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A Gaussian Inequality for Expected Absolute Products

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Abstract We prove the inequality that $\mathbb{E}|X_1X_2\cdots X_n| \le \sqrt{\mathrm{per}(\Sigma)}$, for any centered Gaussian random variables $X_1, \ldots, X_n$ with the covariance matrix Σ, followed by several applications and examples. We also discuss a conjecture on the lower bound of the expectation.

Keywords Multivariate Gaussian · Permanent · Wick formula · MTP_2 density

Mathematics Subject Classification (2000) Primary 60E15 · Secondary 62H12

1 Introduction

Gaussian integrals involving absolute value function arise in a variety of contexts, ranging from roots of random functions to convex geometry. In [13], the author used the expected absolute determinant of a certain Gaussian matrix to represent the number of zeros of random multihomogeneous polynomial system. In [11, 12], the absolute value of a Gaussian quadratic function was studied and related to roots of random harmonic functions. The intrinsic volume of a convex body can also be represented by $\mathbb{E}|\det M|$ where M is a random matrix with independent standard Gaussian entries; see [20] and [21]. See also [2–4] for other applications.

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In this paper, we focus on $\mathbb{E}|X_1X_2\cdots X_n|$, the expected absolute value of the product of Gaussian variables. Explicit formulas for small n's were given in a series of papers [16–18]. For a special case when $X_{j,k}=\xi_j-\xi_k$ where ξ's are i.i.d. Gaussians, $\mathbb{E}|\prod_{1\le j<k\le n}X_{j,k}|$ can be evaluated by Mehta's integral, which is a probabilistic analog of Selberg's integral; see [14] for details. When n is large, the complexity of computation prevents people from finding the exact expression of $\mathbb{E}|X_1X_2\cdots X_n|$ for general Gaussian variables. In this case, estimation of such an expectation becomes essential. In [10], the authors explored the product of Gauss–Markov variables and provided a lower bound of the expectation by representing the expectation as an operator norm. However, their method is not designed to find upper bounds. In general, finding useful bounds for $\mathbb{E}|X_1X_2\cdots X_n|$ is a challenging problem.

In this paper, we present an elegant inequality on $\mathbb{E}|X_1X_2\cdots X_n|$ for general Gaussian variables, which provides an upper bound of the expectation:

Theorem 1 *Assume that $X_1,X_2,\dots,X_n$ are real centered jointly Gaussian random variables, and $\Sigma=(\mathbb{E}X_jX_k)_{n\times n}$ is the covariance matrix, then*

$$\mathbb{E}|X_1X_2\cdots X_n|\le\sqrt{\mathrm{per}(\Sigma)}. \tag{1.1}$$

Here the permanent of matrix $\Sigma=(\sigma_{jk})_{n\times n}$ is defined as $\mathrm{per}(\Sigma)=\sum_{\pi\in S_n}\prod_{j=1}^n\sigma_{j,\pi(j)}$ where the sum is over all of the permutations $\pi=(\pi(1),\pi(2),\dots,\pi(n))$ in the symmetric group S_n. It should be pointed out that this upper bound of $\mathbb{E}|X_1X_2\cdots X_n|$ is always better than the one given by the Cauchy–Schwarz inequality, i.e.,

$$\mathbb{E}|X_1X_2\cdots X_n|\le\sqrt{\mathrm{per}(\Sigma)}\le\left(\mathbb{E}X_1^2X_2^2\cdots X_n^2\right)^{1/2}. \tag{1.2}$$

The second inequality in (1.2) is due to P.E. Frenkel; see [6].

This paper is organized as follows: The proof and some applications of Theorem 1 are given in Sects. 2 and 3. In Sect. 4, we propose a conjecture on a lower bound of the expectation in this section, which is supported by known results.

2 Proof of Theorem 1

To prove Theorem 1, we need help from complex Gaussian variables (which have Gaussian real and imaginary parts). As given in [8] and [7], we call $\mathbf{Z}=(Z_1,Z_2,\dots,Z_n)^T$ a (circularly-)symmetric complex Gaussian random vector if $e^{\mathrm{i}\phi}\mathbf{Z}$ has the same probability distribution as $\mathbf{Z}$ for any real ϕ. Equivalently, a centered complex jointly-Gaussian vector is (circularly-)symmetric if and only if $\mathbb{E}\mathbf{Z}=\mathbb{E}\mathbf{Z}\mathbf{Z}^T=\mathbf{0}$. Next we recall a well known result on the symmetric complex Gaussian variables (cf. [1] and [19]):

Lemma 1 *Let $Z_1,Z_2,\dots,Z_n$ and $W_1,W_2,\dots,W_n$ be centered and correlated symmetric complex Gaussian variables, then*

$$\mathbb{E}\left(Z_1\cdots Z_n\overline{W}_1\cdots\overline{W}_n\right)=\mathrm{per}\left(\mathbb{E}Z_j\overline{W}_k\right)_{n\times n}, \tag{2.1}$$

where $\overline{W}_k$ is the conjugate of W_k.

Remark 1 Different from the proofs given in [1] and [19], we use Wick formula to obtain (2.1). According to [6] and [8], for a sequence of centered real or complex Gaussian random variables $X_1, X_2, \ldots, X_{2n}$, we have

$$\mathbb{E}(X_1 X_2 \cdots X_{2n}) = \mathrm{haf}(\mathbb{E}X_j X_k)_{2n\times 2n},$$

where *haf* denotes the Hafnian of the $2n \times 2n$ matrix. The Hafnian of a matrix $A = (a_{j,k})$ is defined to be

$$\mathrm{haf}(A) := \sum_{\sigma\in F_{2n}} a_{\sigma(1),\sigma(2)} a_{\sigma(3),\sigma(4)} \cdots a_{\sigma(2n-1),\sigma(2n)},$$

where F_{2n} is the set of all permutation σ satisfying $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$ and $\sigma(2i-1) < \sigma(2i)$, for $1 \le i \le n$. As $Z_1, Z_2, \ldots, Z_n$ and $W_1, W_2, \ldots, W_n$ are symmetric complex Gaussians, we always have $\mathbb{E}Z_j Z_k = \mathbb{E}W_j W_k = 0$ for all j and k. Therefore,

$$\mathbb{E}\big(Z_1 \cdots Z_n \overline{W}_1 \cdots \overline{W}_n\big) = \mathrm{haf}\begin{pmatrix} \mathbf{0} & (\mathbb{E}Z_j\overline{W}_k)_{n\times n} \\ (\mathbb{E}Z_j\overline{W}_k)_{n\times n} & \mathbf{0}\end{pmatrix} = \mathrm{per}\big(\mathbb{E}Z_j\overline{W}_k\big)_{n\times n},$$

by the definition of the Hafnian.

Proof of Theorem 1 Let $(Y_1, \ldots, Y_n)$ be an independent copy of $(X_1, \ldots, X_n)$ and $Z_j = X_j + \mathrm{i}Y_j$. Therefore, the (j,k)th entry of the covariance matrix of $(Z_1, \ldots, Z_n)$ is

$$\mathbb{E}Z_j\overline{Z}_k = \mathbb{E}(X_j + \mathrm{i}Y_j)(X_k - \mathrm{i}Y_k) = \mathbb{E}X_jX_k + \mathbb{E}Y_jY_k = 2\sigma_{jk}.$$

Then according to (2.1) we have

$$\mathbb{E}|Z_1 \cdots Z_n|^2 = \mathrm{per}\big(\mathbb{E}Z_j\overline{Z}_k\big)_{n\times n} = \mathrm{per}(2\sigma_{jk})_{n\times n} = 2^n \mathrm{per}(\Sigma).$$

On the other hand, by Cauchy–Schwarz inequality,

$$\begin{aligned}\left(\mathbb{E}\left|\prod_{j=1}^n X_j\right|\right)^2 &\le 2^{-n}\mathbb{E}\big(X_1^2+Y_1^2\big)\big(X_2^2+Y_2^2\big)\cdots\big(X_n^2+Y_n^2\big) \\ &= 2^{-n}\mathbb{E}|Z_1\cdots Z_n|^2 = \mathrm{per}(\Sigma),\end{aligned}$$

which finishes the proof. □

Remark 2 From the Cauchy–Schwarz inequality used above, we can see that the equality condition for (1.1) is $X_j = Y_j$ for all j's. Note that two independent continuous random variables are equal to each other with probability zero, i.e., the equality in (1.1) doesn't hold almost surely.

3 Applications

In this section, we apply Theorem 1 to different types of Gaussian random vectors to estimate the expected absolute value of corresponding Gaussian products.

3.1 Gauss–Markov Variables

This particular type of Gaussian random variables was first analyzed in [10]. The covariance function is given by $\mathbb{E}X_jX_k = \rho^{|j-k|}$ for all $j, k = 1, 2, \ldots, n$. In their paper, the authors showed that $\mathbb{E}|X_1X_2\cdots X_n| \sim \lambda^n$, where λ is the maximal eigenvalue of the Hilbert–Schmidt kernel

$$J(x,y) = \frac{\sqrt{|xy|}}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1+\rho^2}{4(1-\rho^2)}\left(x^2+y^2\right) + \frac{\rho xy}{1-\rho^2}\right).$$

Therefore, λ can be written as

$$\lambda = \sup_f \frac{(Jf, f)}{(f, f)}, \quad \text{where } Jf(\cdot) = \int_{-\infty}^{\infty} J(\cdot, y) f(y)\, dy.$$

In particular, setting $f_\alpha(x) = \sqrt{x}\exp(-ax^2/4)$ with positive a's, we can bound λ below by $(Jf_a, f_a)/(f_a, f_a)$. As given in [10],

$$\lim_{n\to\infty} \left(\mathbb{E}|X_1\cdots X_n|\right)^{1/n} = \lambda > \frac{2a}{\sqrt{2\pi(1-\rho^2)}}\left(\frac{4}{\Delta} + \frac{4\beta}{\Delta^{3/2}}\tan^{-1}\frac{\beta}{\sqrt{\Delta}}\right), \quad (3.1)$$

where $\beta = 2\rho/(1-\rho^2)$, $c = ((1+\rho^2)/(1-\rho^2)+a)/2$ and $\Delta = 4c^2 - \beta^2$.

Due to the supreme form of the variation representation, it is difficult to use their method to provide upper bounds for $\mathbb{E}|X_1\cdots X_n|$. Applying Theorem 1, we are able to obtain an upper bound of this expectation. We start with the permanent of the covariance matrix Σ, which can be represented in a combinatorial way in terms of distances between permutations:

$$\operatorname{per}(\Sigma) = \sum_{\pi\in S_n} \rho^{\sum_{j=1}^n |j-\pi(j)|} = \sum_{k=0}^{2[n^2/4]} \rho^k \cdot \sharp\left\{\pi \in S_n : \operatorname{dist}(\pi, I) = k\right\},$$

where $\sharp$ denotes the number of elements in the set and $\operatorname{dist}(\pi, I) := \sum_{j=1}^n |j - \pi(j)|$ is the distance between the permutation π and the identical permutation $I = (1, 2, \ldots, n)$. However, it appears that there is no explicit or useful formulas to find the number of permutations within the same distance from the identical permutation. Therefore, we consider upper bounds of $\operatorname{per}(\Sigma)$. According to [15], $\operatorname{per}(A) \le \prod_{j=1}^n r_j$ for $n\times n$ nonnegative matrix A, where r_j is the sum of the jth row of A. Set $\tilde{\Sigma} = (|\rho|^{|j-k|})_{n\times n}$, then

$$\begin{aligned}\left(\operatorname{per}(\Sigma)\right)^{1/n} &\le \left(\operatorname{per}(\tilde{\Sigma})\right)^{1/n} \le \prod_{k=1}^n \left(\sum_{j=1}^n |\rho|^{|j-k|}\right)^{1/n} \\ &\le \sum_{j=1}^n |\rho|^{|j-[(n+1)/2]|} \to \frac{1+|\rho|}{1-|\rho|},\end{aligned}$$

as $n\to\infty$. Thus

$$\limsup_{n\to\infty} \left(\mathbb{E}|X_1X_2\cdots X_n|\right)^{1/n} \le \left(\frac{1+|\rho|}{1-|\rho|}\right)^{1/2}. \quad (3.2)$$

Let us compare the lower and upper bounds by an example. Assume $\rho = 0.55$, we can see that $\lambda > 1.012$ by choosing $a = 0.5$. Combining (3.2) and (3.1) together, we have

$$1.012 < \lim_{n\to\infty} \left(\mathbb{E}|X_1 X_2 \cdots X_n|\right)^{1/n} \le 1.856, \quad \text{when } \rho = 0.55. \tag{3.3}$$

We can obtain various lower and upper bounds by choosing different ρ's. But numerical analysis suggests that if we expect an unbounded $\mathbb{E}|X_1 X_2 \cdots X_n|$, then the lower and upper bounds have the smallest gap when $\rho = 0.55$.

3.2 Same Correlations Case

Suppose we have a sequence of standard real Gaussian variables, and the correlations between each two of them are the same. Then as a consequence of Theorem 1, we have

Proposition 1 *Let $X_1, X_2, \ldots, X_n$ be a sequence of real centered Gaussian random variables with $\mathbb{E}X_j^2 = 1$ and $\mathbb{E}X_j X_k = \rho \in [0,1]$ for all $j \ne k$, then we have*

$$\lim_{n\to\infty} n^{-1/2}\left(\mathbb{E}|X_1 X_2 \cdots X_n|\right)^{1/n} = e^{-1/2}\rho^{1/2}.$$

Proof First, note that the $k \times k$ principal minor of the covariance matrix Σ under the above setting is $(k\rho + 1)(1-\rho)^{k-1}$, for $k = 1, 2, \ldots, n$. So the domain of ρ is $[-1/n, 1]$ due to Σ being positive semidefinite. As $n \to \infty$, the domain approaches $[0, 1]$.

In this setting, the permanent of Σ can be expressed as:

$$\operatorname{per}(\Sigma) = \sum_{\sigma\in S_n} \prod_{j=1}^{n} 1^{\mathbf{1}_{j=\sigma(j)}} \rho^{\mathbf{1}_{j\ne\sigma(j)}} = \sum_{\sigma\in S_n} \rho^{\sharp\{j: j\ne\sigma(j), j=1,2,\ldots,n\}} = \sum_{j=0}^{n} d_j \rho^j,$$

where $\sharp$ denotes the number of elements in the set and d_j is the number of permutations in which the longest derangement string is of length j. Here derangement means that none of the elements in the string appears at its original position before permutation. It is known that the number of different derangement strings of length j is $j! \cdot \sum_{k=0}^{j} (-1)^k/k!$ (see, e.g., [5]). Therefore, we have

$$\operatorname{per}(\Sigma) = \sum_{j=0}^{n} \binom{n}{j} j! \left(\sum_{k=0}^{j} \frac{(-1)^k}{k!}\right) \rho^j = n! \rho^n \sum_{l=0}^{n} \frac{1}{l!} \left(\sum_{k=0}^{n-l} \frac{(-1)^k}{k!}\right) \rho^{-l}.$$

Since $\sum_{k=0}^{\infty} (-1)^k/k!$ is bounded due to the convergence of the series, and the infinite series $\sum_{l=0}^{\infty} \rho^{-l}/l!$ is also convergent, we can see that

$$\lim_{n\to\infty} \sum_{l=0}^{n} \frac{1}{l!} \left(\sum_{k=0}^{n-l} \frac{(-1)^k}{k!}\right) \rho^{-l} = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\right) \rho^{-l} = e^{1/\rho - 1}.$$

Therefore, we have

$$\limsup_{n\to\infty} n^{-1/2}\big(\mathbb{E}|X_1X_2\cdots X_n|\big)^{1/n} \le e^{-1/2}\rho^{1/2}. \tag{3.4}$$

Now let us prove that $e^{-1/2}\rho^{1/2}$ is also a lower bound. When n is even, we have

$$\mathbb{E}|X_1\cdots X_n| \ge \big|\mathbb{E}(X_1\cdots X_n)\big| = \frac{n!}{2^{n/2}(n/2)!}\rho^{n/2}.$$

This is because we have $2^{-n/2}n!/(n/2)!$ different ways to pair $X_1, X_2, \dots, X_n$ up, and the number of pairs is always $n/2$. As a consequence,

$$\liminf_{n\to\infty} n^{-1/2}\big(\mathbb{E}|X_1X_2\cdots X_n|\big)^{1/n} \ge e^{-1/2}\rho^{1/2}. \tag{3.5}$$

Combining (3.4) and (3.5), we prove the proposition for even n's. When n is odd, we first observe that the inverse of the covariance matrix in this case is

$$\Sigma^{-1} = (1-\rho)^{-1}\big(1+(n-1)\rho\big)^{-1}\begin{pmatrix} 1+(n-2)\rho & -\rho & \cdots & -\rho \\ -\rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\rho \\ -\rho & \cdots & -\rho & 1+(n-2)\rho \end{pmatrix}.$$

According to Corollary 1.1 and Theorem 3.1 of [9], the density of $(|X_1|, |X_2|, \dots, |X_n|)$ is therefore Multivariate Totally Positive of order 2 (MTP_2) and as a consequence

$$\mathbb{E}\big[\phi_1\big(|X_1|,\dots,|X_n|\big)\cdots\phi_k\big(|X_1|,\dots,|X_n|\big)\big] \ge \prod_{j=1}^{k}\mathbb{E}\phi_j\big(|X_1|,\dots,|X_n|\big),$$

due to ϕ_j's being nonnegative and increasing. Now we set $\phi_1(|X_1|,\dots,|X_n|) = |X_1X_2\cdots X_{m-1}|$ and $\phi_2(|X_1|,\dots,|X_n|) = |X_m|$, for $m = n$ and $n+1$, and obtain

$$\sqrt{2/\pi}\,\mathbb{E}|X_1\cdots X_{n-1}| \le \mathbb{E}|X_1\cdots X_n| \le \sqrt{\pi/2}\,\mathbb{E}|X_1\cdots X_{n+1}|,$$

which implies that the odd and even n cases are equivalent. □

3.3 Tridiagonal Covariance Matrix Case

Considering the case when $\mathbb{E}X_j^2 = 1$ and $\mathbb{E}X_jX_k = \rho\mathbf{1}_{\{|j-k|=1\}}$ for $j \ne k$ with ρ positive, we can evaluate the permanent of the tridiagonal covariance matrix explicitly. By the definition of the permanent, let Σ_n be the covariance matrix of $(X_1, X_2, \dots, X_n)$, then we have the following recursive relation:

$$\operatorname{per}(\Sigma_{n+2}) = \operatorname{per}(\Sigma_{n+1}) + \rho^2\operatorname{per}(\Sigma_n), \quad n = 1, 2, \dots,$$

which indicates that the general terms are

$$\begin{aligned}\mathrm{per}(\Sigma_n) &= \frac{(1+\sqrt{1+4\rho^2})^{n+1} - (1-\sqrt{1+4\rho^2})^{n+1}}{2^{n+1}\sqrt{1+4\rho^2}} \\ &\sim \frac{(1+\sqrt{1+4\rho^2})^{n+1}}{2^{n+1}\sqrt{1+4\rho^2}}, \quad \text{for large } n.\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty}\big(\mathbb{E}|X_1X_2\cdots X_n|\big)^{1/n} \le \Big(1/2+\sqrt{1/4+\rho^2}\Big)^{1/2}.$$

Similar to the argument obtaining (3.5), we can find a lower bound in this setting as well:

$$\liminf_{n\to\infty}\big(\mathbb{E}|X_1X_2\cdots X_n|\big)^{1/n} \ge \rho^{1/2}.$$

4 A Conjecture on the Lower Bound

We can also use Hölder's inequality directly on $\mathbb{E}|X_1\cdots X_n|$ and show that the maximum of this expectation is achieved when $|\mathrm{Corr}(X_j, X_k)| = 1$ for all $j, k = 1, 2, \dots, n$. It is natural to conjecture that the minimum of $\mathbb{E}|X_1\cdots X_n|$ would be achieved when $X_1, X_2, \dots, X_n$ are independent. Based on this idea, we propose the following conjecture,

Conjecture *For the centered real jointly Gaussian random variable $X_1, X_2, \dots, X_n$ and nonnegative α_j, $j = 1, 2, \dots, n$, we have*

$$\mathbb{E}|X_1|^{\alpha_1}|X_2|^{\alpha_2}\cdots|X_n|^{\alpha_n} \ge \prod_{j=1}^{n}\mathbb{E}|X_j|^{\alpha_j}. \tag{4.1}$$

Remark 3 It is easy to check the case $\alpha_j = 1, n = 2$. When $\alpha_j = 1, n = 3$, an explicit formula for $\mathbb{E}|X_1X_2X_3|$ was given in [17]. Numerical analysis suggests that $\mathbb{E}|X_1X_2X_3|$ reaches its minimum when X_1, X_2 and X_3 are independent. The case $\alpha_j = 2$ was proved in [6]. Actually, when the joint density of $(|X_1|, |X_2|, \dots, |X_n|)$ is Multivariate Totally Positive of order 2 (MTP_2), the conjecture (4.1) is true for all $\alpha_j \ge 0$, which is supported by Corollary 1.1 and Theorem 3.1 of [9].

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