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Comparison Results for the Lower Tail of Gaussian Seminorms

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Let $\xi = (\xi_n)$ be i.i.d. $N(0, 1)$ random variables and $q(x)$, $q'(x)$: $R^\infty \to [0, \infty)$ be seminorms. We investigate necessary and sufficient conditions that the ratio of $P(q(\xi) < \varepsilon)$ and $P(q'(\xi) < \varepsilon)$ goes to a positive constant as $\varepsilon \to 0^+$. We give satisfactory answers for l_2-norms and also some results for sup-norms and l_p-norms. Some applications are given to the rate of escape of infinite dimensional Brownian motion, and we give the lower tail of the Ornstein–Uhlenbeck process and a weighted Brownian bridge under the L_2-norms.

KEY WORDS: Lower tail; Gaussian seminorms; Gaussian processes; infinite dimensional Brownian motion.

1. INTRODUCTION

Let $\xi = (\xi_n)$ or $\xi = (\xi_{ij})$ be a sequence of independent Gaussian, mean 0, variance 1, random variables throughout this paper. We shall study the so called lower tail or small deviations of seminorms, that is, the asymptotic behavior of $P(q(\xi) \leqslant \varepsilon)$ as $\varepsilon \to 0^+$ where q: $R^\infty \to R_+ = [0, \infty)$ is a seminorm. There are several reasons that we are interested in this problem. For example, finding the rate of escape of infinite dimensional Brownian motion comes down to finding the lower tail of $q(x)$ (see Erickson[8]). Beyond the study of $P(q(\xi) \leqslant \varepsilon)$ as $\varepsilon \to 0^+$, there has been considerable work regarding $P(q(\xi) > y)$ as $y \to \infty$. For example, the precise asymptotic behavior is given in Zolotarev[21] and Hwang[11] when $q(\cdot)$ is an l_2-norm, and using the exponential integrability result of Fernique[9] one always obtains upper bounds of a Gaussian nature for $P(q(\xi) > y)$ regardless of the form of $q(\cdot)$. Furthermore, the asymptotic behavior of $P(q(\xi) > y)$ as

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$y \to \infty$ at the logarithmic level can be found in Marcus and Shepp[18] and Borell.[4]

Let us consider l_2-norms first, that is,

$$q(x) = \left(\sum_{n \geq 1} a_n x_n^2 \right)^{1/2} \qquad \text{for any} \quad x = (x_n) \in R^\infty$$

where (a_n) is a given sequence of positive numbers and $\sum_{n \geq 1} a_n < \infty$. The setting here actually comes from the following general problem. Considering a Gaussian process $\{X(t): a \leqslant t \leqslant b\}$ with mean zero and covariance $r(s, t) = E(X(s)\, X(t))$ for $s, t \in [a, b]$, we need to know the asymptotic behavior of

$$P\left(\int_a^b X^2(t)\, dt \leqslant \varepsilon^2 \right) \qquad \text{as} \quad \varepsilon \to 0$$

in some problems of probability theory and mathematical statistics. By the Karhunen–Loéve expansion, we have in distribution $\int_a^b X^2(t)\, dt = \sum_{n \geq 1} \lambda_n \xi_n^2$ where $\lambda_n > 0$ for $n \geq 1$, $\sum_{n \geq 1} \lambda_n < \infty$, is the eigenvalue of the equation

$$\lambda f(t) = \int_a^b r(s, t)\, f(s)\, ds \qquad a \leqslant t \leqslant b$$

Thus the problem reduces to finding the asymptotic behavior of $P(\sum_{n \geq 1} a_n \xi_n^2 \leqslant \varepsilon^2)$ as $\varepsilon \to 0$ where $a_n > 0$ and $\sum_{n \geq 1} a_n < \infty$. Theoretically, the problem has been solved by Sytaya.[19] Namely,

Theorem 1. (See Ref. 19.) If $a_n > 0$ and $\sum_{n \geq 1} a_n < \infty$, then

$$P\left(\sum_{n \geq 1} a_n \xi_n^2 \leqslant \varepsilon^2 \right) \sim Q_a(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0^+$$

where

$$Q_a(\varepsilon) = \left(4\pi \sum_{n \geq 1} \left(\frac{a_n \gamma_a}{1 + 2a_n \gamma_a} \right)^2 \right)^{-1/2} \cdot \exp\left(\varepsilon^2 \gamma_a - \frac{1}{2} \sum_{n \geq 1} \log(1 + 2a_n \gamma_a) \right)$$

and $\gamma_a = \gamma_a(\varepsilon)$ is uniquely determined by the following equation for $\varepsilon > 0$ small enough:

$$\varepsilon^2 = \sum_{n \geq 1} \frac{a_n}{1 + 2a_n \gamma_a}$$

Here and throughout this paper, the notation $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ as $x \to 0$ or $x \to \infty$. Note that the given asymptotic behavior is still an implicit expression that is highly inconvenient for concrete computations and applications. This is primarily due to the series form for $Q_a(\varepsilon)$ and the implicit relation between ε and γ in Theorem 1.

A number of authors, Hoffman-Jørgensen et al.,[10] Csáki,[7] Cox,[6] Ibragimov,[12] and Zolotarev,[22] have tried to find the asymptotic behavior of $P(\sum_{n\geqslant 1} a_n \xi_n^2 \leqslant \varepsilon^2)$ as $\varepsilon \to 0$, or upper and lower bounds for ε small for some particular a_n, after the work of Sytaya[19] because of the difficulties in applying Theorem 1. Most of the results of these papers involve difficult calculations that most often depend very much on special properties of the sequence (a_n). As a result, we thought it would be of interest if a comparison result could be established, and this is one of our main considerations in this paper. We also investigate comparison results for the sup-norms and the l_p-norms $(1 \leqslant p < \infty)$, but here we can say less. However, as the examples in Section 4 demonstrate, the comparison results in l_2 are a very useful computational tool.

In Section 2, we give one of our main results of this paper, which is to establish a comparison of $P(\sum_{n\geqslant 1} a_n \xi_n^2 \leqslant \varepsilon^2)$ and $P(\sum_{n\geqslant 1} b_n \xi_n^2 \leqslant \varepsilon^2)$ as $\varepsilon \to 0$ where a_n, b_n are positive and $\sum_{n\geqslant 1} a_n < \infty$, $\sum_{n\geqslant 1} b_n < \infty$. Namely:

Theorem 2. If $\sum_{n\geqslant 1} |1 - a_n/b_n| < \infty$, then

$$P\left(\sum_{n\geqslant 1} a_n \xi_n^2 \leqslant \varepsilon^2\right) \sim \left(\prod b_n/a_n\right)^{1/2} P\left(\sum_{n\geqslant 1} b_n \xi_n^2 \leqslant \varepsilon^2\right) \qquad \text{as} \quad \varepsilon \to 0$$

Furthermore, if $a_n \geqslant b_n$ for n large, then $P(\sum_{n\geqslant 1} a_n \xi_n^2 \leqslant \varepsilon^2)$ and $P(\sum_{n\geqslant 1} b_n \xi_n^2 \leqslant \varepsilon^2)$ have the same order of magnitude as $\varepsilon \to 0$ if and only if $\sum_{n\geqslant 1} |1 - a_n/b_n| < \infty$.

As a direct consequence, if the two sets of norming constants for the independent coordinate l_2-valued Brownian motions satisfy the conditions of Theorem 2, then these two Brownian motions have the same rate of escape function (see Erickson,[8] Kuelbs,[16] and Cox[6]). Also we show that (Corolllary 4) for any positive integer N,

$$\log P\left(\sum_{n\geqslant 1} a_n \xi_n^2 \leqslant \varepsilon^2\right) \sim \log P\left(\sum_{n\geqslant N} a_n \xi_n^2 \leqslant \varepsilon^2\right) \qquad \text{as} \quad \varepsilon \to 0$$

which gives a positive answer to a problem proposed by Cox,[5] that is, for the independent coordinate l_2-valued Brownian motion, the rate of escape function will not change if we delete a finite number of the coordinates.

In Section 3, we give a computable formula in many cases for $\log P(\sum_{n\geqslant 1} a_n \xi_n^2 \leqslant \varepsilon^2)$. By using this formula we calculate some particular interesting cases. In particular, Example 2 extends some results of Csáki[7] for multiparameter Wiener process. Also we list some known concrete examples. These examples serve as the standard base for the comparison theorems in Section 2.

In Section 4, by using the comparison results in Section 2 and examples in Section 3, we demonstrate how to find the lower tail of the Ornstein–Uhlenbeck process and a weighted Brownian bridge under L_2-norms even though we cannot find the eigenvalues of these processes explicitly.

In Section 5, we consider sup-norms first, that is,

$$q(x) = \sup_{n \geqslant 1} \{|x_n|/a_n\} \qquad \text{for any} \quad x = (x_n) \in R^\infty$$

where (a_n) is a given sequence of positive numbers. These were studied in Hoffmann-Jørgensen et al.,[10] and our results provide additional information.

For the l_p-norms, $1 \leqslant p < \infty$, that is,

$$q(x) = \Big(\sum_{n \geqslant 1} a_n \, |x_n|^p \Big)^{1/p} \qquad \text{for any} \quad x = (x_n) \in R^\infty$$

where (a_n) is a given sequence of positive numbers and $\sum_{n \geqslant 1} a_n < \infty$, we give two basic facts that were motivated by comparison results for l_2-norms and sup-norms. All of them strongly suggest that the same type of comparison results for $p = 2$ in Section 2 and $p = \infty$ (i.e., sup) in this section should hold for $1 \leqslant p < \infty$. Also we give some conjectures and suggestions for further research.

2. COMPARISON RESULTS FOR THE l_2-NORMS

Throughout this section, we assume a_n, b_n are positive, $\sum_{n \geqslant 1} a_n < \infty$, $\sum_{n \geqslant 1} b_n < \infty$, and $\varepsilon > 0$.

Lemma 1. If ε is small enough and γ_a and γ_b are unique defined by ε such that

$$\varepsilon^2 = \sum_{n \geqslant 1} \frac{a_n}{1 + 2a_n \gamma_a} = \sum_{n \geqslant 1} \frac{b_n}{1 + 2b_n \gamma_b} \tag{2.1}$$

and

$$\sum_{n \geqslant 1} \left| 1 - \frac{a_n}{b_n} \right| < \infty \tag{2.2}$$

then

$$\lim_{\varepsilon \to 0} \left(\frac{\gamma_a}{\gamma_b} - 1 \right) \sum_{n \geqslant 1} \frac{b_n^2 \gamma_a \gamma_b}{(1 + 2b_n \gamma_a)(1 + 2b_n \gamma_b)} = 0 \tag{2.3}$$

In particular

$$\lim_{\varepsilon \to 0} \frac{\gamma_a}{\gamma_b} = 1 \tag{2.4}$$

Proof. First observe that

$$\begin{aligned} &\left| \left(\frac{\gamma_a}{\gamma_b} - 1 \right) \sum_{n \geqslant 1} \frac{b_n^2 \gamma_a \gamma_b}{(1 + 2b_n \gamma_a)(1 + 2b_n \gamma_b)} \right| \\ &= \left| \frac{\gamma_a}{2} \sum_{n \geqslant 1} \left(\frac{a_n}{1 + 2a_n \gamma_a} - \frac{b_n}{1 + 2b_n \gamma_a} \right) \right| \\ &\leqslant \frac{1}{2} \sum_{n \geqslant 1} \left| \frac{a_n - b_n}{a_n} \right| \cdot \left| \frac{a_n \gamma_a}{1 + 2a_n \gamma_a} \right| \cdot \left| \frac{1}{1 + 2b_n \gamma_a} \right| \\ &\leqslant \frac{1}{2} \sum_{n \geqslant 1} \left| 1 - \frac{b_n}{a_n} \right| < \infty \end{aligned}$$

Hence we obtain (2.3) by the D.C.T. since $\gamma_a \to +\infty$ and $\gamma_b \to +\infty$ as $\varepsilon \to 0$.

In particular (2.4) holds since

$$\liminf_{\varepsilon \to 0} \sum_{n \geqslant 1} \frac{b_n^2 \gamma_a \gamma_b}{(1 + 2b_n \gamma_a)(1 + 2b_n \gamma_b)} \geqslant \liminf_{\varepsilon \to 0} \frac{b_1^2 \gamma_a \gamma_b}{(1 + 2b_1 \gamma_a)(1 + 2b_1 \gamma_b)} = \frac{1}{4} \qquad \square$$

Lemma 2. If ε is small enough and

$$\varepsilon^2 = \sum_{n \geqslant 1} \frac{a_n}{1 + 2a_n \tau_1} = \sum_{n \geqslant N} \frac{a_n}{1 + 2a_n \tau_N} \tag{2.5}$$

then

$$\lim_{\varepsilon \to 0} \left(\frac{\tau_1}{\tau_N} - 1 \right) \sum_{n \geqslant N} \frac{a_n^2 \tau_1 \tau_N}{(1 + 2a_n \tau_1)(1 + 2a_n \tau_N)} = \frac{N}{4} \tag{2.6}$$

In particular

$$\lim_{\varepsilon \to 0} \frac{\tau_1}{\tau_N} = 1 \tag{2.7}$$

The proof is similar to the proof of Lemma 1. We omit it here.

Lemma 3. Under the assumptions of Lemma 1, we have

$$\lim_{\varepsilon \to 0} \sum_{n \geqslant 1} \left(\frac{b_n - a_n}{(1 + 2b_n \gamma_b)\, a_n} \right)^2 = 0 \tag{2.8}$$

$$\lim_{\varepsilon \to 0} \sum_{n \geqslant 1} \frac{b_n - a_n}{(1 + 2b_n \gamma_b)\, a_n} \cdot \frac{b_n(\gamma_a - \gamma_b)}{1 + 2b_n \gamma_b} = 0 \tag{2.9}$$

$$\lim_{\varepsilon \to 0} \sum_{n \geqslant 1} \left(\frac{b_n(\gamma_a - \gamma_b)}{1 + 2b_n \gamma_b} \right)^2 = 0 \tag{2.10}$$

Proof. First observe

$$\sum_{n \geqslant 1} \left(\frac{b_n - a_n}{(1 + 2b_n \gamma_b)\, a_n} \right)^2 \leqslant \sum_{n \geqslant 1} \left(1 - \frac{b_n}{a_n} \right)^2 < \infty$$

and hence (2.8) is proved by the D.C.T.

To prove (2.9), note that

$$\left| \sum_{n \geqslant 1} \frac{b_n - a_n}{(1 + 2b_n \gamma_b)\, a_n} \cdot \frac{b_n(\gamma_a - \gamma_b)}{1 + 2b_n \gamma_b} \right|$$

$$= \left| \left(\frac{\gamma_a}{\gamma_b} - 1 \right) \sum_{n \geqslant 1} \frac{b_n - a_n}{a_n} \cdot \frac{1}{1 + 2b_n \gamma_b} \cdot \frac{b_n \gamma_b}{1 + 2b_n \gamma_b} \right|$$

$$\leqslant \left| \frac{\gamma_a}{\gamma_b} - 1 \right| \sum_{n \geqslant 1} \left| \frac{b_n - a_n}{a_n} \right|$$

Hence by (2.4) of Lemma 1, we obtain (2.9).

Now

$$\sum_{n \geqslant 1} \left(\frac{b_n(\gamma_a - \gamma_b)}{1 + 2b_n \gamma_b} \right)^2$$

$$= \left(1 - \frac{\gamma_a}{\gamma_b} \right)^2 \sum_{n \geqslant 1} \frac{b_n^2 \gamma_b^2}{(1 + 2b_n \gamma_b)^2}$$

$$= \left(1 - \frac{\gamma_a}{\gamma_b} \right)^2 \cdot \frac{\gamma_b}{\gamma_a} \sum_{n \geqslant 1} \frac{b_n^2 \gamma_a \gamma_b}{(1 + 2b_n \gamma_a)(1 + 2b_n \gamma_b)} \cdot \left(1 + \frac{2b_n \gamma_b}{1 + 2b_n \gamma_b} \cdot \frac{\gamma_a - \gamma_b}{\gamma_b} \right)$$

$$\leqslant \left(1 - \frac{\gamma_a}{\gamma_b} \right)^2 \cdot \frac{\gamma_b}{\gamma_a} \cdot \left(1 + \left| 1 - \frac{\gamma_a}{\gamma_b} \right| \right) \sum_{n \geqslant 1} \frac{b_n^2 \gamma_a \gamma_b}{(1 + 2b_n \gamma_a)(1 + 2b_n \gamma_b)}$$

and hence by Lemma 1, we obtain (2.10). □

Lemma 4. Under the assumptions of Lemma 2, we have

$$\lim_{\varepsilon \to 0} \sum_{n \geq 1} \left(\frac{a_n(\tau_1 - \tau_N)}{1 + 2a_n \tau_N} \right)^2 = 0$$

Proof. It is easy to see by following the proof of (2.10) and using Lemma 2 and

$$\sum_{n=1}^{N-1} \frac{a_n^2 \tau_1 \tau_N}{(1 + 2a_n \tau_1)(1 + 2a_n \tau_N)} \leqslant N - 1 < \infty \qquad \square$$

Lemma 5. Under the assumption of Lemma 1, we have

$$\lim_{\varepsilon \to 0} \left(\sum_{n \geq 1} \left(\frac{b_n \gamma_b}{1 + 2b_n \gamma_b} \right)^2 \Big/ \sum_{n \geq 1} \left(\frac{a_n \gamma_a}{1 + 2a_n \gamma_a} \right)^2 \right) = 1 \tag{2.11}$$

Proof. First note that

$$\begin{aligned}
&\left| \sum_{n \geq 1} \left(\frac{a_n \gamma_a}{1 + 2a_n \gamma_a} \right)^2 - \sum_{n \geq 1} \left(\frac{b_n \gamma_b}{1 + 2b_n \gamma_b} \right)^2 \right| \\
&\quad \leqslant \left| \sum_{n \geq 1} \left(\frac{a_n \gamma_a}{1 + 2a_n \gamma_a} \right)^2 - \sum_{n \geq 1} \left(\frac{a_n \gamma_b}{1 + 2a_n \gamma_b} \right)^2 \right| \\
&\qquad + \left| \sum_{n \geq 1} \left(\frac{a_n \gamma_b}{1 + 2a_n \gamma_b} \right)^2 - \sum_{n \geq 1} \left(\frac{b_n \gamma_b}{1 + 2b_n \gamma_b} \right)^2 \right| \\
&\quad = \left| \frac{\gamma_b}{\gamma_a} - 1 \right| \sum_{n \geq 1} \left(\frac{a_n \gamma_a}{1 + 2a_n \gamma_a} \right)^2 \\
&\qquad \cdot \left(\left(1 + \frac{\gamma_b}{\gamma_a} \right) \cdot \frac{1}{(1 + 2a_n \gamma_b)^2} + \frac{4a_n \gamma_b}{(1 + 2a_n \gamma_b)^2} \right) \\
&\qquad + \sum_{n \geq 1} \left| 1 - \frac{a_n}{b_n} \right| \cdot \left(\frac{b_n \gamma_b}{1 + 2b_n \gamma_b} \right)^2 \\
&\qquad \cdot \left(\left(1 + \frac{a_n}{b_n} \right) \cdot \frac{1}{(1 + 2a_n \gamma_b)^2} + \frac{4a_n \gamma_b}{(1 + 2a_n \gamma_b)^2} \right) \\
&\quad \leqslant \left| \left(\frac{\gamma_b}{\gamma_a} - 1 \right) \right| \sum_{n \geq 1} \left(\frac{a_n \gamma_a}{1 + 2a_n \gamma_a} \right)^2 \cdot \left(\left(1 + \frac{\gamma_b}{\gamma_a} \right) + 1 \right) \\
&\qquad + \sum_{n \geq 1} \left| 1 - \frac{a_n}{b_n} \right| \cdot \left(\left(1 + \frac{a_n}{b_n} \right) + 1 \right)
\end{aligned}$$

and observe that

$$\sum_{n\geqslant 1}\left|1-\frac{a_n}{b_n}\right|\left(2+\frac{a_n}{b_n}\right)<\infty \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\sum_{n\geqslant 1}\left(\frac{a_n\gamma_a}{1+2a_n\gamma_a}\right)^2=\infty$$

Hence we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\left|1-\left(\sum_{n\geqslant 1}\left(\frac{b_n\gamma_b}{1+2b_n\gamma_b}\right)^2\Big/\sum_{n\geqslant 1}\left(\frac{a_n\gamma_a}{1+2a_n\gamma_a}\right)^2\right)\right|\\&\qquad\leqslant\limsup_{\varepsilon\to 0}\left|\frac{\gamma_b}{\gamma_a}-1\right|\cdot\frac{\gamma_b}{\gamma_a}\cdot\left(2+\frac{\gamma_b}{\gamma_a}\right)=0\end{aligned}$$

which gives (2.11). □

Lemma 6. Under the assumptions of Lemma 2, we have

$$\lim_{\varepsilon\to 0}\left(\sum_{n\geqslant 1}\left(\frac{a_n\tau_1}{1+2a_n\tau_1}\right)^2\Big/\sum_{n\geqslant N}\left(\frac{a_n\tau_N}{1+2a_n\tau_N}\right)^2\right)=1$$

The proof is similar to the proof of Lemma 5. We omit it here.

Proof of Theorem 2. Let

$$\begin{aligned}R(\varepsilon)&=\left(\varepsilon^2\gamma_a-\frac{1}{2}\sum_{n\geqslant 1}\log(1+2a_n\gamma_a)\right)\\&\quad-\left(\varepsilon^2\gamma_b-\frac{1}{2}\sum_{n\geqslant 1}\log(1+2b_n\gamma_b)\right)-\frac{1}{2}\sum_{n\geqslant 1}\log\frac{b_n}{a_n}\\&=\varepsilon^2(\gamma_a-\gamma_b)-\frac{1}{2}\sum_{n\geqslant 1}\log(1+x_n)\end{aligned}$$

where

$$x_n=\frac{(1+2a_n\gamma_a)\,b_n}{(1+2b_n\gamma_b)\,a_n}-1=\frac{b_n-a_n}{(1+2b_n\gamma_b)\,a_n}+\frac{2b_n(\gamma_a-\gamma_b)}{1+2b_n\gamma_b}$$

and γ_a, γ_b satisfy Eq. (2.1).

Now using the inequality $x-x^2/2\leqslant\log(1+x)\leqslant x$ for $x>-1$, we obtain

$$\begin{aligned}&\varepsilon^2(\gamma_a-\gamma_b)-\frac{1}{2}\sum_{n\geqslant 1}x_n\\&\qquad\leqslant R(\varepsilon)\leqslant\varepsilon^2(\gamma_a-\gamma_b)-\frac{1}{2}\sum_{n\geqslant 1}\left(x_n-\frac{1}{2}x_n^2\right)\end{aligned}\tag{2.12}$$

Note that

$$\left|\varepsilon^2(\gamma_a-\gamma_b)-\frac{1}{2}\sum_{n\geqslant 1} x_n\right|=\left|\frac{1}{2}\sum_{n\geqslant 1}\left(1-\frac{b_n}{a_n}\right)\cdot\frac{1}{1+2b_n\gamma_b}\right|$$
$$\leqslant\frac{1}{2}\sum_{n\geqslant 1}\left|1-\frac{b_n}{a_n}\right|<\infty$$

Hence by the D.C.T., we have

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2(\gamma_a-\gamma_b)-\frac{1}{2}\sum_{n\geqslant 1} x_n\right)=0$$

Using Lemma 3 and

$$\sum_{n\geqslant 1} x_n^2=\sum_{n\geqslant 1}\left(\frac{b_n-a_n}{(1+2b_n\gamma_b)\,a_n}\right)^2+4\sum_{n\geqslant 1}\frac{b_n-a_n}{(1+2b_n\gamma_b)\,a_n}$$
$$\cdot\frac{b_n(\gamma_a-\gamma_b)}{(1+2b_n\gamma_b)}+4\sum_{n\geqslant 1}\left(\frac{b_n(\gamma_a-\gamma_b)}{1+2b_n\gamma_b}\right)^2$$

we have $\lim_{\varepsilon\to 0}\sum_{n\geqslant 1} x_n^2=0$. Thus $\lim_{\varepsilon\to 0} R(\varepsilon)=0$ by (2.12), and together with Lemma 5 and Theorem 1, we finish the proof of the theorem. □

Corollary 1. Let $B_1, B_2,\ldots$ be independent standard one dimensional Brownian motions. Let

$$X_a(t)=(\sqrt{a_1}\,B_1(t), \sqrt{a_2}\,B_2(t),\ldots)\qquad X_b(t)=(\sqrt{b_1}\,B_1(t), \sqrt{b_2}\,B_2(t),\ldots)$$

be processes taking values in R^∞ and assume $\sum_{n\geqslant 1} a_n<\infty$, $\sum_{n\geqslant 1} b_n<\infty$. If $\sum_{n\geqslant 1}|1-a_n/b_n|$ is finite, then $X_a(t)$ and $X_b(t)$ have the same rate of escape function with respect to the l_2-norm.

Proof. Note that by Theorem 3.5 of Cox,[6] the rate of escape functions with respect to the l_2-norm for $X_a(t)$ and $X_b(t)$ both exist. Then by Theorem 2 of Erickson[8] and our Theorem 2, the claim is true. □

Corollary 2. Let $b_n=a_n$ for $n\geqslant N+1$ (N is fixed) in Theorem 2; then

$$P\left(\sum_{n\geqslant 1} a_n\xi_n^2\leqslant\varepsilon^2\right)\sim\left(\prod_{n=1}^{N} b_n/a_n\right)^{1/2} P\left(\sum_{n\geqslant 1} b_n\xi_n^2\leqslant\varepsilon^2\right)\qquad\text{as}\quad\varepsilon\to 0$$

i.e., the first N terms of a_n do not change the order of magnitude of $P(\sum_{n\geqslant 1} a_n\xi_n^2\leqslant\varepsilon^2)$ as $\varepsilon\to 0$.

Corollary 3. If $\sum_{n\geqslant 1}|1-a_n/b_n|\, I_{\{a_n<b_n\}}(n)<\infty$ and $P(\sum_{n\geqslant 1} a_n\xi_n^2 \leqslant \varepsilon^2)$ has the same order of magnitude as $P(\sum_{n\geqslant 1} b_n\xi_n^2\leqslant\varepsilon^2)$, that is,

$$\lim_{\varepsilon\to 0} P\left(\sum_{n\geqslant 1} a_n\xi_n^2\leqslant\varepsilon^2\right)\Big/ P\left(\sum_{n\geqslant 1} b_n\xi_n^2\leqslant\varepsilon^2\right)=C \qquad C>0$$

then

$$\sum_{n\geqslant 1}\left|1-\frac{a_n}{b_n}\right|<\infty \qquad \text{and} \qquad C=\left(\prod_{n\geqslant 1} b_n/a_n\right)^{1/2}$$

where $I(x)$ is indicator function.

Proof. By using Theorem 2 for integer $N>0$, we have

$$\begin{aligned}
&P\left(\sum_{n\geqslant 1} a_n\xi_n^2\leqslant\varepsilon^2\right)\\
&\quad = P\left(\sum_{n\geqslant 1} a_n I_{\{a_n<b_n\}}(n)\,\xi_n^2+\sum_{1\leqslant n\leqslant N} a_n I_{\{a_n\geqslant b_n\}}(n)\,\xi_n^2\right.\\
&\qquad \left. +\sum_{n\geqslant N+1} a_n I_{\{a_n\geqslant b_n\}}(n)\,\xi_n^2\leqslant\varepsilon^2\right)\\
&\quad \leqslant P\left(\sum_{n\geqslant 1} a_n I_{\{a_n<b_n\}}(n)\,\xi_n^2+\sum_{1\leqslant n\leqslant N} a_n I_{\{a_n\geqslant b_n\}}(n)\,\xi_n^2\right.\\
&\qquad \left. +\sum_{n\geqslant N+1} b_n I_{\{a_n\geqslant b_n\}}(n)\,\xi_n^2\leqslant\varepsilon^2\right)\\
&\quad \sim\left(\prod_{n:\,\{a_n<b_n\}}^{\infty} b_n/a_n\right)^{1/2}\cdot\left(\prod_{n:\,\{a_n\geqslant b_n\}}^{N} b_n/a_n\right)^{1/2}\cdot P\left(\sum_{n\geqslant 1} b_n\xi_n^2\leqslant\varepsilon^2\right)
\end{aligned}$$

that is,

$$\begin{aligned}
C&=\lim_{\varepsilon\to 0} P\left(\sum_{n\geqslant 1} a_n\xi_n^2\leqslant\varepsilon^2\right)\Big/ P\left(\sum_{n\geqslant 1} b_n\xi_n^2\leqslant\varepsilon^2\right)\\
&\leqslant\left(\prod_{n:\,\{a_n<b_n\}} b_n/a_n\right)^{1/2}\cdot\left(\prod_{n\leqslant N:\,\{a_n\geqslant b_n\}} b_n/a_n\right)^{1/2}
\end{aligned}$$

Let $N\to\infty$; we have $\prod_{n:\,\{a_n\geqslant b_n\}}(b_n/a_n)$ converge to something bigger than zero and finite, i.e., $\sum_{n\geqslant 1}|1-a_n/b_n|\, I_{\{a_n\geqslant b_n\}}(n)<\infty$. And together with the assumption, we get $\sum_{n\geqslant 1}|1-a_n/b_n|<\infty$. The second part is true by Theorem 2. □

Remark 1. Combining Theorem 2 and Corollary 3 gives us the following: If

$$\sum_{n\geq 1} |1-a_n/b_n|\, I_{\{a_n<b_n\}}(n)<\infty \qquad \text{(in particular if } a_n\geq b_n \text{ for } n \text{ large)}$$

then $P(\sum_{n\geq 1} a_n\xi_n^2 \leq \varepsilon^2)$ has the same order of magnitude as $P(\sum_{n\geq 1} b_n\xi_n^2\leq\varepsilon^2)$ if and only if $\sum_{n\geq 1}|1-a_n/b_n|$ is finite. On the other hand, we can conclude that no matter how we arrange a_n and b_n, the condition $\sum_{n\geq 1}|1-a_n/b_n|<\infty$ is not necessary by comparing (3.1) and (3.4) for $d=\frac{1}{2}$.

Our next result shows the change of tail behavior if the first $N-1$ terms of $\sum_{n\geq 1} a_n\xi_n^2$ are omitted.

Theorem 3. For positive integer N, we have

$$P\left(\sum_{n\geq 1} a_n\xi_n^2\leq\varepsilon^2\right) \sim\left(\prod_{n=1}^{N-1} 2a_n\right)^{-1/2} \tau_N^{-(N-1)/2} P\left(\sum_{n\geq N} a_n\xi_n^2\leq\varepsilon^2\right) \qquad \text{as } \varepsilon\to 0$$

where $\tau_N=\tau_N(\varepsilon)$ satisfies the equation

$$\varepsilon^2=\sum_{n\geq N}\frac{a_n}{1+2a_n\tau_N}$$

Proof. Let

$$\begin{aligned} S(\varepsilon)&=\left(\varepsilon^2\tau_1-\frac{1}{2}\sum_{n\geq 1}\log(1+2a_n\tau_1)\right) - \left(\varepsilon^2\tau_N-\frac{1}{2}\sum_{n\geq 1}\log(1+2a_n\tau_N)\right)\\ &=\varepsilon^2(\tau_1-\tau_N)-\frac{1}{2}\sum_{n\geq 1}\log(1+x_n) \end{aligned}$$

where $x_n=2(\tau_1-\tau_N)a_n/(1+2a_n\tau_N)$ and τ_1, τ_N satisfy Eq. (2.5).

Now using the inequality $x-x^2/2\leq\log(1+x)\leq x$ for $x>-1$, we obtain

$$\varepsilon^2(\tau_1-\tau_N)-\frac{1}{2}\sum_{n\geq 1} x_n\leq S(\varepsilon)\leq\varepsilon^2(\tau_1-\tau_N)-\frac{1}{2}\sum_{n\geq 1}\left(x_n-\frac{1}{2}x_n^2\right) \tag{2.13}$$

Noting that by (2.5) and (2.7)

$$\left|\varepsilon^2(\tau_1-\tau_N)-\frac{1}{2}\sum_{n\geq 1}x_n\right|=\left|\frac{\tau_1}{\tau_N}-1\right|\cdot\sum_{n=1}^{N-1}\frac{a_n\tau_N}{1+2a_n\tau_N}$$

$$\leqslant\frac{N}{2}\cdot\left|\frac{\tau_1}{\tau_N}-1\right|\to 0 \quad \text{as} \quad \varepsilon\to 0$$

and $\lim_{\varepsilon\to 0}x_n^2=0$ by Lemma 4, we have $\lim_{\varepsilon\to 0}S(\varepsilon)=0$ by (2.13). Hence

$$\begin{aligned}
&P\left(\sum_{n\geq 1}a_n\xi_n^2\leqslant\varepsilon^2\right)\\
&\sim\left(4\pi\sum_{n\geq 1}\left(\frac{a_n\tau_1}{1+2a_n\tau_1}\right)^2\right)^{-1/2}\exp\left(\varepsilon^2\tau_1-\frac{1}{2}\sum_{n\geq 1}\log(1+2a_n\tau_1)\right)\\
&\sim\left(4\pi\sum_{n\geq N}\left(\frac{a_n\tau_N}{1+2a_n\tau_N}\right)^2\right)^{-1/2}\\
&\qquad\cdot\exp\left(S(\varepsilon)+\varepsilon^2\tau_N-\frac{1}{2}\sum_{n\geq 1}\log(1+2a_n\tau_N)\right)\\
&\sim\exp\left(S(\varepsilon)-\frac{1}{2}\sum_{n=1}^{N-1}\log(1+2a_n\tau_N)\right)P\left(\sum_{n\geq N}a_n\xi_n^2\leqslant\varepsilon^2\right)\\
&\sim\prod_{n=1}^{N-1}(1+2a_n\tau_N)^{-1/2}P\left(\sum_{n\geq N}a_n\xi_n^2\leqslant\varepsilon^2\right)\\
&\sim\left(\prod_{n=1}^{N-1}2a_n\right)^{-1/2}\tau_N^{-(N-1)/2}P\left(\sum_{n\geq N}a_n\xi_n^2\leqslant\varepsilon^2\right)
\end{aligned}$$

where the first $\sim$ and the third $\sim$ are by Theorem 1, the second $\sim$ is by Lemma 6, the fourth $\sim$ is by $\lim_{\varepsilon\to 0}S(\varepsilon)=0$ and the fifth follows since $\tau_N\to\infty$ as $\varepsilon\to 0$. □

Corollary 4. For positive integer N, we have

$$\log P\left(\sum_{n\geq 1}a_n\xi_n^2\leqslant\varepsilon^2\right)\sim\log P\left(\sum_{n\geq N}a_n\xi_n^2\leqslant\varepsilon^2\right) \quad \text{as} \quad \varepsilon\to 0$$

Proof. It can be seen by noting that from Theorem 3

$$\begin{aligned}
\log P\left(\sum_{n\geq 1}a_n\xi_n^2\leqslant\varepsilon^2\right)\sim&-\frac{1}{2}\sum_{n=1}^{N-1}\log(2a_n)-\frac{N-1}{2}\log\tau_N\\
&+\log P\left(\sum_{n\geq N}a_n\xi_n^2\leqslant\varepsilon^2\right)
\end{aligned}$$

and from (3.7) and (3.8) in Section 3, $\log\tau_N=o(\log P(\sum_{n\geq N}a_n\xi_n^2\leqslant\varepsilon^2))$. □

Corollary 5. For the independent coordinate l_2-valued Brownian motion, the rate of escape function will not change if we delete a finite number of the coordinates.

Proof. It is easy to see by Corollary 4 here and Theorem 2 of Erickson.(8) □

3. THE ASYMPTOTIC BEHAVIOR

As mentioned in the Introduction, many authors have computed the exact asymptotic behavior of $P(\sum_{n\geq 1} a_n\xi_n^2 \leq \varepsilon^2)$ for some particular a_n, $\sum_{n\geq 1} a_n < \infty$ and $a_n > 0$. Here we list some of the most important ones for the use of our comparison results.

Hoffmann-Jørgensen et al.(10) showed the following examples in their Theorem 5.1 and 5.2:

$$P\left(\sum_{n\geq 1}\left(2\left[\frac{n+1}{2}\right]\right)^{-2}\xi_n^2\leq\varepsilon^2\right)\sim\sqrt{2\pi}\,\varepsilon^{-1}\exp\left(-\frac{\pi^2}{8}\cdot\frac{1}{\varepsilon^2}\right)\quad\text{as}\quad\varepsilon\to 0 \tag{3.1}$$

and

$$P\left(\sum_{n\geq 1}\left(2\left[\frac{n+1}{2}\right]-1\right)^{-2}\xi_n^2\leq\varepsilon^2\right)\sim 4\sqrt{2}\,\pi^{-3/2}\varepsilon\exp\left(-\frac{\pi^2}{8}\cdot\frac{1}{\varepsilon^2}\right)\quad\text{as}\quad\varepsilon\to 0$$

where $[x]$ denotes the greatest integer function.

Zolotarev(22) evaluated the following examples by modifying Theorem 1. For $\alpha > 1$,

$$P\left(\sum_{n\geq 1}\frac{1}{n^\alpha}\xi_n^2\leq\varepsilon^2\right)\sim(2\pi)^{4(2-\alpha)}K^4(1-1/\alpha)^8\,\gamma^{4(2-\alpha)/(\alpha-1)}\varepsilon^{(2-\alpha)/2(\alpha-1)}\cdot\exp\left(-(\alpha-1)K\cdot\left(\frac{\gamma}{\varepsilon^2}\right)^{1/(\alpha-1)}\right) \tag{3.2}$$

as $\varepsilon\to 0$ where $K=(2^{-1}\Gamma(1-\alpha^{-1})\,\Gamma(1+\alpha^{-1}))^{\alpha/(\alpha-1)}$ and γ is the Euler constant. The case $\alpha=2$ of this result was noted by Anderson and Darling(1) and Sytaya.(19) Also

$$P\left(\sum_{n\geq 1}\frac{1}{e^{n-1}}\xi_n^2\leq\varepsilon^2\right)\sim\exp\left(\frac{1}{16}-\frac{\pi^2}{12}-\frac{1}{2}\log\pi\right)T^{-1}\exp\left(-\frac{1}{4}T^2\right) \tag{3.3}$$

as $\varepsilon \to 0$ where $T = \log(1/2\varepsilon^2) + \log\log(1/2\varepsilon^2) - \frac{1}{2}$. Furthermore, it is not too hard by using results in Zolotarev[22] to get the following. For $d > -1$,

$$P\left(\sum_{n \geq 1} (n+d)^{-2}\, \xi_n^2 \leq \varepsilon^2\right) \sim C_d \varepsilon^{-2d} \exp\left(-\frac{\pi^2}{8} \cdot \frac{1}{\varepsilon^2}\right) \tag{3.4}$$

where C_d is a constant.

For the multiple index case, Csáki[7] showed by using Theorem 1 that

$$\log P\left(\sum_{i \geq 1} \sum_{j \geq 1} \left(i - \frac{1}{2}\right)^{-2} \left(j - \frac{1}{2}\right)^{-2} \xi_{ij}^2 \leq \varepsilon^2\right)$$
$$\sim -\frac{1}{8\pi^2} \left(\frac{1}{\varepsilon}\right)^2 \left(\log \frac{1}{\varepsilon^2}\right)^2 \tag{3.5}$$

which is good enough for certain applications (see Remark 4).

For the rest of this section, we want to give a way of computing $\log P(\sum_{n \geq 1} a_n \xi_n^2 \leq \varepsilon^2)$ for a large class of a_n by modifying Theorem 1. We use it to compute an example that generalizes (3.5).

Theorem 4. If there exists a differentiable function $h(x)$ on $[A, \infty)$ such that

$$\frac{d}{dx}\left(\sum_{n \geq 1} \frac{a_n}{1 + 2xa_n}\right) = -\sum_{n \geq 1} \frac{2a_n^2}{(1 + 2xa_n)^2} \sim h'(x) \qquad \text{as} \quad x \to \infty \tag{3.6}$$

then

$$\log P\left(\sum_{n \geq 1} a_n \xi_n^2 \leq \varepsilon^2\right) \sim -\left(\int_A^{\gamma} h(x)\, dx - \gamma h(\gamma)\right) \qquad \text{as} \quad \varepsilon \to 0$$

where $\varepsilon^2 = \sum_{n \geq 1} a_n/(1 + 2\gamma a_n)$ and $\sum_{n \geq 1} a_n < \infty$.

Proof. From Theorem 1 we have

$$\log P\left(\sum_{n \geq 1} a_n \xi_n^2 \leq \varepsilon^2\right) \sim -\frac{1}{2}\log(4\pi) - \frac{1}{2}\log \sum_{n \geq 1} \left(\frac{a_n \gamma}{1 + 2a_n \gamma}\right)^2$$
$$+ \gamma \sum_{n \geq 1} \frac{a_n}{1 + 2a_n \gamma} - \int_0^{\gamma} \sum_{n \geq 1} \frac{a_n}{1 + 2a_n x}\, dx \tag{3.7}$$

Note that

$$\left| -\frac{1}{2}\log(4\pi) - \frac{1}{2}\log \sum_{n \geq 1} \left(\frac{a_n \gamma}{1 + 2a_n \gamma}\right)^2 \right|$$
$$\leq \frac{1}{2}\log(4\pi) + \frac{1}{2}\left|\log \sum_{n \geq 1} a_n^2\right| + \log \gamma$$

and

$$\log \gamma = o\left(\gamma \sum_{n \geq 1} \frac{a_n}{1+2a_n\gamma} - \int_0^\gamma \sum_{n \geq 1} \frac{a_n}{1+2a_n x}\, dx\right) \tag{3.8}$$

Hence we obtain

$$\log P\left(\sum_{n \geq 1} a_n \xi_n^2 \leq \varepsilon^2\right) \sim \gamma \sum_{n \geq 1} \frac{a_n}{1+2a_n\gamma} - \int_0^\gamma \sum_{n \geq 1} \frac{a_n}{1+2a_n x}\, dx$$

$$\sim \gamma h(\gamma) - \int_A^\gamma h(x)\, dx$$

where the first $\sim$ is by (3.7) and (3.8), the second $\sim$ is by L'Hôpital's rule. □

Remark 2. Assuming (3.6) and using L'Hôpital's rule, we can see that

$$\varepsilon^2 = \sum_{n \geq 1} \frac{a_n}{1+2\gamma a_n} \sim h(\gamma) \qquad \text{as} \quad \gamma \to \infty \tag{3.9}$$

Thus in practice when we use Theorem 4, we use (3.9) to find a function $h(x)$ instead of (3.6) and then check that the function $h(x)$ satisfies (3.6). In general, the way to check that $h(x)$ satisfies (3.6) is the same as the way to find $h(x)$ in (3.9). It is not clear that the $h(x)$ in (3.9) always satisfies (3.6), but the examples we considered in this section have this property since a_n is nice enough.

Remark 3. The result used to obtain (3.2), (3.3), and (3.4) in Zolotarev[22] gives more information, and is much more complicated than Theorem 4. It is also notable that the result there is not suitable for the multiple index case in general.

The lemma below provides the first step in finding the function $h(x)$, and the examples in this section demonstrate how everything works for some special cases.

Lemma 7. If $f\colon (0, \infty) \to R_+$ is an increasing function such that $f(n) = a_n^{-1}$ and $\int_1^\infty (f(x))^{-1}\, dx < \infty$, then for any constant $A > 0$

$$\varepsilon^2 = \sum_{n \geq 1} \frac{a_n}{1+2\gamma a_n} = \sum_{n \geq 1} \frac{1}{2\gamma + f(n)} \sim \int_A^\infty \frac{1}{2\gamma + f(x)}\, dx \qquad \text{as} \quad \varepsilon \to 0 \tag{3.10}$$

and

$$\sum_{n \geq 1} \frac{2a_n^2}{(1+2\gamma a_n)^2} \sim \int_A^\infty \frac{1}{(2\gamma + f(x))^2}\, dx \qquad \text{as} \quad \varepsilon \to 0 \tag{3.11}$$

Proof. First observe that for $n_0+1>A\geqslant n_0$

$$\varepsilon^2=\sum_{n\geqslant 1}\frac{1}{2\gamma+f(n)}\geqslant\sum_{n=n_0+1}^{\infty}\int_n^{n+1}\frac{1}{2\gamma+f(x)}\,dx$$

$$=\int_A^{\infty}\frac{1}{2\gamma+f(x)}\,dx-\int_A^{n_0+1}\frac{1}{2\gamma+f(x)}\,dx$$

$$\varepsilon^2=\sum_{n\geqslant 1}\frac{1}{2\gamma+f(n)}\leqslant\sum_{n_0+2}^{\infty}\int_{n-1}^{n}\frac{1}{2\gamma+f(x)}\,dx+\sum_{n=1}^{n_0+1}\frac{1}{2\gamma+f(n)}$$

$$\leqslant\int_A^{\infty}\frac{1}{2\gamma+f(x)}\,dx+\sum_{n=1}^{n_0+1}\frac{1}{2\gamma+f(n)}$$

and hence (3.10) is true by the fact that for each $k\geqslant 1$

$$\sum_{n=k}^{\infty}\frac{\gamma}{2\gamma+f(n)}\to\infty\qquad\text{as}\quad\gamma\to\infty$$

Similarly, (3.11) holds. □

Example 1. Let $a_n=(n+a)^{-\alpha}$, $\alpha>1$, $a>-1$, and $f(x)=(x+a)^{\alpha}$; then from (3.10) we have the following:

$$\varepsilon^2=\sum_{n\geqslant 1}\frac{a_n}{1+2\gamma a_n}\sim\int_1^{\infty}\frac{1}{2\gamma+(x+a)^{\alpha}}\,dx$$

$$\sim\int_0^{\infty}\frac{1}{2\gamma+x^{\alpha}}\,dx=\left(\frac{\pi}{\alpha}\right)\left(\sin\frac{\pi}{\alpha}\right)^{-1}(2\gamma)^{1/\alpha-1}=h(\gamma)\tag{3.12}$$

and hence

$$\gamma\sim 2^{-1}\left(\frac{\pi}{\alpha}\Big/\sin\frac{\pi}{\alpha}\right)^{\alpha/(\alpha-1)}\left(\frac{1}{\varepsilon}\right)^{2\alpha/(\alpha-1)}\tag{3.13}$$

It can be readily checked by (3.11) that $h(x)=(\pi/\alpha)(\sin\pi/\alpha)^{-1}(2x)^{(1-\alpha)/\alpha}$ satisfies (3.6); hence we have by Theorem 4 and (3.13) that

$$\log P\left(\sum_{n\geqslant 1}(n+a)^{-\alpha}\xi_n^2\leqslant\varepsilon^2\right)$$

$$\sim-\left(\frac{\pi}{\alpha}\Big/\sin\frac{\pi}{\alpha}\right)\left(\int_0^{\gamma}(2x)^{(1-\alpha)/\alpha}\,dx-\gamma(2\gamma)^{(1-\alpha)/\alpha}\right)$$

$$=-2^{(1-\alpha)/\alpha}\left(\frac{\pi}{\alpha}\Big/\sin\frac{\pi}{\alpha}\right)(\alpha-1)\,\gamma^{1/\alpha}$$

$$\sim-2^{-1}(\alpha-1)\left(\frac{\pi}{\alpha}\Big/\sin\frac{\pi}{\alpha}\right)^{\alpha/(\alpha-1)}\left(\frac{1}{\varepsilon}\right)^{2/(\alpha-1)}$$

The following example shows how to deal with multiple index cases. The idea is try to use the estimates for one index less. Previously, Csáki[7] investigated a special case [see (3.5)] of the following example using entirely different methods. His methods do not seem to generalize to all the cases we consider.

Example 2. Let $a_{ij}=(i+a)^{-\alpha}(j+b)^{-\beta}$, $\alpha\geqslant\beta>1$, and $a>-1$, $b>-1$. As noted in Remark 2, we need to use (3.9) to find a function $h(x)$ and then check that the function $h(x)$ satisfies (3.6). Since we are considering the multiple index, Lemma 7 cannot be employed directly as in Example 1. Hence we have to estimate the series in (3.9) in order to find the function $h(x)$.

For any positive integers $i_0>1$, $j_0>1$, we have

$$\begin{aligned}\int_{i_0+1}^{\infty}\int_{j_0+1}^{\infty}&\frac{dx\,dy}{2\gamma+(x+a)^{\alpha}(y+b)^{\beta}}+\sum_{i=1}^{i_0}\sum_{j\geqslant 1}\frac{1}{2\gamma+a_{ij}^{-1}}\\ &\leqslant\sum_{i\geqslant 1}\sum_{j\geqslant 1}\frac{1}{2\gamma+(i+a)^{\alpha}(j+b)^{\beta}}=\sum_{i\geqslant 1}\sum_{j\geqslant 1}\frac{1}{2\gamma+a_{ij}^{-1}}\\ &\leqslant\int_{i_0}^{\infty}\int_{j_0}^{\infty}\frac{dx\,dy}{2\gamma+(x+a)^{\alpha}(y+b)^{\beta}}+\sum_{i=1}^{i_0}\sum_{j\geqslant 1}\frac{1}{2\gamma+a_{ij}^{-1}}\\ &\quad+\sum_{j=1}^{j_0}\sum_{i\geqslant 1}\frac{1}{2\gamma+a_{ij}^{-1}}\end{aligned}\tag{3.14}$$

For $m>1$ and $n>1$, let

$$\begin{aligned}H_{\alpha,\beta}^{m,n}(\gamma)&=\int_{m}^{\infty}\int_{n}^{\infty}\frac{dx\,dy}{2\gamma+(x+a)^{\alpha}(y+b)^{\beta}}\\ &=\int_{n+b}^{\infty}\int_{m+a}^{\infty}\frac{1}{2\gamma+x^{\alpha}y^{\beta}}\,dx\,dy\end{aligned}\tag{3.15}$$

Substituting $x=y^{-\beta/\alpha}(2\gamma)^{1/\alpha}s$ first, then $y=(m+a)^{-\alpha/\beta}(2\gamma)^{1/\beta}t^{\alpha/\beta}$ into (3.15), yields

$$\begin{aligned}H_{\alpha,\beta}^{m,n}(\gamma)&=(m+a)^{1-\alpha/\beta}(2\gamma)^{1/\beta-1}\alpha\beta^{-1}\\ &\quad\times\int_{(m+a)(n+b)^{\beta/\alpha}(2\gamma)^{-1/\alpha}}^{\infty}\left(t^{\alpha/\beta-2}\int_{t}^{\infty}\frac{ds}{1+s^{\alpha}}\right)dt\end{aligned}\tag{3.16}$$

Case (i). If $\alpha=\beta$, we obtain as $\gamma\to\infty$ that

$$H_{\beta,\beta}^{m,n}(\gamma) = (2\gamma)^{1/\beta-1}\int_\delta^\infty \frac{1}{t}\int_t^\infty \frac{1}{1+s^\beta}\,ds\,dt$$

$$\sim (2\gamma)^{1/\beta-1}\int_\delta^1 \frac{1}{t}\int_t^\infty \frac{1}{1+s^\beta}\,ds\,dt$$

$$= (2\gamma)^{1/\beta-1}\left(\int_\delta^1 \frac{1}{t}\int_0^\infty \frac{1}{1+s^\beta}\,ds\,dt - \int_\delta^1 \frac{1}{t}\int_0^t \frac{1}{1+s^\beta}\,ds\,dt\right)$$

$$\sim (2\gamma)^{1/\beta-1}\int_0^\infty \frac{1}{1+s^\beta}\,ds\int_\delta^1 \frac{1}{t}\,dt$$

$$= (2\gamma)^{1/\beta-1}\frac{\pi}{\beta}\left(\sin\frac{\pi}{\beta}\right)^{-1}\log\frac{1}{\delta}$$

$$\sim \frac{\pi}{\beta^2}\left(\sin\frac{\pi}{\beta}\right)^{-1}(2\gamma)^{1/\beta-1}\log\gamma = C\gamma^{1/\beta-1}\log\gamma \tag{3.17}$$

by observing that $\int_1^\infty t^{-1}\int_t^\infty (1+s^\beta)^{-1}\,ds\,dt < \infty$ for the first $\sim$ and $\int_\delta^1 t^{-1}\int_0^t (1+s^\beta)^{-1}\,ds\,dt \leqslant 1$ for the second $\sim$, where $\delta = (m+a)(n+b)(2\gamma)^{-1/\beta}$ and $C = (\pi/\beta^2)\cdot(\sin\pi/\beta)^{-1}\,2^{(1-\beta)/\beta}$.

Now note that (3.12) tells us that

$$\sum_{i=1}^{i_0}\sum_{j\geqslant 1}\frac{1}{2\gamma+(i+a)^\beta(j+b)^\beta} = O(\gamma^{1/\beta-1}) \tag{3.18}$$

$$\sum_{j=1}^{j_0}\sum_{i\geqslant 1}\frac{1}{2\gamma+(i+a)^\beta(j+b)^\beta} = O(\gamma^{1/\beta-1}) \tag{3.19}$$

Hence from (3.14), (3.17), (3.18), and (3.19), we obtain

$$\varepsilon^2 = \sum_{i\geqslant 1}\sum_{j\geqslant 1}\frac{1}{2\gamma+(i+a)^\beta(j+b)^\beta} \sim H_{\beta,\beta}^{m,n}(\gamma) \sim C\gamma^{1/\beta-1}\log\gamma = h(\gamma) \tag{3.20}$$

By a similar argument, we can now check that $h(x) = Cx^{(1-\beta)/\beta}\log x$ also satisfies (3.6). Hence by Theorem 4 for $h(x) = Cx^{(1-\beta)/\beta}\log x$, we have as $\varepsilon \to 0$

$$\log P\left(\sum_{i\geqslant 1}\sum_{j\geqslant 1}(i+a)^{-\beta}(j+b)^{-\beta}\xi_{ij}^2 \leqslant \varepsilon^2\right)$$

$$\sim -\frac{\pi}{\beta^2}\left(\sin\frac{\pi}{\beta}\right)^{-1}\left(\int_1^\gamma (2x)^{1/\beta-1}\log x\,dx - \gamma(2\gamma)^{1/\beta-1}\log\gamma\right)$$

$$\sim -\frac{\pi}{\beta^2}\left(\sin\frac{\pi}{\beta}\right)^{-1} 2^{1/\beta-1}(\beta-1)\,\gamma^{1/\beta}\log\gamma \tag{3.21}$$

From (3.20), we have $2\log\varepsilon \sim (\beta^{-1}-1)\log\gamma + \log\log\gamma \sim (\beta^{-1}-1)\log\gamma$, which is

$$\log\gamma \sim 2\beta(\beta-1)^{-1}\log\frac{1}{\varepsilon} \tag{3.22}$$

Hence $\varepsilon^2 \sim 2C\beta(\beta-1)^{-1}\gamma^{(1-\beta)/\beta}\log\varepsilon^{-1}$, which gives us

$$\gamma \sim \left(2C\frac{\beta}{\beta-1}\right)^{\beta/(\beta-1)}\left(\frac{1}{\varepsilon}\right)^{2\beta/(\beta-1)}\left(\log\frac{1}{\varepsilon}\right)^{\beta/(\beta-1)} \tag{3.23}$$

Putting (3.22) and (3.23) into (3.21) yields

$$\begin{aligned}\log P&\left(\sum_{i\geqslant 1}\sum_{j\geqslant 1}(i+a)^{-\beta}(j+b)^{-\beta}\xi_{ij}^2\leqslant\varepsilon^2\right)\\ &\sim -\left(\frac{\pi}{\beta}\left(\sin\frac{\pi}{\beta}\right)^{-1}\right)^{\beta/(\beta-1)}2^{1/(\beta-1)}(\beta-1)^{-1/(\beta-1)}\\ &\quad\times\left(\frac{1}{\varepsilon}\right)^{2/(\beta-1)}\left(\log\frac{1}{\varepsilon}\right)^{\beta/(\beta-1)}\end{aligned} \tag{3.24}$$

Case (ii). If $\alpha>\beta$, then (3.16) gives us

$$\begin{aligned}H_{\alpha,\beta}^{m,n}(\gamma) &\sim (m+a)^{1-\alpha/\beta}(2\gamma)^{1/\beta-1}\cdot\alpha\beta^{-1}\int_0^\infty\left(t^{\alpha/\beta-2}\int_t^\infty\frac{1}{1+s^\alpha}\,ds\right)dt\\ &=(m+a)^{1-\alpha/\beta}(2\gamma)^{1/\beta-1}\frac{\pi}{\alpha-\beta}\left(\sin\frac{\pi}{\beta}\right)^{-1}\end{aligned} \tag{3.25}$$

From (3.12), we have

$$\begin{aligned}&\sum_{i=1}^{i_0}\sum_{j\geqslant 1}\frac{1}{2\gamma+(i+a)^\alpha(j+b)^\beta}\\ &\qquad\sim\left(\frac{\pi}{\beta}\right)\left(\sin\frac{\pi}{\beta}\right)^{-1}(2\gamma)^{1/\beta-1}\sum_{i=1}^{i_0}\frac{1}{(i+a)^{\alpha/\beta}}\end{aligned} \tag{3.26}$$

and

$$\sum_{j=1}^{j_0}\sum_{i\geqslant 1}\frac{1}{2\gamma+(i+a)^\alpha(j+b)^\beta}=O(\gamma^{1/\alpha-1})=o(\gamma^{1/\beta-1}) \tag{3.27}$$

Thus by (3.14), (3.25), and (3.26), we obtain

$$\sum_{i=1}^{i_0} \frac{1}{(i+a)^{\alpha/\beta}} + \frac{\beta}{\alpha-\beta} (i_0+1+a)^{1-\alpha/\beta}$$
$$\leqslant \liminf_{\gamma\to\infty} \left(\frac{\pi}{\beta}\right)^{-1} \left(\sin\frac{\pi}{\beta}\right) (2\gamma)^{1-1/\beta} \sum_{i\geqslant 1} \sum_{j\geqslant 1} \frac{1}{2\gamma+(i+a)^{\alpha}(j+b)^{\beta}}$$
$$\leqslant \limsup_{\gamma\to\infty} \left(\frac{\pi}{\beta}\right)^{-1} \left(\sin\frac{\pi}{\beta}\right) (2\gamma)^{1-1/\beta} \sum_{i\geqslant 1} \sum_{j\geqslant 1} \frac{1}{2\gamma+(i+a)^{\alpha}(j+b)^{\beta}}$$
$$\leqslant \sum_{i=1}^{i_0} \frac{1}{(i+a)^{\alpha/\beta}} + \frac{\beta}{\alpha-\beta} (i_0+a)^{1-\alpha/\beta} \tag{3.28}$$

Let $i_0 \to \infty$; we obtain

$$\varepsilon^2 = \sum_{i\geqslant 1} \sum_{j\geqslant 1} \frac{1}{2\gamma+(i+a)^{\alpha}(j+b)^{\beta}}$$
$$\sim (2\gamma)^{1/\beta-1} \left(\frac{\pi}{\beta}\right)\left(\sin\frac{\pi}{\beta}\right)^{-1} \sum_{i\geqslant 1} \frac{1}{(i+a)^{\alpha/\beta}} = D\gamma^{1/\beta-1} = h(\gamma) \tag{3.29}$$

where

$$D = 2^{1/\beta-1} \left(\frac{\pi}{\beta}\right)\left(\sin\frac{\pi}{\beta}\right)^{-1} \sum_{i\geqslant 1} \frac{1}{(i+a)^{\alpha/\beta}}$$

By a similar argument as above, we can now check that $h(x) = Dx^{(1-\beta)/\beta}$ also satisfies (3.6). Now by Theorem 4 for $h(x) = Dx^{(1-\beta)/\beta}$, we have

$$\log P\left(\sum_{i\geqslant 1} \sum_{j\geqslant 1} (i+a)^{-\alpha}(j+b)^{-\beta} \xi_{ij}^2 \leqslant \varepsilon^2\right)$$
$$\sim -\left(\int_0^{\gamma} Dx^{(1-\beta)/\beta}\, dx - \gamma\, D\gamma^{(1-\beta)/\beta}\right) = -D(\beta-1)\,\gamma^{1/\beta} \tag{3.30}$$

And furthermore, from (3.29), we have

$$\gamma \sim D^{\beta/(\beta-1)} \left(\frac{1}{\varepsilon}\right)^{2\beta/(\beta-1)} \tag{3.31}$$

Substituting (3.31) into (3.30) yields

$$\log P\left(\sum_{i\geqslant 1} \sum_{j\geqslant 1} (i+a)^{-\alpha}(j+b)^{-\beta} \xi_{ij}^2 \leqslant \varepsilon^2\right)$$
$$\sim -(\beta-1)\, D^{\beta/(\beta-1)} \left(\frac{1}{\varepsilon}\right)^{2/(\beta-1)} \tag{3.32}$$

Example 3. By looking at what we did in Example 2 and induction, we can also obtain the following result for $\beta > 1$ and $a_n > -1$, $n = 1, 2,..., d$:

$$\log P\left(\sum_{i_1 \geq 1} \cdots \sum_{i_d \geq 1} (i_1 + a_1)^{-\beta} \cdots (i_d + a_d)^{-\beta} \xi^2_{i_1 \cdots i_d} \leq \varepsilon^2\right)$$
$$\sim 2^{-1} \left(\frac{\pi}{(d-1)!\, \beta} \bigg/ \sin\frac{\pi}{\beta}\right)^{\beta/(\beta-1)} (\beta - 1)^{1-(d-1)\beta/(\beta-1)}$$
$$\times \left(\frac{1}{\varepsilon^2}\right)^{1/(\beta-1)} \left(\log\frac{1}{\varepsilon^2}\right)^{(d-1)\beta/(\beta-1)}$$

where $\xi_{i_1 \cdots i_d}$ are i.i.d. $N(0, 1)$.

Remark 4. It follows from the Karhunen–Loéve expansion that the following representation holds in distribution (see Kuelbs[15]):

$$\int_0^1 \int_0^1 W^2(s, t)\, ds\, dt = \sum_{i \geq 1} \sum_{j \geq 1} \left(i - \frac{1}{2}\right)^{-2} \left(j - \frac{1}{2}\right)^{-2} \pi^{-4} \xi^2_{ij}$$

where $W(s, t)$ is standard Brownian sheet. Hence as a particular case of (3.24) [also from (3.5)], we obtain

$$\log P\left(\int_0^1 \int_0^1 W^2(s, t)\, ds\, dt \leq \varepsilon^2\right)$$
$$= \log P\left(\sum_{i \geq 1} \sum_{j \geq 1} \left(i - \frac{1}{2}\right)^{-2} \left(j - \frac{1}{2}\right)^{-2} \xi^2_{ij} \leq \pi^4 \varepsilon^2\right)$$
$$\sim -\frac{1}{8\pi^2} \left(\frac{1}{\varepsilon}\right)^2 \left(\log\frac{1}{\varepsilon^2}\right)^2 \qquad (3.33)$$

This estimate is related to the extension of Chung's law of the iterated logarithm for $W(s, t)$ and can be used to show that if there exist a function $\phi(T)$ such that

$$\varliminf_{T \to \infty} \sup_{0 \leq s, t \leq T} |W(s, t)| / \phi(T) = \text{constant} \quad \text{a.s.}$$

then $\phi(T)$ has to be bigger than $T \log\log\log T / (\log\log T)^{1/2}$, which is the best known lower bound for $\phi(T)$ and is conjectured to be the right one. The best known upper bound for $\phi(T)$ is given by Bass,[3] and is $T(\log\log\log T)^{3/2} / (\log\log T)^{1/2}$.

4. LOWER TAIL OF SOME GAUSSIAN PROCESSES

We shall now demonstrate how to find the lower tail of some Gaussian processes under the L_2-norms by using our comparison results in Section 2 and the examples in Section 3. More applications can be found in Li,[17] where limit theorems are proved.

Let $\{U(t): -\infty < t < \infty\}$ be the Ornstein–Uhlenbeck process, i.e., the Gaussian process with mean zero and covariance function $r(s, t) = E(U(s)\, U(t)) = \exp(-|s-t|)$ for $s, t \in (-\infty, \infty)$. Then we have the following.

Theorem 5. For $-\infty < a < b < \infty$,

$$\log P\left(\int_a^b U^2(t)\, dt \leqslant \varepsilon^2\right) \sim -\frac{(b-a)^2}{4} \cdot \frac{1}{\varepsilon^2} \qquad \text{as} \quad \varepsilon \to 0$$

Proof. By the Karhunen–Loéve expansion (see Ash and Gardner,[2] Kac and Siegert[13]), we have in distribution $\int_a^b U^2(t)\, dt = \sum_{n \geqslant 1} \lambda_n \xi_n^2$, where λ_n are the eigenvalues of the equation

$$\lambda f(t) = \int_a^b \exp(-|s-t|)\, f(s)\, ds \qquad a \leqslant t \leqslant b$$

We need to find the eigenvalues λ_n. From the above equation, we obtain

$$\lambda f(t) = e^{-t} \int_a^t e^s f(s)\, ds + e^t \int_t^b e^{-s} f(s)\, ds \qquad a \leqslant t \leqslant b \tag{4.1}$$

and a pair of boundary condition is

$$\lambda f(a) = e^a \int_a^b e^{-s} f(s)\, ds \qquad \lambda f(b) = e^{-b} \int_a^b e^s f(s)\, ds \tag{4.2}$$

Differentiating (4.1), we obtain

$$\lambda f'(t) = -e^{-t} \int_a^t e^s f(s)\, ds + e^t \int_t^b e^{-s} f(s)\, ds \qquad a \leqslant t \leqslant b \tag{4.3}$$

and another pair of boundary condition is

$$\lambda f'(a) = e^a \int_a^b e^{-s} f(s)\, ds \qquad \lambda f'(b) = -e^{-b} \int_a^b e^s f(s)\, ds \tag{4.4}$$

Differentiating (4.3) again and combining the boundary conditions (4.2) and (4.4), we have

$$\lambda f''(t)=(\lambda-2)f(t) \qquad a\leqslant t\leqslant b$$

and (4.5)

$$f(a)=f'(a) \qquad f(b)=-f'(b)$$

From (4.5), we have

$$(\lambda-2)\int_a^b f^2(t)\,dt=\lambda\int_a^b f(t)\,f''(t)\,dt$$
$$=-\lambda\left(f^2(a)+f^2(b)+\int_a^b (f'(t))^2\,dt\right)$$

Hence $0<\lambda\leqslant 2$. Let $\eta=(2\lambda^{-1}-1)^{1/2}\geqslant 0$. We obtain $f(t)=c_1\sin\eta t+c_2\cos\eta t$ where c_1 and c_2 are constants. Substituting this $f(t)$ into the two boundary conditions in (4.5) and simplifying them yields

$$c_1(\sin\eta a-\eta\cos\eta a)+c_2(\cos\eta a+\eta\sin\eta a)=0$$
$$c_1(\sin\eta b+\eta\cos\eta b)+c_2(\cos\eta b-\eta\sin\eta b)=0$$

To find nontrivial constants c_1 and c_2, i.e., $c_1^2+c_2^2\neq 0$, the determinant of above equations has to be zero, that is,

$$2\eta\cos(b-a)\eta=(\eta^2-1)\sin(b-a)\eta \tag{4.6}$$

Clearly (4.6) has a finite number of solution on bounded intervals and has exactly one solution in $[(k\pi-\pi/2)/(b-a),\ (k\pi+\pi/2)/(b-a)]$ for $k\geqslant k_0$ where k_0 is large enough. Let η_k be the solution of (4.6) in $[(k\pi-\pi/2)/(b-a),\ (k\pi+\pi/2)/(b-a)]$ for $k\geqslant k_0$ [η_k is not necessary the kth solution of (4.6)]. We can see that $\eta_k=k\pi/(b-a)+O(k^{-1})$ by using the inequality $\tan x>x$ on $(0,\pi/2)$. Thus

$$\sum_{k\geqslant k_0}\left|\frac{k^2\pi^2(b-a)^{-2}}{\eta_k^2+1}-1\right|<\infty$$

Hence we have by Theorem 2, Corollary 4, and Example 1 or (3.4), as $\varepsilon\to 0$

$$\log P\left(\int_a^b U^2(t)\,dt \leqslant \varepsilon^2\right)$$

$$= \log P\left(\sum_{n\geqslant 1} \lambda_n \xi_n^2 \leqslant \varepsilon^2\right) \sim \log P\left(\sum_{n\geqslant n_0} \lambda_n \xi_n^2 \leqslant \varepsilon^2\right)$$

$$\sim \log P\left(\sum_{k\geqslant k_0} \frac{2}{\eta_k^2+1} \xi_k^2 \leqslant \varepsilon^2\right) \sim \log P\left(\sum_{k\geqslant k_0} \frac{2(b-a)^2}{k^2\pi^2} \xi_k^2 \leqslant \varepsilon^2\right)$$

$$\sim \log P\left(\sum_{k\geqslant 1} \frac{1}{k^2} \xi_k^2 \leqslant \frac{\pi^2}{2(b-a)^2} \varepsilon^2\right) \sim -\frac{(b-a)^2}{4}\cdot\frac{1}{\varepsilon^2}$$

This finishes the proof. □

Remark 5. If we want to find the asymptotic behavior of $P(\int_a^b U^2(t)\,dt \leqslant \varepsilon^2)$ as $\varepsilon \to 0$ without log, we just need to count the number of zeros of equation (4.6) on $[0, (k_0\pi - \pi/2)/(b-a)]$ and use Theorem 3 instead of Corollary 4.

Let $\{B(t): 0 \leqslant t \leqslant 1\}$ be the Brownian bridge. We next consider a weighted L_2-norms for $B(t)$.

Theorem 6. For $\alpha > 0$ and $\beta = 1 - \alpha^{-1} < 1$,

$$P\left(\int_0^1 B^2(t^\alpha)\,dt \leqslant \varepsilon^2\right) \left(= P\left(\int_0^1 \frac{1}{t^\beta} B^2(t)\,dt \leqslant \alpha\varepsilon^2\right)\right)$$

$$\sim C_\alpha \varepsilon^{-(\alpha-1)/2(\alpha+1)} \exp\left(-\frac{\alpha}{2(\alpha+1)^2}\cdot\frac{1}{\varepsilon^2}\right) \qquad \text{as} \quad \varepsilon \to 0$$

where C_α is a positive constant.

Proof. For $\alpha > 0$, $\{B(t^\alpha): 0 \leqslant t \leqslant 1\}$ is a Gaussian process with mean zero and covariance function $r(s, t) = E(B(s^\alpha)\,B(t^\alpha)) = \min(s^\alpha, t^\alpha) - s^\alpha t^\alpha$ for $s, t \in [0, 1]$. Similar to what we did in the proof of Theorem 5. We need to find the eigenvalues λ_n of the equation $\lambda f(t) = \int_0^1 r(s, t) f(s)\,ds$, which is

$$\lambda f(t) = (1 - t^\alpha) \int_0^t s^\alpha f(s)\,ds + t^\alpha \int_t^1 (1 - s^\alpha) f(s)\,ds \qquad (4.7)$$

with boundary condition $f(0) = 0$ and $f(1) = 0$. Differentiating (4.7), we obtain $\lambda f'(t) = \alpha t^{\alpha-1} \int_t^1 f(s)\,ds$. Moving $t^{\alpha-1}$ to the left and differentiating the equation yields

$$\lambda t f''(t) - \lambda(\alpha - 1) f'(t) + \alpha t^\alpha f(t) = 0$$

The general solution of this equation is (see Kamke,[14] page 440)

$$f(t)=c_1 t^{\alpha/2} J_{\alpha/(\alpha+1)}(2(\alpha+1)^{-1}\sqrt{\alpha/\lambda}\, t^{(\alpha+1)/2}) + c_2 t^{\alpha/2} J_{-\alpha/(\alpha+1)}(2(\alpha+1)^{-1}\sqrt{\alpha/\lambda}\, t^{(\alpha+1)/2})$$

where $J_\nu(x)$ is the Bessel function. Using the boundary condition $f(0)=0$ and $f(1)=0$, we have $J_{\alpha/(\alpha+1)}(2(\alpha+1)^{-1}\sqrt{\alpha/\lambda})=0$. Hence by the asymptotic formula for zeros of the Bessel function (see Watson[20]), we have

$$\frac{2}{\alpha+1}\sqrt{\frac{\alpha}{\lambda_n}}=\left(n+\frac{\alpha-1}{4(\alpha+1)}\right)\pi+O\left(\frac{1}{n}\right)$$

which shows that

$$\sum_{n\geqslant 1}\left|\frac{4\alpha}{(\alpha+1)^2\pi^2}\left(n+\frac{\alpha-1}{4(\alpha+1)}\right)^{-2}\cdot\frac{1}{\lambda_n}-1\right|<\infty$$

Thus by Theorem 2 and (3.4), we obtain

$$\begin{aligned}
&P\left(\int_0^1 B^2(t^\alpha)\,dt\leqslant\varepsilon^2\right)\\
&=P\left(\sum_{n\geqslant 1}\lambda_n\xi_n^2\leqslant\varepsilon^2\right)\\
&\sim D_\alpha P\left(\sum_{n\geqslant 1}\frac{4\alpha}{(\alpha+1)^2\pi^2}\cdot\left(n+\frac{\alpha-1}{4(\alpha+1)}\right)^{-2}\xi_n^2\leqslant\varepsilon^2\right)\\
&\sim C_\alpha\varepsilon^{-(\alpha-1)/2(\alpha+1)}\exp\left(-\frac{\alpha}{2(\alpha+1)^2}\cdot\frac{1}{\varepsilon^2}\right)\qquad\text{as}\quad\varepsilon\to 0
\end{aligned}$$

This completes our proof. □

5. THE SUP-NORMS AND THE l_p-NORMS

We first study the relations between the two sup-norms q_a, q_b: $R^\infty\to[0,\infty]$, where $q_a(x)=\sup_{n\geqslant 1}|x_n|/a_n$, $q_b(x)=\sup_{n\geqslant 1}|x_n|/b_n$, and (a_n) and (b_n) are given sequence of positive numbers.

Suppose $q_a(\xi)<\infty$ a.s. $q_b(\xi)<\infty$ a.s. and put

$$F_a(t)=P(q_a(\xi)\leqslant t)\qquad F_b(t)=P(q_b(\xi)\leqslant t)\qquad t\geqslant 0$$

$$C(a)=\inf\{t>0\mid F_a(t)>0\}\qquad C(b)=\inf\{t>0\mid F_b(t)>0\}$$

Theorem 7. If $\lim_{n\to\infty} a_n/b_n = 1$, then $C(a) = C(b)$.

Proof. First observe that

$$F_a(t) = \prod_{n\geqslant 1} P(|\xi_n| \leqslant ta_n) = \prod_{n\geqslant 1} (1 - R(ta_n))$$

where $R(t) = (2/\pi)^{1/2} \int_t^\infty \exp(-x^2/2)\, dx$. Hence $F_a(t) > 0$ iff $\sum_{n\geqslant 1} R(ta_n) < \infty$, and, using the inequality as in Hoffmann-Jørgensen et al.,[10]

$$C_1 \exp(-t^2/2)/(1+t) < R(t) < C_2 \exp(-t^2/2)/(1+t) \qquad t > 0$$

for some finite positive constants C_1 and C_2, we see that $F_a(t) > 0$ iff

$$\sum_{n\geqslant 1} \exp(-t^2 a_n^2/2)/(1 + ta_n) < \infty \tag{5.1}$$

Now (5.1) converges for all $t > \delta$ iff

$$\sum_{n\geqslant 1} \exp(-t^2 a_n^2/2) < \infty \qquad \text{for every} \quad t > \delta$$

Hence we have

$$C(a) = \inf\left\{ t > 0 \,\middle|\, \sum_{n\geqslant 1} \exp(-t^2 a_n^2/2) < \infty \right\}$$

$$C(b) = \inf\left\{ t > 0 \,\middle|\, \sum_{n\geqslant 1} \exp(-t^2 b_n^2/2) < \infty \right\}$$

Now for any $\varepsilon > 0$, there exists N such that $(b_n/a_n)^2 \geqslant 1 - 2^{-1}\varepsilon^2/(C(a)+\varepsilon)^2$ for $n \geqslant N$. Hence

$$\begin{aligned}
&\sum_{n\geqslant 1} \exp(-(C(a)+\varepsilon)^2\, b_n^2) \\
&\quad = \sum_{n=1}^{N} \exp(-(C(a)+\varepsilon)^2\, b_n^2) + \sum_{n>N} \exp(-(C(a)+\varepsilon)^2\, a_n^2 (b_n/a_n)^2) \\
&\quad \leqslant N + \sum_{n>N} \exp(-(C(a)+\varepsilon)^2\, a_n^2 (1 - 2^{-1}\varepsilon^2/(C(a)+\varepsilon)^2)) \\
&\quad \leqslant N + \sum_{n>N} \exp\left(-\left(C(a) + \frac{\varepsilon}{2}\right)^2 a_n^2\right) < \infty
\end{aligned}$$

Thus $C(a) \geqslant C(b)$. And $C(a) = C(b)$ by symmetry. □

In Hoffmann-Jørgensen et al.,[10] it is shown that if $C(a) > 0$, then $F_a(t)$ can have a jump of any size in $[0, 1)$ at $C(a)$. Our next result shows that if (a_n) and (b_n) are fairly close, then $F_a(t)$ and $F_b(t)$ will jump at the same point.

Theorem 8. If $\sum_{n\geq 1} |1 - a_n/b_n| < \infty$, then $F_a(C(a)) = 0$ iff $F_b(C(b)) = 0$.

Proof. First observe that, for $t > 0$,

$$\min\left(1, \frac{a_n}{b_n}\right) \leqslant \int_0^{ta_n} \exp(-x^2/2)\,dx \Big/ \int_0^{tb_n} \exp(-x^2/2)\,dx$$
$$\leqslant \max\left(1, \frac{a_n}{b_n}\right) \tag{5.2}$$

and $0 < \prod_{n\geq 1} \min(1, a_n/b_n) \leqslant \prod_{n\geq 1} \max(1, a_n/b_n) < \infty$. Now from Theorem 7, we have $C(a) = C(b)$ and therefore, for $t > C(a) = C(b)$,

$$F_a(t) = \prod_{n\geq 1} \sqrt{\frac{2}{\pi}} \int_0^{ta_n} \exp(-x^2/2)\,dx > 0$$

$$F_b(t) = \prod_{n\geq 1} \sqrt{\frac{2}{\pi}} \int_0^{tb_n} \exp(-x^2/2)\,dx > 0$$

Hence for $t > C(a) = C(b)$,

$$F_a(t)/F_b(t)$$
$$= \left(\prod_{n\geq 1} \sqrt{\frac{2}{\pi}} \int_0^{ta_n} \exp(-x^2/2)\,dx\right) \Big/ \left(\prod_{n\geq 1} \sqrt{\frac{2}{\pi}} \int_0^{tb_n} \exp(-x^2/2)\,dx\right)$$
$$= \prod_{n\geq 1} \left(\int_0^{ta_n} \exp(-x^2/2)\,dx \Big/ \int_0^{tb_n} \exp(-x^2/2)\,dx\right)$$

and by (5.2)

$$\prod_{n\geq 1} \min\left(1, \frac{a_n}{b_n}\right) \leqslant F_a(t)/F_b(t) \leqslant \prod_{n\geq 1} \max\left(1, \frac{a_n}{b_n}\right)$$

Note that $F_a(t)$ and $F_b(t)$ are absolutely continuous for $t > C(a) = C(b)$ (see Hoffmann-Jørgensen et al.[10]). Letting $t \to C(a) = C(b)$ from the right, we obtain $F_a(C(a)) = 0$ if and only if $F_b(C(b)) = 0$. □

The next result tells us when $F_a(t)$ and $F_b(t)$ have the same order of magnitude as $t \to 0$ provided $C(a) = C(b) = 0$.

Theorem 9. If $\sum_{n\geq 1} |1 - a_n/b_n| < \infty$ and $C(a) = C(b) = 0$, then

$$\lim_{t\to 0^+} F_a(t)/F_b(t) = \prod_{n\geq 1} a_n/b_n$$

Proof. Following the proof of Theorem 8 and noting that by L'Hôpital's rule, we have

$$\lim_{t\to 0^+}\int_0^{ta_n}\exp(-x^2/2)\,dx\Big/\int_0^{tb_n}\exp(-x^2/2)\,dx$$

$$=\lim_{t\to 0^+}\frac{a_n\exp(-a_n^2t^2/2)}{b_n\exp(-b_n^2t^2/2)}=\frac{a_n}{b_n}$$

Hence by the D.C.T. for $\prod_{n\geqslant 1}(\int_0^{ta_n}\exp(-x^2/2)\,dx/\int_0^{tb_n}\exp(-x^2/2)\,dx)$, we conclude the result. □

Corollary 6. If $\sum_{n\geqslant 1}|1-a_n/b_n|\,I_{\{a_n\leqslant b_n\}}(n)<\infty$, $C(a)=C(b)=0$ and $F_a(t)/F_b(t)$ exists and is not equal to zero as $t\to 0^+$, then

$$\sum_{n\geqslant 1}\left|1-\frac{a_n}{b_n}\right|<\infty \qquad\text{and}\qquad \lim_{t\to 0^+}F_a(t)/F_b(t)=\prod_{n\geqslant 1}a_n/b_n$$

Proof. The proof of Corollary 6 is similar to Corollary 3 with Remark 1 also being applied here. □

Motivated by the comparison results for the l_2-norms and the sup-norms, we consider the l_p-norms where $1\leqslant p<\infty$.

Proposition 1. For any positive integer N

$$\lim_{\varepsilon\to 0}P\left(\sum_{n=1}^N a_n|\xi_n|^p\leqslant\varepsilon^p\right)\Big/P\left(\sum_{n=1}^N b_n|\xi_n|^p\leqslant\varepsilon^p\right)=\left(\prod_{n=1}^N\frac{a_n}{b_n}\right)^{-1/p}$$

Proof. Let

$$B_{N,p}=\left\{\bar{x}=(x_1,\dots,x_N)\in R^N:\ \sum_{n=1}^N|x_n|^p\leqslant 1\right\}$$

We have

$$P\left(\sum_{n=1}^N a_n|\xi_n|^p\leqslant\varepsilon^p\right)$$

$$=(2\pi)^{-N/2}\left(\prod_{n=1}^N a_n\right)^{-1/p}\varepsilon^N\int_{B_{N,p}}\exp\left(-\frac{1}{2}\sum_{n=1}^m\varepsilon^2a_n^{-2/p}\right)d\bar{x}$$

$$\sim(2\pi)^{-N/2}\left(\prod_{n=1}^N a_n\right)^{-1/p}\varepsilon^N\int_{B_{N,p}}1\,d\bar{x}\qquad\text{as}\quad\varepsilon\to 0^+$$

Hence the claim follows. □

Proposition 2. If $a_n \geqslant b_n$ and $\sum_{n\geqslant 1} |1 - a_n/b_n| < \infty$, then

$$\left(\prod_{n\geqslant 1} \frac{a_n}{b_n}\right)^{-1/p} \leqslant P\left(\sum_{n\geqslant 1} a_n |\xi_n|^p \leqslant \varepsilon^p\right) \Big/ P\left(\sum_{n\geqslant 1} b_n |\xi_n|^p \leqslant \varepsilon^p\right) \leqslant 1$$

Proof. For any positive integer m, we have

$$P\left(\sum_{n\geqslant 1} a_n |\xi_n|^p \leqslant \varepsilon^p\right) \leqslant P\left(\sum_{n\geqslant 1} b_n |\xi_n|^p \leqslant \varepsilon^p\right)$$

and

$$\begin{aligned}
&\left(\prod_{n=1}^{m} a_n\right)^{1/p} \cdot P\left(\sum_{n=1}^{m} a_n |\xi_n|^p \leqslant \varepsilon^p\right) \\
&= (2\pi)^{-m/2} \int \cdots \int_{\sum_{n=1}^{m} |y_n|^p \leqslant \varepsilon^p} \exp\left(-\frac{1}{2} \sum_{n=1}^{m} y_n^2 a_n^{-2/p}\right) dy_1 \cdots dy_m \\
&\geqslant (2\pi)^{-m/2} \int \cdots \int_{\sum_{n=1}^{m} |y_n|^p \leqslant \varepsilon^p} \exp\left(-\frac{1}{2} \sum_{n=1}^{m} y_n^2 b_n^{-2/p}\right) dy_1 \cdots dy_m \\
&= \left(\prod_{n=1}^{m} b_n\right)^{1/p} \cdot P\left(\sum_{n=1}^{m} b_n |\xi_n|^p \leqslant \varepsilon^p\right)
\end{aligned}$$

Hence

$$\left(\prod_{n=1}^{m} \frac{a_n}{b_n}\right)^{-1/p} \leqslant P\left(\sum_{n=1}^{m} a_n |\xi_n|^p \leqslant \varepsilon^p\right) \Big/ P\left(\sum_{n=1}^{m} b_n |\xi_n|^p \leqslant \varepsilon^p\right) \leqslant 1$$

Let $m \to \infty$, and we obtain desired result. □

Now we state some conjectures and open problems that arise from this work.

(i) We conjecture that if $\sum_{n\geqslant 1} |1 - a_n/b_n| < \infty$, then for $1 \leqslant p < \infty$

$$\lim_{\varepsilon \to 0} P\left(\sum_{n\geqslant 1} a_n |\xi_n|^p \leqslant \varepsilon^p\right) \Big/ P\left(\sum_{n\geqslant 1} b_n |\xi_n|^p \leqslant \varepsilon^p\right) = \left(\prod_{n\geqslant 1} \frac{a_n}{b_n}\right)^{-1/p}$$

(ii) We conjecture that for some positive constant C,

$$P\left(\sum_{n\geqslant 1} a_n \xi_n^2 \leqslant \varepsilon^2\right) \sim C P\left(\sum_{n\geqslant 1} b_n \xi_n^2 \leqslant \varepsilon^2\right) \quad \text{as} \quad \varepsilon \to 0$$

if and only if $\prod_{n\geqslant 1} a_n/b_n$ exists and $C = (\prod_{n\geqslant 1} a_n/b_n)^{-1/2}$.

Note that this conjecture is true by Corollary 3 if we assume

$$\sum_{n\geqslant 1} |1-a_n/b_n|\, I_{\{a_n\leqslant b_n\}}(n) < \infty$$

Also the two examples in Remark 1 are covered in this conjecture. If this conjecture is true, the we can ask whether it is true for the l_p-norms as in (i).

(iii) We can ask under what conditions on a_n and b_n does it follow that

$$\log P\left(\sum_{n\geqslant 1} a_n\, |\xi_n|^p \leqslant \varepsilon^p\right) \sim C \log P\left(\sum_{n\geqslant 1} b_n\, |\xi_n|^p \leqslant \varepsilon^p\right) \quad \text{as} \quad \varepsilon \to 0 \tag{5.3}$$

where C is a positive constant and $1 \leqslant p \leqslant \infty$.

This type of result will tell us more about the tail behavior of the l_p-norms and help us to find the rate of escape function. Our Corollary 4 gives that if $b_n = a_{n+N}$, then (5.3) is true for $p=2$ and $C=1$.

(iv) We conjecture that if $b_n = a_{n+N}$, then (5.3) is true for $1 \leqslant p \leqslant \infty$ and $C=1$.

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REFERENCES

1. Anderson, T. W., and Darling, D. A. (1952). Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes. *Ann. Math. Stat.* **23**, 193–212.
2. Ash, R. B., and Gardner, M. F. (1975). *Topics in Stochastic Processes*, Academic Press, New York.
3. Bass, R. F. (1988). Probability estimates for multiparameter Brownian processes. *Ann. Prob.* **16**, 251–264.
4. Borell, C. (1975). The Brunn-Minkowski inequality in Gauss space. *Invent. Math.* **30**, 207–216.
5. Cox, D. D. (1980). Normalized Brownian motion on Banach spaces. Ph.D thesis, University of Washington, Seattle.
6. Cox, D. D. (1982). On the existence of natural rate of escape functions for infinite dimensional Brownian motions. *Ann. Prob.* **10**, 623–638.

7. Csáki, E. (1982). On small values of the square integral of a multiparameter Wiener process. *Statistics and Probability. Proc. of the 3rd Pannonian Symp. on Math. Stat.* D. Reidel, Boston, pp. 19–26.
8. Erickson, K. B. (1980). Rates of escape of infinite dimensional Brownian motion. *Ann. Prob.* **8**, 325–338.
9. Fernique, X. (1970). Intégrabilité des vecteurs gaussiens. *C. R. Acad. Sci. Paris* **270**, 1698–1699.
10. Hoffmann-Jørgensen, J., Shepp, L. A., and Dudley, R. M. (1979). On the lower tail of Gaussian seminorms. *Ann. Prob.* **7**, 319–342.
11. Hwang, C. (1980). Gaussian measures of large balls in a Hilbert space. *Proc. AMS* **78**, 107–110.
12. Ibragimov, I. A. (1982). On the probability that a Gaussian vector with values in a Hilbert space hits a sphere of small radius. *J. Sov. Math.* **20**, 2164–2174.
13. Kac, M., and Siegert, A. J. F. (1947). An explicit representation of a stationary Gaussian process. *Ann. Math. Stat.* **18**, 438–442.
14. Kamke, E. (1977). *Differentialgleichungen.* Teubner, Stuttgart.
15. Kuelbs, J. (1968). The invariance principle for a lattice of random variables. *Ann. Math. Stat.* **39**, 382–389.
16. Kuelbs, J. (1978). Rates of growth for Banach space valued independent increments processes. *Proc. 2nd Oberwolfach Conference on Probability on Banach Spaces.* Lecture Notes in Mathematics, Vol. 709, pp. 151–169, Springer, Berlin.
17. Li, W. V. (1992). Lim inf results for the Wiener process and its increments under the L_2-norm. *Probability Theory and Related Fields*, to appear.
18. Marcus, M. B., and Shepp, L. A. (1972). Sample behavior of Gaussian processes. *Proc. Sixth. Berkeley Symp. Math. Stat. Prob.* **2**, 423–441.
19. Sytaya, G. N. (1974). On some asymptotic representations of the Gaussian measure in a Hilbert space. In *Theory of Stochastic Processes.* Publication No. 2, pp. 93–104, Ukrainian Academy of Sciences, Republican Interdepartmental Collection (in Russian).
20. Watson, G. N. (1966). *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England.
21. Zolotarev, V. M. (1961). Concerning a certain probability problem. *Theor. Probab. Appl.* **6**, 201–204.
22. Zolotarev, V. M. (1986). Asymptotic behavior of the Gaussian measure in l_2. *J. Sov. Math.* **24**, 2330–2334.