\left(t + \frac{d}{b-a}\right) - W(t)\right|^2 \mathrm{d}t \geqslant \frac{s}{(b-a)^2}\right). \qquad \square$$

3. Proof of the theorems

We use C to denote a finite positive constant whose precise value is unimportant and which may be different in each statement or equation in this section. Before we prove Theorem 1 and Theorem 2, let us observe the following reduction. First, without loss of generality we can assume $b(T)$ is a continuous and strictly increasing function since clearly we can find two continuous and strictly increasing functions $b_1(T)$ and $b_2(T)$ such that

$$b_1(T) \leq b(T) \leq b_2(T) \quad \text{and} \quad \lim_{T\to\infty} a(T)/b_i(T) = \theta, \quad i = 1, 2.$$

Second, we only need to consider $b(T) = T$. That is, since $b(T)$ is a continuous and strictly increasing function, we have

$$\limsup_{T\to\infty} \frac{1}{b^2(T)\log\log b(T)} \int_{a(T)}^{b(T)} W^2(t)\,dt$$
$$= \limsup_{S\to\infty} \frac{1}{S^2 \log\log S} \int_{a(b^{-1}(S))}^{S} W^2(t)\,dt \quad \text{a.s.}$$

and

$$\lim_{S\to\infty} \frac{a(b^{-1}(S))}{S} = \lim_{S\to\infty} \frac{a(b^{-1}(S))}{b(b^{-1}(S))} = \theta.$$

Third, for $0 < \theta < 1$, we only need to show the Theorem 1 for $b(T) = T$, $a(T) = \theta T$ since for any $\delta > 0$, $1 > (\theta+\delta) > (\theta-\delta) > 0$, we then have

$$\limsup_{T\to\infty} \frac{1}{T^2 \log\log T} \int_{(\theta+\delta)T}^{T} W^2(t)\,dt = 2\lambda(\theta+\delta)$$
$$\leq \limsup_{T\to\infty} \frac{1}{T^2 \log\log T} \int_{a(T)}^{T} W^2(t)\,dt$$
$$\leq \limsup_{T\to\infty} \frac{1}{T^2 \log\log T} \int_{(\theta-\delta)T}^{T} W^2(t)\,dt = 2\lambda(\theta-\delta) \quad \text{a.s.}$$

Proof of Theorem 1. Let us first show for $0 < \theta < 1$,

$$\limsup_{T\to\infty} \frac{1}{T^2 \log\log T} \int_{\theta T}^{T} W^2(t)\,dt = 2\lambda(\theta) \quad \text{a.s.} \tag{3.1}$$

For any $\varepsilon > 0$, let $T_k = b^k$, $b > 1$. In order to show

$$\limsup_{T\to\infty} \frac{1}{T^2 \log\log T} \int_{\theta T}^{T} W^2(t)\,dt \leq 2\lambda(\theta)\cdot(1+\varepsilon) \quad \text{a.s.} \tag{3.2}$$

it suffices to show that

$$\sum_{k\geq 1} P\left(\int_{\theta T_k}^{T_{k+1}} W^2(t)\,dt \geq 2\lambda(\theta)(1+\varepsilon)T_k^2 \log\log T_k\right) < \infty. \tag{3.3}$$

By Lemma 2, we can estimate as following for k large:

$$P\left(\int_{\theta T_k}^{T_{k+1}} W^2(t)\,dt \geq 2\lambda(\theta)(1+\varepsilon)T_k^2 \log\log T_k\right)$$
$$= P\left(T_{k+1}^2 \int_{\theta T_k/T_{k+1}}^{1} W^2(t)\,dt \geq 2\lambda(\theta)(1+\varepsilon)T_k^2 \log\log T_k\right)$$
$$= P\left(\int_{\theta/b}^{1} W^2(t)\,dt \geq 2\lambda(\theta)(1+\varepsilon)b^{-2} \log\log b^k\right)$$
$$\leq C \exp\left(-\frac{\lambda(\theta)}{b^2\lambda(\theta/b)}(1+\varepsilon)\log\log b^k\right).$$

From (2) of Lemma 4, we can choose $b>1$ close to 1 such that

$$\frac{\lambda(\theta)}{b^2\lambda(\theta/b)}(1+\varepsilon) > 1+\tfrac{1}{2}\varepsilon.$$

Hence we conclude (3.3).

Now for $\varepsilon>0$, define $T_k = b^k$ where $b\theta>1$ and consider the events

$$A_k = \left\{\int_{\theta T_k}^{T_k} W^2(t)\,dt \geq 2\lambda(\theta)(1-\varepsilon)T_k^2 \log\log T_k\right\}.$$

We show by Lemma 6 that $P(A_k \text{ i.o.})=1$ which, together with (3.2), obviously implies (3.1). Let F_k be the σ-field generated by $W(t)$, $t\leq T_k$, then $A_k\in F_k$. And by Lemma 8, Lemma 2 and the fact that $(\theta T_k - T_{k-1})/(T_k - T_{k-1}) \leq \theta$, we have

$$P(A_k \mid F_{k-1}) = P(A_k \mid W(T_{k-1}))$$
$$= P\left(\int_{\theta T_k}^{T_k} W^2(t)\,dt \geq 2\lambda(\theta)(1-\varepsilon)T_k^2 \log\log T_k \,\Big|\, W(T_{k-1})\right)$$
$$\geq P\left(\int_{\theta T_k - T_{k-1}}^{T_k - T_{k-1}} W^2(t)\,dt \geq 2\lambda(\theta)(1-\varepsilon)T_k^2 \log\log T_k\right) \quad \text{a.s.}$$
$$= P\left(\int_{(\theta T_k - T_{k-1})/(T_k - T_{k-1})}^{1} W^2(t)\,dt \geq 2\lambda(\theta)(1-\varepsilon)\frac{T_k^2}{(T_k - T_{k-1})^2}\log\log T_k\right)$$
$$\geq P\left(\int_{\theta}^{1} W^2(t)\,dt \geq 2\lambda(\theta)(1-\varepsilon)\frac{b^2}{(b-1)^2}\log\log b^k\right)$$
$$\geq C\left(2\lambda(\theta)(1-\varepsilon)\frac{b^2}{(b-1)^2}\log\log b^k\right)^{-1/2}$$
$$\times \exp\left(-(1-\varepsilon)\frac{b^2}{(b-1)^2}\log\log b^k\right).$$

Hence $\sum_{k\geq 1} P(A_k/F_{k-1}) = \infty$ a.s. if we choose $b>\theta^{-1}$ large such that $(1-\varepsilon)b^2/(b-1)^2 < 1-\frac{1}{2}\varepsilon$. This in turn shows our Theorem 1 for $0<\theta<1$.

If $\theta=1$, using the result for $0<\theta<1$, we have for any $\varepsilon>0$,

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T^2\log\log T}\int_{a(T)}^{T}W^2(t)\,dt\\
&\leqslant \limsup_{T\to\infty}\frac{(1+\varepsilon)^2\log\log(1+\varepsilon)T}{\log\log T}\\
&\quad\times\frac{1}{((1+\varepsilon)T)^2\log\log(1+\varepsilon)T}\int_{a(T)}^{(1+\varepsilon)T}W^2(t)\,dt\\
&=(1+\varepsilon)^2\limsup_{T\to\infty}\frac{1}{((1+\varepsilon)T)^2\log\log(1+\varepsilon)T}\int_{a(T)}^{(1+\varepsilon)T}W^2(t)\,dt\\
&=(1+\varepsilon)^2\cdot 2\lambda\left(\frac{1}{1+\varepsilon}\right)\quad\text{a.s.}
\end{aligned}$$

Let $\varepsilon\to 0$, we proved Theorem 1 for $\theta=1$ by (2) of Lemma 4.

If $\theta=0$, we have for $\varepsilon>0$,

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T^2\log\log T}\int_{0}^{T}W^2(t)\,dt=\frac{8}{\pi^2}\\
&\geqslant\limsup_{T\to\infty}\frac{1}{T^2\log\log T}\int_{a(T)}^{T}W^2(t)\,dt\\
&\geqslant\limsup_{T\to\infty}\frac{1}{T^2\log\log T}\int_{\varepsilon T}^{T}W^2(t)\,dt=2\lambda(\varepsilon)\quad\text{a.s.}
\end{aligned}$$

where the first equality is from Strassen (1964) and the second equality is from (3.1). Let $\varepsilon\to 0$, we proved Theorem 1 for $\theta=0$ by (2) of Lemma 4. □

Proof of Theorem 2. As pointed out at the beginning of this section, we only need to consider $b(T)=T$ and $a(T)=T-l_T$ where T/l_T is monotonically non-decreasing and $T/l_T\to\infty$ as $T\to\infty$. First show that

$$\limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\left|\sup_{T-l_T\leqslant t\leqslant T}|W(t)|-\inf_{T-l_T\leqslant t\leqslant T}|W(t)|\right|=0\quad\text{a.s.}\tag{3.4}$$

Note that

$$\left|\sup_{T-l_T\leqslant t\leqslant T}|W(t)|-\inf_{T-l_T\leqslant t\leqslant T}|W(t)|\right|\leqslant\sup_{0\leqslant t\leqslant(T+l_T)-l_T}\ \sup_{0\leqslant s\leqslant l_T}|W(t+s)-W(t)|$$

and by Theorem A,

$$\limsup_{T\to\infty}\gamma_T\sup_{0\leqslant t\leqslant(T+l_T)-l_T}\ \sup_{0\leqslant s\leqslant l_T}|W(t+s)-W(t)|=1\quad\text{a.s.}$$

where

$$\gamma_T=(2l_T(\log(Tl_T^{-1}+1)+\log\log(T+l_T)))^{-1/2}.$$

Hence we obtain (3.4) easily by observing $\sqrt{T\log\log T}\cdot\gamma_T\to\infty$ as $T\to\infty$.

Now we can conclude Theorem 2 by (1.10), (3.4) and the following estimates:

$$\limsup_{T\to\infty}\left(\frac{1}{l_T T\log\log T}\int_{T-l_T}^{T} W^2(t)\,dt\right)^{1/2}$$

$$\geq \limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\inf_{T-l_T\leq t\leq T}|W(t)|$$

$$\geq \limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\sup_{T-l_T\leq t\leq T}|W(t)|$$

$$-\limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\left|\sup_{T-l_T\leq t\leq T}|W(t)|-\inf_{T-l_T\leq t\leq T}|W(t)|\right|$$

$$=\limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\sup_{T-l_T\leq t\leq T}|W(t)|=\sqrt{2}\quad\text{a.s.}$$

and

$$\limsup_{T\to\infty}\left(\frac{1}{l_T T\log\log T}\int_{T-l_T}^{T} W^2(t)\,dt\right)^{1/2}$$

$$\leq \limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\sup_{T-l_T\leq t\leq T}|W(t)|=\sqrt{2}\quad\text{a.s.}\qquad\square$$

Proof of Theorem 3. Let us first show

$$\limsup_{T\to\infty}\frac{1}{T^2\log\log T}\int_0^{(1-\alpha)T}|W(t+\alpha T)-W(t)|^2\,dt$$

$$\leq (1-\alpha)^2\zeta\left(\frac{\alpha}{1-\alpha}\right)\quad\text{a.s.}\tag{3.5}$$

Considering a subsequence $T_k=b^k$, $b>1$ and $\varepsilon>0$, we have by Lemma 3,

$$P\left(\int_0^{(1-\alpha)T_k}|W(t+\alpha T_k)-W(t)|^2\,dt\right.$$

$$\left.\geq (1-\alpha)^2\zeta\left(\frac{\alpha}{1-\alpha}\right)(1+\varepsilon)T_k^2\log\log T_k\right)$$

$$=P\left(\int_0^1\left|W\left(t+\frac{\alpha}{1-\alpha}\right)-W(t)\right|^2dt\geq\zeta\left(\frac{\alpha}{1-\alpha}\right)(1+\varepsilon)\log\log T_k\right)$$

$$\leq C\exp(-(1+\varepsilon)\log\log T_k)=C(k\log b)^{-(1+\varepsilon)}.$$

Hence by the Borel-Cantelli Lemma,

$$\limsup_{k\to\infty}\frac{1}{T_k^2\log\log T_k}\int_0^{(1-\alpha T_k}|W(t+\alpha T_k)-W(t)|^2\,dt$$

$$\leq (1-\alpha)^2\zeta\left(\frac{\alpha}{1-\alpha}\right)\quad\text{a.s.}\tag{3.6}$$

Now we can conclude Theorem 2 by (1.10), (3.4) and the following estimates:

$$\limsup_{T\to\infty}\left(\frac{1}{l_T T\log\log T}\int_{T-l_T}^{T} W^2(t)\,\mathrm{d}t\right)^{1/2}$$

$$\geqslant \limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\inf_{T-l_T\leqslant t\leqslant T}|W(t)|$$

$$\geqslant \limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\sup_{T-l_T\leqslant t\leqslant T}|W(t)|$$

$$-\limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\left|\sup_{T-l_T\leqslant t\leqslant T}|W(t)|-\inf_{T-l_T\leqslant t\leqslant T}|W(t)|\right|$$

$$=\limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\sup_{T-l_T\leqslant t\leqslant T}|W(t)|=\sqrt{2}\quad \text{a.s.}$$

and

$$\limsup_{T\to\infty}\left(\frac{1}{l_T T\log\log T}\int_{T-l_T}^{T} W^2(t)\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant \limsup_{T\to\infty}\frac{1}{\sqrt{T\log\log T}}\sup_{T-l_T\leqslant t\leqslant T}|W(t)|=\sqrt{2}\quad \text{a.s.} \qquad \square$$

Proof of Theorem 3. Let us first show

$$\limsup_{T\to\infty}\frac{1}{T^2\log\log T}\int_0^{(1-\alpha)T}|W(t+\alpha T)-W(t)|^2\,\mathrm{d}t$$

$$\leqslant (1-\alpha)^2\zeta\left(\frac{\alpha}{1-\alpha}\right)\quad \text{a.s.} \tag{3.5}$$

Considering a subsequence $T_k=b^k$, $b>1$ and $\varepsilon>0$, we have by Lemma 3,

$$P\left(\int_0^{(1-\alpha)T_k}|W(t+\alpha T_k)-W(t)|^2\,\mathrm{d}t\right.$$

$$\left.\geqslant (1-\alpha)^2\zeta\left(\frac{\alpha}{1-\alpha}\right)(1+\varepsilon)T_k^2\log\log T_k\right)$$

$$=P\left(\int_0^1\left|W\left(t+\frac{\alpha}{1-\alpha}\right)-W(t)\right|^2\mathrm{d}t\geqslant \zeta\left(\frac{\alpha}{1-\alpha}\right)(1+\varepsilon)\log\log T_k\right)$$

$$\leqslant C\exp(-(1+\varepsilon)\log\log T_k)=C(k\log b)^{-(1+\varepsilon)}.$$

Hence by the Borel-Cantelli Lemma,

$$\limsup_{k\to\infty}\frac{1}{T_k^2\log\log T_k}\int_0^{(1-\alpha T_k}|W(t+\alpha T_k)-W(t)|^2\,\mathrm{d}t$$

$$\leqslant (1-\alpha)^2\zeta\left(\frac{\alpha}{1-\alpha}\right)\quad \text{a.s.} \tag{3.6}$$

Now note the fact that if $b^k = T_k \leqslant T_{k+1} = b^{k+1}$, $b > 1$, then

$$\begin{aligned} Z(t) &= |W(t+\alpha T) - W(t)| \\ &\leqslant |W(t+\alpha T_{k+1}) - W(t)| + |W(t+\alpha T_{k+1}) - W(t+\alpha T)| \\ &\leqslant |W(t+\alpha T_{k+1}) - W(t)| + \sup_{0\leqslant s\leqslant \alpha(T_{k+1}-T)} |W(t+\alpha T+s) - W(t+\alpha T)| \\ &\leqslant |W(t+\alpha T_{k+1}) - W(t)| + \sup_{0\leqslant s\leqslant \alpha(b-1)T} |W(t+\alpha T+s) - W(t+\alpha T)| \\ &= X(t) + Y(t+\alpha T) \end{aligned} \tag{3.7}$$

where

$$X(t) = |W(t+\alpha T_{k+1}) - W(t)| \quad \text{and} \quad Y(t) = \sup_{0\leqslant s\leqslant \alpha(b-1)T} |W(t+s) - W(t)|.$$

By Theorem A and observing that for $b^k = T_k \leqslant T < T_{k+1} = b^{k+1}$, $b > 1$,

$$\limsup_{T\to\infty} \frac{T_{k+1}\log\log T_{k+1}}{T\log\log T} \leqslant \limsup_{T\to\infty} \frac{T_{k+1}\log\log T_{k+1}}{T_k\log\log T_k} = b, \tag{3.8}$$

we have

$$\limsup_{T\to\infty} \frac{1}{\sqrt{T\log\log T}} \sup_{0\leqslant t\leqslant(1-\alpha)T} Z(t) = \sqrt{2\alpha} \quad \text{a.s.}, \tag{3.9}$$

$$\limsup_{T\to\infty} \frac{1}{\sqrt{T\log\log T}} \sup_{0\leqslant t\leqslant(1-\alpha)T_{k+1}} X(t) \leqslant \sqrt{2\alpha b} \quad \text{a.s.} \tag{3.10}$$

and

$$\limsup_{T\to\infty} \frac{1}{\sqrt{T\log\log T}} \sup_{0\leqslant t\leqslant T} Y(t) = \sqrt{2\alpha(b-1)} \quad \text{a.s.} \tag{3.11}$$

From (3.77), we have $Z^2(t) \leqslant X^2(t) + (X(t)+Z(t))Y(t+\alpha T)$. Hence

$$\begin{aligned} \int_0^{(1-\alpha)T} &|W(t+\alpha T) - W(t)|^2\,\mathrm{d}t = \int_0^{(1-\alpha)T} Z^2(t)\,\mathrm{d}t \\ &\leqslant \int_0^{(1-\alpha)T} X^2(t)\,\mathrm{d}t + \int_0^{(1-\alpha)T} (X(t)+Z(t))Y(t+\alpha T)\,\mathrm{d}t \\ &\leqslant \int_0^{(1-\alpha)T_{k+1}} X^2(t)\,\mathrm{d}t \\ &\quad + T\left(\sup_{0\leqslant t\leqslant(1-\alpha)T} X(t) + \sup_{0\leqslant t\leqslant(1-\alpha)T} Z(t)\right) \sup_{0\leqslant t\leqslant(1-\alpha)T} Y(t+\alpha T) \\ &\leqslant \int_0^{(1-\alpha)T_{k+1}} X^2(t)\,\mathrm{d}t \\ &\quad + T\left(\sup_{0\leqslant t\leqslant(1-\alpha)T_{k+1}} X(t) + \sup_{0\leqslant t\leqslant(1-\alpha)T} Z(t)\right) \sup_{0\leqslant t\leqslant T} Y(t). \end{aligned} \tag{3.12}$$

Thus we conclude (3.5) by combining (3.6), (3.8), (3.9), (3.10), (3.11) and (3.12).

Turning to the other half, we define $T_k = b^k$, $b(1-\alpha) > 1$ and for $\varepsilon > 0$ the events

$$A_k = \left\{ \int_0^{(1-\alpha)T_k} |W(t+\alpha T_k) - W(t)|^2 \, dt \right.$$

$$\left. \geqslant (1-\alpha)^2 \zeta\left(\frac{\alpha}{1-\alpha}\right)(1-\varepsilon) T_k^2 \log\log T_k \right\}.$$

We show by Lemma 6 that $P(A_k \text{ i.o.}) = 1$ which, together with (3.5), obviously implies (1.8). Let F_k be the σ-field generated by $W(t)$, $t \leqslant T_k$. Then $A_k \in F_k$. By Lemma 9, Lemma 3 and the fact that $\alpha b/((1-\alpha)b-1) > \frac{1}{2}$, we have

$$\begin{aligned} P(A_k \mid F_{k-1}) &= P(A_k \mid W(T_{k-1})) \\ &\geqslant P\left(\int_0^1 \left| W\left(t + \frac{\alpha T_k}{(1-\alpha)T_k - T_{k-1}}\right) - W(t)\right|^2 dt \right. \\ &\qquad \left. \geqslant \zeta\left(\frac{\alpha}{1-\alpha}\right) \frac{(1-\varepsilon)T_k^2 \log\log T_k}{(T_k - (1-\alpha)^{-1}T_{k-1})^2}\right) \quad \text{a.s.} \\ &= P\left(\int_0^1 \left| W\left(t + \frac{\alpha b}{(1-\alpha)b-1}\right) - W(t)\right|^2 dt \right. \\ &\qquad \left. \geqslant \zeta\left(\frac{\alpha}{1-\alpha}\right)(1-\varepsilon) \frac{((1-\alpha)b)^2}{((1-\alpha)b-1)^2} \log\log b^k\right) \\ &\geqslant C \exp\left(-(1-\varepsilon) \cdot \left(\zeta\left(\frac{\alpha}{1-\alpha}\right) \Big/ \zeta\left(\frac{\alpha b}{(1-\alpha)b-1}\right)\right) \right. \\ &\qquad \left. \cdot \frac{((1-\alpha)b)^2}{((1-\alpha)b-1)^2} \log\log b^k\right). \end{aligned}$$

From (2) of Lemma 5, we can choose $b > (1-\alpha)^{-1}$ large such that

$$(1-\varepsilon) \cdot \left(\zeta\left(\frac{\alpha}{1-\alpha}\right) \Big/ \zeta\left(\frac{\alpha b}{(1-\alpha)b-1}\right)\right) \cdot \frac{((1-\alpha)b)^2}{((1-\alpha)b-1)^2} < 1 - \tfrac{1}{2}\varepsilon.$$

Hence $\sum_{k \geqslant 1} P(A_k \mid F_{k-1}) = \infty$ a.s. which in turn shows our Theorem 3. □

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