PRICING AND HEDGING OF CLIQUET OPTIONS AND LOCALLY-CAPPED CONTRACTS

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Abstract. This paper provides a new approach for pricing and hedging popular highly path-dependent equity-linked contracts. We illustrate our technique with two examples: the locally-capped contracts (a popular design on the exchange-listed retail investment contracts on the American Stock Exchange) and the Cliquet option (extensively sold by insurance companies). Wilmott [17] describes these types of contracts as the "height of fashion in the world of equity derivatives". Existing literature proposes methods based on partial differential equations, Monte Carlo techniques or Fourier analysis. We show that there exist semi-closed-form expressions of their prices as well as of the hedging parameters.

Key words. pricing, hedging, cliquet option, locally-capped payoff.

AMS subject classifications.

1. Introduction. This paper provides analytic formulas for the prices and hedging parameters of popular equity-linked contracts that are highly path-dependent. We illustrate our technique with two examples: the locally-capped contracts (popular design on the exchange-listed retail investment contracts on the American Stock Exchange, see Bernard, Boyle and Gornall [2]) and the Cliquet option (extensively sold by insurance companies, see (Palmer [16])). Wilmott [17] describes these types of contracts as the "height of fashion in the world of equity derivatives". Numerical partial differential equations techniques have been developed to price cliquet options by Windcliff, Forsyth, and Vetzal [18] and Wilmott [17]. Monte Carlo techniques have been used by Bernard, Boyle and Gornall [2]. In this paper, we derive semi-closed-form expressions of the prices and Greeks of these types of contracts. We make use of techniques that have been applied in other contexts by Haagerup [6], Konig, Schutt and Tomczak-Jaegermann [7] and Li and Wei [11, 12].

In this paper we study a "cliquet option" and a specific locally capped contract called "monthly sum cap". These contracts are used for the sake of illustration and our techniques apply to a wide range of path-dependent contracts with payoffs based on periodical returns over the contracts' term. A "monthly sum cap" contract is a type of equity indexed annuities (EIA) which are customized investment products sold by insurance companies that provide savings and insurance benefits. In a typical equity indexed annuity (also called structured product when sold by banks), the investor pays an initial amount to the financial institution or the insurance company. At the maturity date, the payoff to the investor is based on the performance of some designated reference index. The contract participates in the gains (if any) in the reference index during this period. The detailed arrangements of how this participation is calculated vary but invariably there is some limit. For example, the limit may be expressed in terms of a participation rate (say 60%) in the return of the underlying index. Alternatively, the return on the reference portfolio can be capped where the

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cap is imposed periodically or globally. In addition, these contracts generally provide a floor of protection if the market does poorly.

There are many different ways of computing the crediting rate. In the so-called "point-to-point design", the payoff is linked to the rate of return on the underlying index over the term of the contract. The payoff is not path-dependent and closed-form formulas can often be obtained. Our study focuses on the locally-capped design. For example in a "monthly sum cap", the credited rate is based on the sum of the monthly-capped rates. This product is a typical example of globally-floored locally-capped contract since it is monthly-capped and usually has a guarantee (global floor) over the life of the contract (See for example Bernard, Boyle and Gornall [2] and Bernard and Boyle [1] for more details on these financial securities). The payoff of this index-linked contract is given by

$$X_T = K \max\left(1+g, 1+\sum_{k=1}^{n} \min\left(c, \frac{S(t_k)-S(t_{k-1})}{S(t_{k-1})}\right)\right), \tag{1.1}$$

where $t_k = k\Delta$ with $n\Delta = T$, K is the initial investment net of fees and commissions, g the guaranteed rate at maturity, $S(t)$ denotes the price of the underlying index at time t and $c \geq 0$ is the local cap for each period $t_i - t_{i-1}$.

These monthly sum cap contracts are similar to the cliquet options studied for instance by Wilmott [17]. For example the payoff of a "minimum coupon cliquet" is given by

$$Y_T = K \max\left(1+g, 1+\sum_{k=1}^{n} \max\left(0, \min\left(c, \frac{S(t_k)-S(t_{k-1})}{S(t_{k-1})}\right)\right)\right). \tag{1.2}$$

Note that the return of each period is capped at c but also has a local floor equal to 0. This design is also very popular in the insurance industry since most insurance EIAs include annual guarantees in addition to the designs described earlier. Annual guarantees result in a cliquet-style final payoff.

Section 2 exposes the financial market model and presents the two contracts under study. Section 3 provides the formulas for their prices and hedging parameters in a general arbitrage-free market. Section 4 illustrates our findings through some numerical examples in the Black and Scholes framework and in a Lévy market model.

2. Market Model Assumptions . In this section we introduce the notation for the financial market and for the two contracts under study.

2.1. Financial Model. Consider an arbitrage-free market. Let Q be the risk-neutral probability used for pricing in this market. The price at the initial time 0 of a financial derivative with payoff X_T paid at time T is given by $e^{-rT}\mathbb{E}\left[X_T\right]$, where the expectation is taken under this risk-neutral probability Q and where we suppose that the risk-free rate r is constant. The derivations in the paper require the stationarity and the independence of the increments of the underlying process. Stochastic interest rates may be added as long as their presence does not affect the stationarity and the independence properties. The financial contracts under study are both linked to periodical returns (see payoffs (1.1) and (1.2)). To simplify the notation, we denote by R_k the return of the underlying stock over the k^{th} period.

$$R_k = \frac{S(t_k)-S(t_{k-1})}{S(t_{k-1})}. \tag{2.1}$$

We assume that the discretization steps are equally spaced ($\forall k,\ t_k - t_{k-1} = \Delta$). In addition we suppose that increments of the underlying process are independent and identically distributed, which is the case for instance in the Black and Scholes model but also when the underlying asset S is modeled by a Lévy process. In section 4, we derive an example in the Black and Scholes model and then present how to extend it in a more general Lévy market model.

To derive semi-closed-form expressions for monthly sum cap contracts and cliquet options, we only need the distribution of R_k, specifically the survival function (e.g. Black-Scholes market) or its characteristic function (e.g. Lévy market) as it appears later in the derivations. We further assume that R_k is continuously distributed with density $f_R(\cdot)$ and characteristic function ϕ_R under the risk-neutral probability. Note that the distribution does not depend on k because returns are i.i.d. In a Black and Scholes market, R_k follows a lognormal distribution with constant parameters. With Lévy processes this distribution is more complicated but can still be computed. Examples are given in Section 4.

2.2. Payoffs of the Monthly Sum Cap and the Cliquet Option. We denote by X_T the payoff of a *monthly sum cap*. The contract is of European type since the payoff is paid at a fixed future maturity time T. The payoff X_T is given by

$$\begin{aligned} X_T &= K \max\left(1+g, 1+\sum_{k=1}^{n} \min(c, R_k)\right) \\ &= K(1+g) + K \max\left\{0, \sum_{k=1}^{n} C_k\right\} \end{aligned} \tag{2.2}$$

where the quantity C_k is defined by

$$C_k := \min(c, R_k) - g/n. \tag{2.3}$$

The payoff of the *cliquet option* is denoted by Y_T and can be expressed as

$$\begin{aligned} Y_T &= K \max\left(1+g, 1+\sum_{k=1}^{n} \max(0, \min(c, R_k))\right) \\ &= K(1+g) + K \max\left\{0, \sum_{k=1}^{n} Z_k\right\} \end{aligned} \tag{2.4}$$

where Z_k is defined by

$$Z_k := \max(0, \min(c, R_k)) - g/n. \tag{2.5}$$

As it turns out from the above simplification, the payoff of the locally-capped contract given by (2.2) is now very similar to the payoff of the cliquet option given by (2.4). Thus we can treat them in a similar way.

3. Pricing Formulas and Greeks.

3.1. Prices. We need to calculate the expectation of the discounted payoffs (2.2) and (2.4) under the risk-neutral probability measure to obtain semi-closed-form expressions of the prices of the two financial contracts under study. Precisely, we need to calculate the following expectation:

$$\mathbb{E} \max\left(0, \sum_{k=1}^{n} L_k\right), \tag{3.1}$$

where L_k are i.i.d. random variables. L_k is equal to C_k for the locally-capped contract and to Z_k for the cliquet option. Since L_k are identically distributed, their distribution does not depend on k and we can denote by ϕ_L their characteristic function. We will denote respectively by ϕ_C and ϕ_Z the characteristic functions of C_k and Z_k.

Proposition 3.1. *The price at time 0 of a payoff* $\max(0, \sum_{k=1}^n L_k)$ *paid at time* T *is given by*

$$\frac{ne^{-rT}}{2}\mathbb{E}\, L_1 + \frac{e^{-rT}}{\pi}\int_0^{+\infty} t^{-2}\left(1 - Re\left(\phi_L^n(t)\right)\right) dt.$$

Proof. Note that

$$\max(0, x) = \frac{x + |x|}{2}, \tag{3.2}$$

where we replace x by $\sum_{k=1}^n L_k$. It is thus equivalent to look for the expectation of the absolute value instead of looking directly at the maximum. We make use of the following useful representation for the absolute value

$$|x| = \frac{2}{\pi}\int_0^{+\infty} \frac{1-\cos(xt)}{t^2} dt = \frac{2}{\pi}\int_0^{+\infty} \frac{1 - \mathbb{E}_{\varepsilon}\left[e^{ixt\varepsilon}\right]}{t^2} dt, \tag{3.3}$$

where $\varepsilon = \pm 1$ is a Rademacher random variable, i.e. $P(\varepsilon = \pm 1) = 1/2$. This equality can be easily seen by simple substitution. Other use of (3.3) can be found in Haagerup [6], Konig, Schutt and Tomczak-Jaegermann [7] and Li and Wei [11, 12]. Thus,

$$\Theta_L := \mathbb{E}\left|\sum_{k=1}^n L_k\right| \tag{3.4}$$

$$= \frac{2}{\pi}\int_0^{+\infty} \frac{1}{t^2}\left(1 - \mathbb{E}_{\varepsilon}\mathbb{E}_L\left[e^{it\varepsilon\left(\sum_{k=1}^n L_k\right)}\right]\right) dt$$

$$= \frac{2}{\pi}\int_0^{+\infty} t^{-2}(1 - \mathbb{E}_{\varepsilon}\phi_L^n(t\varepsilon))dt$$

$$= \frac{2}{\pi}\int_0^{+\infty} t^{-2}(1 - Re\left(\phi_L^n(t)\right))dt, \tag{3.5}$$

where interchange of integrations follows from positivity of the integrant and the boundness of $e^{ixt\varepsilon}$. The price of $\max(0, \sum_{k=1}^n L_k)$ is then obtained using (3.2) to replace the maximum by the absolute value. The calculation of the expected absolute value corresponds to the above calculation Θ_L. Proposition 3.1 follows. □

Direct substitution of L by respectively C and Z yields the following result.

Proposition 3.2. *The price at time 0 of the monthly sum cap is given by*

$$K(1+g)e^{-rT} + 2^{-1}Ke^{-rT}(\Theta_C + n\mathbb{E}\, C_1) \tag{3.6}$$

where

$$\Theta_C = \frac{2}{\pi}\int_0^{+\infty} t^{-2}(1 - Re\left(\phi_C^n(t)\right))dt. \tag{3.7}$$

and the price at time 0 of the cliquet option is equal to

$$K(1+g)e^{-rT} + 2^{-1}Ke^{-rT}(\Theta_Z + n\mathbb{E}\, Z_1) \tag{3.8}$$

where

$$\Theta_Z = \frac{2}{\pi}\int_0^{+\infty} t^{-2}(1 - Re\,(\phi_Z^n(t)))dt. \tag{3.9}$$

3.2. Greeks. Knowing formulas (3.6) and (3.8) for the prices of these highly path-dependent securities, we are now able to calculate the derivatives of these prices with respect to the different financial parameters, such as the interest rate r (Theta of the option), the volatility σ (Vega of the option) or with respect to the initial stock price S_0 (Delta of the option). After a careful look at the formulas (3.6) and (3.8), one can notice that prices at time 0 do not depend on S_0, therefore the delta is equal to 0 (as well as the gamma that stands for the second derivative with respect to S_0). In fact cliquet options are very sensitive to the volatility parameter. The vega is therefore an important greek to study.

Vega

. The vega can be obtained by differentiating (with respect to σ) the prices of the monthly sum cap and the Cliquet option.

Proposition 3.3. *The vega of the monthly sum cap and of the Cliquet option are given by differentiating their respective prices with respect to the parameter* σ

$$\textit{Vega at time 0} = \frac{\partial}{\partial\sigma}(\textit{Price}) = \frac{Ke^{rT}}{2}\left(\frac{\partial}{\partial\sigma}\Theta_L + n\frac{\partial}{\partial\sigma}\mathbb{E}\,L\right), \tag{3.10}$$

where L should be replaced by C in the case of the monthly sum cap and by Z in the case of the cliquet option and where

$$\frac{\partial}{\partial\sigma}\Theta_L = \frac{1}{\pi}\int_0^{+\infty}\frac{(-1)}{t^2}\cdot\frac{\partial}{\partial\sigma}\left(\phi_L^n(t) + \phi_L^n(-t)\right)dt. \tag{3.11}$$

Proof. We can write $Re(\phi_L^n(t)) = (\phi_L^n(t) + \phi_L^n(-t))/2$ in (3.5) and then take partial derivative with respect to σ. The interchange of integration and differentiation is justified by the positivity of the integrant. Note that there is no singularity at zero for the integrant since $\phi_L(t) = 1 + h(\sigma)t + O(t^2)$ for some $h(\sigma)$ as $t \to 0$, and

$$\phi_L^n(t) + \phi_L^n(-t) = (1 + nh(\sigma)t + O(t^2)) + (1 - h(\sigma)t + O(t^2)) = 2 + O(t^2).$$

3.3. Implementation. The analytic formulas obtained for these contracts' prices and greeks (see (3.6), (3.8) and (3.10)) are functions of Θ_C and Θ_Z given by (3.7) and (3.9) which involve the characteristic functions ϕ_C of C_k and ϕ_Z of Z_k. Therefore, one needs to know the distribution of C_k and of Z_k. Denote by R, one of the identically distributed returns R_k defined by (2.1).

Lemma 3.4 (Distribution of C_k). *The random variables C_k, $k = 1...n$ are independent and identically distributed with characteristic function ϕ_C,*

$$\phi_C(t) := \mathbb{E}\left[e^{itC_k}\right] = e^{-it(1+g/n)}\left(1 + it\int_0^{1+c} e^{itx}Q\left(R \geqslant x - 1\right)dx\right). \tag{3.12}$$

The expectation of C_k is equal to

$$\mathbb{E}\,C_k = (c - g/n)\,Q\left(R \geqslant c\right) + \int_{-1-g/n}^{c-g/n} xf_R\left(x + g/n\right)dx,$$

where f_R denotes the density of R under the risk-neutral probability.

Proof. The distribution of the random variable C_k defined in (2.3) is given by

$$Q(C_k > x) = \begin{cases} 0 & \text{if } x > c - g/n \\ Q\left(R - g/n > x\right) & \text{if } x \leqslant c - g/n. \end{cases} \tag{3.13}$$

Thus C_k has a mixed distribution, with a density up to $c - g/n$ (assuming the underlying asset price S is continuously distributed) and a mass point at $c - g/n$. Recall that for a non-negative random variable Z with finite expectation, its characteristic function writes as

$$\phi_Z(t) = \mathbb{E}\left[e^{itZ}\right] = 1 + it \int_0^{+\infty} e^{itx} Q(Z > x)dx. \tag{3.14}$$

This equality is well-known and follows from Fubini's theorem. Denote by C one of the identically distributed random variables C_k for $k = 1..n$.

Note that C defined by (2.3) is bounded from below and therefore stays above $-1 - g/n$ because it is almost surely larger than $c - g/n$ from its survival function given in (3.13) and the fact that the local cap, c is non-negative. The r.v. $C + 1 + g/n$ is then non-negative.

$$Q\left(C + 1 + \frac{g}{n} > x\right) = \begin{cases} 0 & \text{if } x > 1 + c \\ Q\left(R > x - 1\right) & \text{if } x \leqslant 1 + c. \end{cases}$$

In addition $\phi_C(t) = \phi_{C+1+g/n}(t)e^{-it(1+g/n)}$. Using (3.14), one obtains

$$\phi_C(t) = e^{-it(1+g/n)}\left(1 + it \int_0^{1+c} e^{itx} Q\left(R > x - 1\right) dx\right).$$

□

Similarly we can get the distribution of $Z_k = \max\left(0, \min\left(c, R_k\right)\right) - g/n$.

LEMMA 3.5 (distribution of Z_k). *The random variables Z_k are independent and identically distributed with a characteristic function*

$$\phi_Z(t) := \mathbb{E}\left[e^{itZ_k}\right] = e^{-itg/n}\left(1 + it \int_0^{c} e^{itx} Q(R > x)dx\right). \tag{3.15}$$

The expectation of Z_k is equal to

$$\mathbb{E}\, Z_k = \left(c - g/n\right) Q\left(R \geqslant c\right) + \int_{-g/n}^{c-g/n} x f_R\left(x + \frac{g}{n}\right) dx - \frac{g}{n} Q(R < 0),$$

where f_R denotes the density of R under the risk-neutral probability Q.

Proof. The survival function of Z_k is given by

$$Q(Z_k > x) = \begin{cases} 0 & \text{if } x > c - g/n \\ Q\left(R - g/n > x\right) & \text{if } -g/n \leqslant x \leqslant c - g/n \\ 1 & \text{if } x < -g/n. \end{cases} \tag{3.16}$$

Here Z_k has also a mixed distribution, it has a density over $[-g/n, c - g/n]$ and two mass points at $-g/n$ and at $c - g/n$. The proof of the characteristic function is similar to Lemma 3.4. We omit it. □

3.4. Theoretical Approximation. In both cases, the characteristic functions (3.12) and (3.15) are expressed as an integral of the distribution of the periodically returns (see (3.12) and (3.15)). In general, this integral cannot be computed exactly and one has to rely on some approximations. For example in the Black and Scholes market, the periodical return follows a lognormal distribution, then the variables C and Z follow truncated lognormal distributions. It is well-known that the characteristic function of the lognormal distribution has no explicit formula and could be difficult to estimate, see Leipnik [10] for a complete discussion on this topic. However in our case, even if no closed-form expressions of ϕ_C and ϕ_Z are available, there is a possible procedure to calculate Θ_C and Θ_Z and the prices of the locally-capped contracts. We make use of the beneficial fact that the periodical returns are capped and that the support of their distribution is therefore bounded.

To calculate Θ_L (defined in (3.4)), we need to calculate the real part of ϕ_L^n. In the case when there is no closed-form expression of ϕ_L, two steps are needed. First we truncate the integral in Θ_L (see (3.4)) and define $\Theta_L(M)$ as follows

$$\Theta_L(M) = \frac{2}{\pi} \int_0^M \frac{1 - Re\left(\phi_L^n(t)\right)}{t^2} dt. \tag{3.17}$$

Second, in the formula of ϕ_L (either (3.12) or (3.15)), we approximate $\exp(itx)$ by a finitely truncated Taylor series, $\sum_{k=0}^m (itx)^k/k!$, and define for any positive integer m

$$\Theta_C(M,m) = \frac{2}{\pi} \int_0^M \frac{1 - Re\left(\phi_{C,m}^n(t)\right)}{t^2} dt. \tag{3.18}$$

where

$$\phi_{C,m}(t) = e^{-it(1+g/n)} \left(1 + it \int_0^{1+c} \sum_{k=0}^m \frac{(itx)^k}{k!} Q\left(R \geqslant x-1\right) dx\right). \tag{3.19}$$

By doing so, we are able to get a polynomial expression of ϕ_L and therefore for $Re(\phi_L^n(t))$ and obtain an interesting expression for $\Theta_L(M)$. Note that the procedure described below may not converge in the absence of a cap level c (as shown by Leipnik [10]). It is therefore especially suited for the pricing and hedging of contracts locally-capped at $c < +\infty$.

PROPOSITION 3.6. *Given any positive integer M,*

$$|\Theta_L - \Theta_L(M)| \leqslant \frac{4}{M\pi}. \tag{3.20}$$

In addition, for fixed M

$$\lim_{m\to+\infty} |\Theta_C(M) - \Theta_C(M,m)| = 0.$$

This proposition shows the convergence of the approximation to Θ_L when M and m are sufficiently large. The proof of the convergence can be found in Appendix A. This result is of theoretical interest as it guarantees the convergence.

A numerical example in the Black and Scholes framework illustrates this approximation and its accuracy in Section 4. *Practical* approximations of prices (3.6), (3.8) and vegas (3.10) should in fact proceed by direct computation of (3.12) and (3.15) with numerical integration techniques and not use the approximation technique given in Proposition 3.6. We will discuss this point in the next section, following Table 4.1.

4. Numerical Examples . We present first our results in the Black and Scholes framework in 4.1 and extend them in a Lévy market model in Section 4.2.

4.1. Black and Scholes setting. In a Black and Scholes market, the financial market is arbitrage-free, perfectly liquid and complete. The risk-free rate is constant and equal to r, the constant yield of dividend of the index is η and its volatility is denoted by σ. Thus

$$\frac{dS_t}{S_t} = (\mu - \eta)dt + \sigma dW_t \tag{4.1}$$

where W is a standard Brownian motion under the historical probability P. In this framework, there exists a unique risk-neutral probability Q, under which the underlying index follows (using Girsanov theorem),

$$\frac{dS_t}{S_t} = (r - \eta)dt + \sigma dZ_t$$

where Z is a standard Brownian motion under Q. Standard calculations give the following underlying price:

$$S_t = S_0 e^{\left(r-\eta-\sigma^2/2\right)t+\sigma Z_t}.$$

The no-arbitrage price at the initial time 0 of a contract with a payoff X_T paid at time T is obtained by taking the expectation under the risk-neutral probability Q of the discounted payoff, that is $e^{-rT}\mathbb{E}\left[X_T\right]$.

4.1.1. Distribution of periodical returns. Under our assumptions, R_k are i.i.d. and given by

$$R_k = \exp\left\{\left(r - \eta - \sigma^2/2\right)\Delta + \sigma\left(Z_{t_k} - Z_{t_{k-1}}\right)\right\} - 1. \tag{4.2}$$

R_k has clearly the same distribution as $e^{\xi} - 1$, where ξ is a normal distribution $\mathcal{N}\left(m_\xi, \sigma_\xi^2\right)$ with the mean m_ξ and the standard deviation σ_ξ given by

$$\xi \sim \mathcal{N}\left(\left(r - \eta - \sigma^2/2\right)\Delta, \sigma^2\Delta\right).$$

We now apply Lemma 3.4 and Lemma 3.5 in turn to derive the distributions of C_k and Z_k in the Black and Scholes setting.

4.1.2. Distribution of C_k. The survival function of C_k in the Black and Scholes model is given in (3.13) with $R = e^{\xi} - 1$ Note that C_k are i.i.d with a mixed distribution. It has a density, denoted by f_C, up to $c - g/n$ (a lognormal distribution over $[-1 - g/n, c - g/n]$) and a point mass at $c - g/n$. More precisely, we have

$$\begin{cases} Q(C_k = c - g/n) & = Q(R \geqslant c) = Q(\xi \geqslant \ln(1+c)) = N\left(\frac{m_\xi - \ln(1+c)}{\sigma\sqrt{\Delta}}\right) \\ f_C(x) & = \frac{1}{\sigma(x+1+g/n)\sqrt{2\pi\Delta}} \exp\left(-\frac{(\ln(x+1+g/n)-m_\xi)^2}{2\sigma^2\Delta}\right) \quad \text{if } x \in (-1-g/n, c-g/n) \end{cases} \tag{4.3}$$

where $m_\xi = \left(r - \eta - \sigma^2/2\right)\Delta$ and $N(\cdot)$ denotes the cdf of a standard normal distribution. The characteristic function ϕ_C has the following compact expression

$$\phi_C(t) := \mathbb{E}\left[e^{itC_k}\right] = e^{-it(1+g/n)}\left(1 + it\int_0^{1+c} e^{itx} N\left(\frac{m_\xi - \ln(x)}{\sigma\sqrt{\Delta}}\right)dx\right). \tag{4.4}$$

The expectation of C_k does not depend on k and is equal to

$$\mathbb{E}\left(C_k\right)=\left(c-\frac{g}{n}\right)N\left(\frac{m_\xi-\ln(1+c)}{\sigma\sqrt{\Delta}}\right)+\int_0^{1+c}\left(y-1-\frac{g}{n}\right)\frac{1}{\sqrt{2\pi\Delta}\sigma y}e^{-\frac{\left(\ln(y)-m_\xi\right)^2}{2\sigma^2\Delta}}dy.$$

4.1.3. Distribution of Z_k. To price the cliquet option in the Black and Scholes model, we need the distribution of Z_k given in (3.16) with $R=e^\xi-1$. Simple calculations imply

$$\begin{cases} Q(Z_k=c-\frac{g}{n}) & =Q(R_k\geqslant c)=Q(\xi\geqslant\ln(1+c))=N\left(\frac{m_\xi-\ln(1+c)}{\sigma\sqrt{\Delta}}\right)\\ f_Z(x) & =\frac{1}{\sigma(x+1+g/n)\sqrt{2\pi\Delta}}\exp\left(-\frac{(\ln(x+1+g/n)-m_\xi)^2}{2\sigma^2\Delta}\right)\quad\text{if } x\in(-g/n,c-g/n)\\ Q(Z_k=-g/n) & =Q(R_k\leqslant 0)=Q(\xi\leqslant 0)=N\left(\frac{-m_\xi}{\sigma\sqrt{\Delta}}\right)\end{cases}\tag{4.5}$$

where $m_\xi=\left(r-\eta-\sigma^2/2\right)\Delta$. A straightforward calculation gives

$$\begin{aligned}\phi_Z(t)&=e^{it(c-g/n)}N\left(\frac{m_\xi-\ln(1+c)}{\sigma\sqrt{\Delta}}\right)\\&\quad+\int_1^{1+c}e^{it(y-1-g/n)}\frac{1}{\sqrt{2\pi\Delta}\sigma y}e^{-\frac{\left(\ln(y)-m_\xi\right)^2}{2\sigma^2\Delta}}dy+e^{-itg/n}N\left(\frac{-m_\xi}{\sigma\sqrt{\Delta}}\right).\end{aligned}$$

The expectation of Z_k is equal to

$$\begin{aligned}\mathbb{E}\left(Z_k\right)&=\left(c-\frac{g}{n}\right)N\left(\frac{m_\xi-\ln(1+c)}{\sigma\sqrt{\Delta}}\right)\\&\quad+\int_1^{1+c}\left(y-1-\frac{g}{n}\right)\frac{1}{\sqrt{2\pi\Delta}\sigma y}e^{-\frac{\left(\ln(y)-m_\xi\right)^2}{2\sigma^2\Delta}}dy-\frac{g}{n}N\left(\frac{-m_\xi}{\sigma\sqrt{\Delta}}\right).\end{aligned}$$

4.1.4. For the Greeks. Here we provide an expression of the differentiation of the characteristic function with respect to σ in the particular case of the Black and Scholes market. The ultimate goal is to derive the vega of the monthly sum cap by applying the formula in Proposition 3.3. The differentiation of ϕ_C for the monthly sum cap is given by

$$\begin{aligned}\frac{\partial\left(\phi_C(t)\right)}{\partial\sigma}&=\frac{\ln\left(1+c\right)-m_\xi-\sigma^2\Delta}{\sigma^2\sqrt{2\pi T}}e^{it(c-g/n)}e^{-\frac{\left(m_\xi-\ln(1+c)\right)^2}{2\sigma\sqrt{\Delta}}}\\&+\int_0^{1+c}\frac{1}{\sigma y\sqrt{2\pi\,\Delta}}e^{it(y-1-g/n)}\left(-\frac{1}{\sigma}-\frac{\ln\left(y\right)-m_\xi}{\sigma}+\frac{\left(\ln\left(y\right)-m_\xi\right)^2}{\sigma^3\Delta}\right)e^{-\frac{\left(\ln(y)-m_\xi\right)^2}{2\sigma^2\Delta}}dy,\end{aligned}\tag{4.6}$$

and the differentiation of ϕ_Z for the cliquet option is given by

$$\begin{aligned}\frac{\partial\left(\phi_Z(t)\right)}{\partial\sigma}&=\frac{\ln\left(1+c\right)-m_\xi-\sigma^2\Delta}{\sigma^2\sqrt{2\pi T}}e^{it(c-g/n)}e^{-\frac{\left(m_\xi-\ln(1+c)\right)^2}{2\sigma^2\Delta}}+\frac{m_\xi+\sigma^2\Delta}{\sigma^2\sqrt{2\pi T}}e^{-itg/n}e^{-\frac{m_\xi^2}{2\sigma^2\Delta}}\\&+\int_1^{1+c}\frac{1}{\sigma y\sqrt{2\pi\,\Delta}}e^{it(y-1-g/n)}\left(-\frac{1}{\sigma}-\frac{\ln\left(y\right)-m_\xi}{\sigma}+\frac{\left(\ln\left(y\right)-m_\xi\right)^2}{\sigma^3\Delta}\right)e^{-\frac{\left(\ln(y)-m_\xi\right)^2}{2\sigma^2\Delta}}dy.\end{aligned}\tag{4.7}$$

4.1.5. Approximation. In the Black and Scholes model,

$$R_k = e^{\xi} - 1 \sim LN(m_\xi, \sigma\sqrt{\Delta}).$$

Define for $k > 0$ and given c,

$$\mu_k := \mu_k^{(c)} = \int_0^{1+c} k x^{k-1} N\left(\frac{m_\xi - \ln(x)}{\sigma\sqrt{\Delta}}\right) dx, \tag{4.8}$$

and assume $\mu_0^{(c)} = 1$. For all k, $\mu_k = \mu_k^{(c)}$ is a positive real number. Note that when $c = +\infty$, $\mu_k^{(+\infty)} = \mathbb{E}\left[Z^k\right]$ which is the k-th moments of a r.v. Z with a lognormal distribution. Then, for $m \geqslant 1$,

$$\begin{aligned}\phi_{C,m}(t) &= e^{-it(1+g/n)}\left(1 + it\int_0^{1+c}\sum_{k=0}^{m-1}\frac{(itx)^k}{k!}N\left(\frac{m_\xi - \ln(x)}{\sigma\sqrt{\Delta}}\right)dx\right)\\ &= e^{-it(1+g/n)}\sum_{j=0}^{m}\frac{(it)^j}{j!}\mu_j \end{aligned}\tag{4.9}$$

and thus

$$\phi_{C,m}^n(t) = e^{-it(n+g)}\left(\sum_{j=0}^{m}\frac{(it)^j}{j!}\mu_j\right)^n = e^{-it(n+g)}\sum_{\ell=0}^{nm}\alpha_\ell (it)^\ell \tag{4.10}$$

where

$$\alpha_\ell := \sum_{\{(j_1,...,j_n)\ |\ j_1+...+j_n=\ell\}}\left(\prod_{k=1}^{n}\frac{\mu_{j_k}}{j_k!}\right).$$

In particular,

$$\begin{cases}\alpha_0 = \mu_0^n = 1\\ \alpha_1 = n\mu_1,\\ \alpha_2 = n(n-1)\mu_1^2/2 + n\mu_2/2,\\ \alpha_3 = n(n-1)(n-2)\mu_1^3/6 + n(n-1)\mu_2\mu_1/4 + n\mu_3/6.\end{cases}$$

Thus, $\Theta_C(M)$ in (3.17) can be approximated by

$$\Theta_C(M,m) = \frac{2}{\pi}\int_0^M \frac{1 - Re\left(\phi_{C,m}^n(t)\right)}{t^2}dt. \tag{4.11}$$

given in formula (3.18). Using the expression of $\phi_{C,m}^n(t)$ in (4.10) we obtain the following expression of the real part

$$Re\left(\phi_{C,m}^n(t)\right) = \cos((n+g)t)\sum_{\ell=0}^{\lfloor nm/2\rfloor}\alpha_{2\ell}(-1)^\ell t^{2\ell} + \sin((n+g)t)\sum_{\ell=0}^{\lfloor (nm-1)/2\rfloor}\alpha_{2\ell+1}(-1)^\ell t^{2\ell+1}. \tag{4.12}$$

We are now able to explicitly calculate the initial terms of

$$\Theta_C(M,m) = E_0 + E_1 + E_2 + E_S + E_C \tag{4.13}$$

where

$$\begin{cases} E_0 &= \frac{2}{\pi}\int_0^M \ t^{-2}(1-\cos((n+g)t))dt \\ &= n+g-\frac{2}{\pi}\int_M^{+\infty} \ t^{-2}(1-\cos((n+g)t))dt \sim n+g \\ E_1 &= \frac{-2}{\pi}\int_0^M \ t^{-1}\sin((n+g)t)\alpha_1 dt \\ &= -\alpha_1+\frac{2}{\pi}\int_M^{+\infty} \ t^{-1}\sin((n+g)t)\alpha_1 dt \sim -\alpha_1 \\ E_2 &= \frac{2}{\pi}\int_0^M \ \cos((n+g)t)\alpha_2 dt = \frac{2\sin((n+g)M)\alpha_2}{\pi(n+g)} \\ E_S &= \frac{2}{\pi}\sum_{\ell=1}^{\lfloor (nm-1)/2\rfloor}\int_0^M \sin((n+g)t)\alpha_{2\ell+1}(-1)^{\ell+1}t^{2\ell-1}dt \\ E_C &= \frac{2}{\pi}\sum_{\ell=1}^{\lfloor nm/2\rfloor}\int_0^M \cos((n+g)t)\alpha_{2\ell}(-1)^{\ell+1}t^{2\ell-2}dt \end{cases}$$

and the asymptotics $\sim$ refer to the limit as $M \to +\infty$. Of course additional terms for α_m, $m \geq 3$, can be given but practical improvements are limited.

4.1.6. Numerical example. We consider a 5 year contract ($T = 5$). The interest rate is $r = 4\%$ and the volatility of the underlying portfolio S is set at $\sigma = 20\%$, the dividend yield is $\eta = 1\%$ and the guaranteed rate $g = 10\%$. The investor brings an initial amount of money $K = 1000$ and invests in a monthly sum cap over the life of the contract. The monthly cap is chosen to be 8.5%.

TABLE 4.1
Comparison of numerical techniques to estimate Θ_C

Monte Carlo		Formula (3.7) in Proposition 3.2 using (4.15)	
$A = 5\cdot 10^5$	$0.3372(4\cdot 10^{-4})$	$M=300,\ \varepsilon_1 = 1\cdot 10^{-4}\ \varepsilon_2 = 5\cdot 10^{-5}$	0.3374
$A = 5\cdot 10^6$	$0.3370(10^{-4})$	$M=300,\ \varepsilon_1 = 1\cdot 10^{-4}\ \varepsilon_2 = 1\cdot 10^{-4}$	0.3373
$A = 5\cdot 10^6$	$0.3370(10^{-4})$	$M=300,\ \varepsilon_1 = 1\cdot 10^{-4}\ \varepsilon_2 = 5\cdot 10^{-4}$	0.3373
$A = 5\cdot 10^6$	$0.3370(10^{-4})$	$M=300,\ \varepsilon_1 = 1\cdot 10^{-4}\ \varepsilon_2 = 1\cdot 10^{-3}$	0.3373
$A = 5\cdot 10^6$	$0.3370(10^{-4})$	$M=300,\ \varepsilon_1 = 1\cdot 10^{-4}\ \varepsilon_2 = 5\cdot 10^{-3}$	0.3371
$A = 5\cdot 10^6$	$0.3370(10^{-4})$	$M=300,\ \varepsilon_1 = 1\cdot 10^{-4}\ \varepsilon_2 = 1\cdot 10^{-2}$	0.3369
$A = 5\cdot 10^6$	$0.3370(10^{-4})$	$M=400,\ \varepsilon_1 = 1\cdot 10^{-4}\ \varepsilon_2 = 1\cdot 10^{-2}$	0.3374
$A = 5\cdot 10^6$	$0.3370(10^{-4})$	$M=1850,\ \varepsilon_1 = 7\cdot 10^{-5}\ \varepsilon_2 = 0\cdot 10^{-2}$	0.3369

In Table 4.1, we compare numerical estimation of Θ_C using different approaches. The first two columns are obtained by Monte Carlo using A simulations. In parenthesis we give the standard deviation. The last two columns correspond to Proposition 3.2 with a direct numerical integration of expression (4.4) (which is (3.12) expressed in the Black and Scholes framework) over $[\varepsilon_1, 1+c]$.

$$\phi_C(t) \approx e^{-it\left(1+\frac{g}{n}\right)}\left(1+it\int_{\varepsilon_1}^{1+c} e^{itx} N\left(\frac{m_\xi - \ln(x)}{\sigma\sqrt{\Delta}}\right)dx\right). \tag{4.14}$$

Formula (3.7) in Proposition 3.2 gives the price of a Monthly Sum Cap contract. It is approximated by

$$\Theta_C \approx \frac{2}{\pi}\int_{\varepsilon_2}^{M} t^{-2}(1 - Re\,(\phi_C^n(t)))dt \tag{4.15}$$

where ϕ_C^n is obtained by (4.14) and M, ε_1 and ε_2 are chosen in Table 4.1.

We use Matlab software to obtain all the following numerical results. The results are reported using the Matlab procedure "*quadl*$(\cdot)$." This procedure approximates the integral of the integrant from the lower bound of integration (here ε_2 or ε_1) to the upper bound (here M or $1+c$), to within an error of 10^{-6} using recursive adaptive Lobatto quadrature (See Gander and Gautschi [4]). Monte Carlo techniques can be very slow (several minutes) whereas our technique gives a number within a few seconds.

A numerical integration of (4.4) converges faster than the approximate results obtained using the approximation (4.13) obtained by truncation. In practice, Proposition 3.2 is a very good alternative to Monte Carlo simulations. The fact that it is a semi-closed-form and that it is not obtained by Monte Carlo simulations imply that the Greeks are less sensitive and easier to estimate.

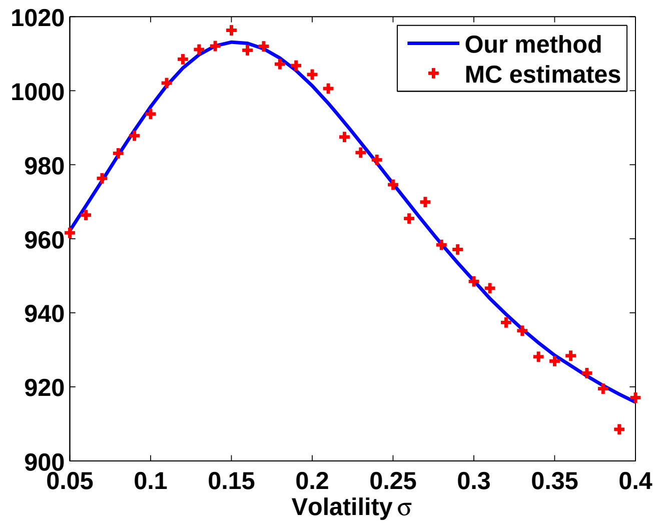

Fig. 4.1. *Prices of monthly sum cap contracts with respect to volatility parameter σ, ranging from 5% to 40%. Monte Carlo estimates are obtained using 5,000 simulations. Parameters are set to $T = 5$, $r = 0.04$, $\eta = 0.01$, $g = 0.1$, $K = 1000$, $c = 0.085$. "Our Method" refers to a numerical integration of* (4.15) *using the expression* (4.4) *for the characteristic function.*

We report in Figure 4.1 the prices as a function of σ as well as their corresponding greeks as a function of σ (for a range of σ between 5% to 40%). We observe that the price of a monthly sum cap contract has a very particular behaviour with respect to the volatility parameter. It is this specific behaviour that can be used to implement a natural hedge against volatility fluctuations in Bernard and Boyle [1].

In equity-linked insurance, the investor (policyholder) typically gives a fixed amount at inception of the contract to the insurer. This corresponds to the full amount he wants to invest. Parameters of the contract are then determined so that the no-arbitrage value (or price) is equal to the initial investment of the investor. If this is the case, the contract is called a "fair" contract. For example on Figure 4.1, we observe that the contract's price is generally not equal to 1,000 (which is the initial investment here). In fact it appears clearly from Figure 4.1 that the contract is fair only when σ is around 11% or around 20%. We chose the parameters of the contract so that it would be fair with the parameters for the financial market chosen in our simulation (in particular $\sigma = 20\%$ for our base parameter choice). However in Figure 4.1, we study the contract as a function of the volatility parameter, all other parameters being fixed, the price of the contract changes and it cannot stay "fair".

Observe also the non-monotonicity of the price with respect to the volatility parameter σ and its very high sensitivity to changes in σ. Recall that vega is the sensitivity to σ (derivative of the price with respect to σ). Therefore vega will obviously change sign between $\sigma = 14\%$ and $\sigma = 17\%$ as it appears in Figure 4.1 already.

Finally Figure 4.2 displays the corresponding vegas. We now used 10,000 simulations to estimate vega. It is well-known that obtaining accurate and stable Greeks by Monte Carlo simulation can be challenging (See for example Korn et al. [8] or Glasserman [5]). To compute the vega by Monte Carlo, there are multiple techniques. We use the so-called "principle of common random numbers (or path recycling)" (see for example page 260 of Korn et al. [8], it is a standard technique which consists of using the same numbers to compute the price of the contract when the volatility is $\sigma + \kappa\sigma$ (where κ denotes a very small percentage) and when the volatility is σ). We then compute the difference between these two prices and divide it by $\kappa\sigma$. We tried several values for κ and picked one sufficiently small for which the vega is not sensible to the choice of $\kappa\sigma$, here 1% for Figure 4.2. This is an advantage of using semi-closed-form expressions and numerical integration. Sensitivities can be calculated, and the curves are sufficiently smooth to be differentiated another time if necessary. It is needed for example to obtain the gamma of options. Also note that the calculation by numerical integration can be done in a few seconds whereas the Monte Carlo simulation needed to obtain the same accuracy requires several minutes.

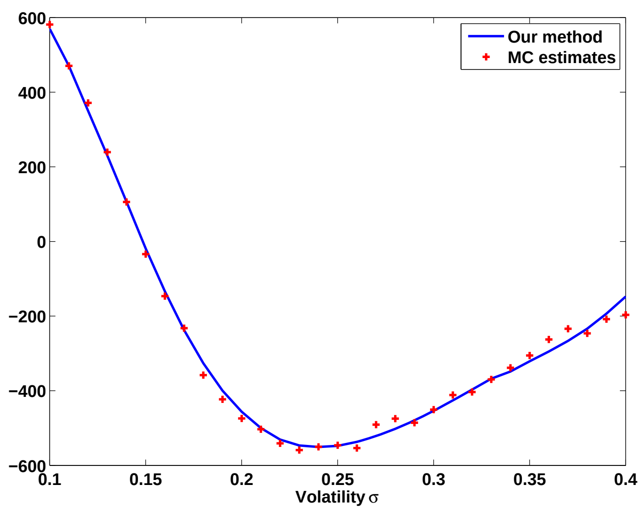

Fig. 4.2. *Vegas of monthly sum cap contracts with respect to volatility parameter σ, ranging from 5% to 40%. Monte Carlo estimates are obtained using 10,000 simulations. Parameters are set to $T = 5$, $r = 0.04$, $\eta = 0.01$, $g = 0.1$, $K = 1000$, $c = 0.085$. "Our Method" refers to a numerical integration of* (4.15) *using the expression* (4.4) *for the characteristic function.*

4.2. Lévy market model . Semi-closed-form expressions for the prices of monthly sum cap contracts and cliquet options are given in Proposition 3.2. It is shown that these prices can be expressed as function of EC_1 and ϕ_C (respectively EZ_1 and ϕ_Z) which can be computed as soon as $Q(R > c)$ and $f_R(\cdot)$ are known as it appears clearly in Lemmas 3.4 and 3.5.

In a Black and Scholes market model, the expressions for $Q(R > c)$ and $f_R(\cdot)$ are known explicitly and formulas can be simplified as it was done in the previous section. In this section we show that our study can be extended to more realistic market models, the Lévy market models.

Lévy market models are popular models used for stock prices. In particular they are able to reflect the presence of jumps in stock prices and they incorporate the Black-Scholes setting as a special case. In a Lévy market, semi-closed-form expressions for $Q(R > c)$ and $f_R(\cdot)$ can also be derived easily. Recall that when time steps are equally spaced, and the underlying process S is stationary with independent increments, R (defined in (2.1)) can be calculated as

$$R := \frac{S(\Delta) - S(0)}{S(0)}$$

where Δ is the time step. In a Lévy market,

$$S(\Delta) = S(0)e^{X(\Delta)}$$

where the characteristic function of $X(\Delta)$, denoted by ϕ_X, is known. Furthermore note that

$$Q(R > x) = Q(X(\Delta) > \ln(x)).$$

Therefore the survival function of R is directly obtained from the survival function of $X(\Delta)$. In particular R is distributed over $(-1, +\infty)$ and has the following probability distribution function

$$f_R(x) = \frac{f_X(\ln(1+x))}{1+x},$$

where the pdf of $X(\Delta)$, $f_X(\cdot)$, is calculated for example as $f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\phi_X(u)du$. To obtain the survival function of $X(\Delta)$, we make use of the following formula. For all $a \in \mathbb{R}$, the cdf of $X(\Delta)$ is equal to

$$F_X(x) = \frac{e^{ax}}{2\pi}\int_{-\infty}^{+\infty} e^{iux}\frac{\phi_X(ia-u)}{a+iu}du. \tag{4.16}$$

This formula is the standard Fourier inversion formula when $a = 0$. It is proved by Le Courtois and Walter [9] that it holds for an arbitrary choice for a. For example it can be useful to remove potential singularities in the integration. This approach is particularly interesting to deal with a large class of Lévy processes. We now illustrate this point with the Variance-Gamma model presented in Madan et al. [13].

The Variance-Gamma Lévy process is a subclass of Lévy process easy to simulate. We thus compare two approaches. The first approach is obtained by Monte Carlo using the simulation of the Variance-Gamma model by subordination of the Brownian motion. The second method consists of using (4.15) to compute (3.7). In the case of a Variance-Gamma process, the characteristic function ϕ_X is given by

$$\phi_X(u) = \left(1 + \sigma^2\nu u^2/2 - i\delta\nu u\right)^{\frac{-\Delta}{\nu}}. \tag{4.17}$$

The simulation technique is very standard and can be found for example in Korn et al. [8], Algorithm 7.2 page 346 (where we slightly change this algorithm to take into account the scale parameter σ: we simulate increments as $X(t_i) = X(t_{i-1}) + \sigma\sqrt{G_i}Y_i + \theta G_i$ instead of $X(t_i) = X(t_{i-1}) + \sqrt{G_i}Y_i + \theta G_i$). Increments are independent and identically distributed, we have that $R = e^X - 1$ where $X = (r+\omega)dt + \theta g + \sigma\sqrt{g}N(0,1)$ where g is a Gamma distribution $\Gamma(dt/\nu, 1)$. The density of X can be obtained easily as the integral of the product of the density of a normal distribution and the density of a Gamma random variable (See Madan and Senata [14], Madan et al. [13]). We use the same parameters as before: $T = 5$, $r = 0.04$, $g = 0.1$, $K = 1000$, $c = 0.085$ and $S_0 = 1$. The Variance-Gamma parameters are such that $\delta = 0.01$, $\sigma = 0.05$, $\theta = 0.001$ and $\nu = 0.0625$. We find that a Monte Carlo estimate $\widehat{\Theta}_C = 0.3163(0.0008)$, using $A = 80,000$ simulations and the standard deviation is given in parenthesis. A numerical integration of (4.15) using the expression of Θ_C given in Formula (3.7) in Proposition 3.2 converges for $M \geq 30$, $\varepsilon_1 < 0.0001$ and $\varepsilon_2 < 0.01$. As the simulation of Variance Gamma Lévy models is very simple, it is easier to obtain this accuracy by direct Monte Carlo simulations rather than using the double numerical integration involved in the method that we describe above. However, it is not always possible to simulate using an exact (and simple) technique the underlying process. For instance, only approximate simulation methods are available to simulate the CGMY process introduced by Carr et al. [15] (see for example Table 6.1 and Section 6.3 in Cont and Tankov [3]).

5. Conclusion. This paper presents a new formula for calculating prices and hedging parameters of locally-capped contracts and cliquet-style options. Examples are derived in the Black and Scholes framework and in Lévy market models. Our approach applies to any financial market where the underlying is stationary with independent increments. The main difficulty in pricing and hedging these contracts in more advanced financial models is to be able to get an expression for the expectation and for the characteristic function of truncated returns of the underlying stock price.

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Appendix. Convergence of the approximation .

Let $\varepsilon > 0$. There are two steps. First we have

$$\forall M \geqslant \frac{4}{\pi\varepsilon}, \quad |\Theta_L - \Theta_L(M)| \leqslant \varepsilon.$$

This follows from

$$\begin{aligned} |\Theta_L - \Theta_L(M)| &\leqslant \left| \frac{2}{\pi} \int_M^{+\infty} \frac{1 - Re(\phi_L^n(t))}{t^2} \right| \\ &\leqslant \frac{2}{\pi} \int_M^{+\infty} \frac{2}{t^2} dt \\ &\leqslant \frac{4}{M\pi}. \end{aligned}$$

Recall that

$$\phi_C(t) = e^{-it\left(1+\frac{g}{n}\right)} \left(1 + it \int_0^{1+c} e^{itx} Q\left(R \geqslant x-1\right) dx\right). \tag{A.1}$$

We define for $m \geqslant 1$,

$$\phi_{C,m}(t) = e^{-it\left(1+\frac{g}{n}\right)} \left(1 + it \int_0^{1+c} \sum_{k=0}^{m} \frac{(itx)^k}{k!} Q\left(R \geqslant x-1\right) dx\right).$$

Then

$$\begin{aligned} |\phi_C(t) - \phi_{C,m}(t)| &\leqslant t \int_0^{1+c} \left| e^{itx} - \sum_{k=0}^{m} \frac{(itx)^k}{k!} \right| Q\left(R \geqslant x-1\right) dx \\ &\leqslant 2 \frac{t^{m+2}}{(m+1)!} \int_0^{1+c} x^{m+1} Q\left(R \geqslant x-1\right) dx \end{aligned}$$

because

$$\left| e^{itx} - \sum_{k=0}^{m} \frac{(itx)^k}{k!} \right| \leqslant 2 \frac{(xt)^{m+1}}{(m+1)!}. \tag{A.2}$$

The upper bound (A.2) can be derived for instance by writing e^{itx} as $\cos(tx)+i\sin(tx)$ and writing the integral remainders of the respective taylor expansion of $\cos(tx)$ and $\sin(tx)$. Thus,

$$|\phi_C(t) - \phi_{C,m}(t)| \leqslant 2 \frac{t^{m+2}}{(m+2)!} (1+c)^{m+2}. \tag{A.3}$$

We now define for given M and n,

$$\Theta_C(M, m) = \frac{2}{\pi} \int_0^M \frac{1 - Re\left(\phi_{C,m}^n(t)\right)}{t^2} dt \tag{A.4}$$

and show that it approximates $\Theta_C(M)$. We start with

$$|\Theta_C(M) - \Theta_C(M, m)| \leqslant \frac{2}{\pi} \int_0^M t^{-2} \left| Re\left(\phi_C^n(t)\right) - Re\left(\phi_{C,m}^n(t)\right) \right| dt.$$

Note that

$$\begin{aligned}\left|Re\left(\phi_C^n(t)\right) - Re\left(\phi_{C,m}^n(t)\right)\right| &\leqslant \left|\phi_C^n(t) - \phi_{C,m}^n(t)\right| \\ &\leqslant |\phi_C(t) - \phi_{C,m}(t)| \left|\sum_{k=0}^{n-1} \phi_C^{n-k}(t)\phi_{C,m}^k(t)\right| \\ &\leqslant |\phi_C(t) - \phi_{C,m}(t)| \sum_{k=0}^{n-1} |\phi_{C,m}(t)|^k \\ &\leqslant |\phi_C(t) - \phi_{C,m}(t)| \sum_{k=0}^{n-1} (1 + |\phi_C(t) - \phi_{C,m}(t)|)^k\end{aligned}$$

because $|\phi_C(t)| = 1$. Using (A.3), one has

$$\begin{aligned}&|\Theta_C(M) - \Theta_C(M,m)| \\ &\leqslant \frac{2}{\pi}\int_0^M \left(2\frac{t^m}{(m+2)!}(1+c)^{m+2}\right)\sum_{k=0}^{n-1}\left(1+\left(2\frac{t^{m+2}}{(m+2)!}(1+c)^{m+2}\right)\right)^k dt.\end{aligned}$$

Since M and n are fixed, then

$$\lim_{m\to+\infty} |\Theta_C(M) - \Theta_C(M,m)| = 0.$$

It shows that this approximation works in theory. Examples in section 4 will illustrate that it does not work as well as direct numerical integration of $\Theta_L(M)$ given by (A.4). However, the approximations here are of theoretical interest in further asymptotic analysis. $\square$