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Karhunen–Loeve expansions for the detrended Brownian motion

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ABSTRACT

The detrended Brownian motion is defined as the orthogonal component of projection of the standard Brownian motion into the subspace spanned by linear functions. Karhunen–Loeve expansion for the process is obtained, together with the explicit formula for the Laplace transform of the squared L_2 norm. Distribution identities are established in connection with the second order Brownian bridge developed by MacNeill (1978). As applications, large and small deviation asymptotic behaviors for the L_2 norm are given.

1. Introduction

Let $X = \{X(t), 0 \le t \le 1\}$ be a mean zero Gaussian process on $C[0, 1]$ with covariance function $K_X(t, s) = \mathbb{E}X(t)X(s)$, $0 \le s, t \le 1$. Then the well-known Karhunen–Loeve (KL) expansion is

$$X(t) = \sum_{k\ge1} \eta_k \sqrt{\lambda_k} f_k(t), \tag{1.1}$$

where $\{\eta_k, k \ge 1\}$ is a sequence of i.i.d. $N(0, 1)$ random variables, $\{f_k(t), k \ge 1\}$ forms an orthogonal sequence in $L^2[0, 1]$ and $\{\lambda_k, k \ge 1\}$ is the set of eigenvalues of the integral operator $T_X f(t) = \int_0^1 K_X(t, s)f(s)ds$. A natural consequence of the KL expansions is the distributional identity

$$\int_0^1 X^2(t)dt \overset{\text{law}}{=} \sum_{k=1}^{\infty} \lambda_k \eta_k^2. \tag{1.2}$$

The KL expansions for the so-called demeaned (or centered) Gaussian process on [0, 1], denoted by $\overline{X}$, have been studied for various X extensively, where one defines

$$\overline{X}(t) = X(t) - \int_0^1 X(s)ds,$$

with mean zero and covariance function $\overline{K}_X(t, s) = \mathbb{E}\overline{X}(t)\overline{X}(s)$, $0 \le s, t \le 1$. In particular, results on the demeaned (or centered) Brownian process $\overline{W}$ and Brownian bridge $\overline{B}$ can be found in Beghin et al. (2005), Deheuvels (2007) and

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Karol et al. (2008). And one has the distribution identities

$$\int_0^1 B^2(t)dt \overset{\text{law}}{=} \int_0^1 \overline{W}^2(t)dt \overset{\text{law}}{=} \int_0^1 W_*^2(t)dt \overset{\text{law}}{=} \int_0^1 \tilde{Y}^2(t)dt,$$

where $W(t), 0 \le t \le 1$, is a standard Brownian motion with the covariance function $K_W(t,s) = \mathbb{E}W(t)W(s) = \min(t,s)$, the process $B(t)$ is a standard Brownian bridge with the covariance function $K_B(t,s) = \mathbb{E}B(t)B(s) = \min(s,t) - st, 0 \le s, t \le 1$, the process $W_*(t) = W(t) - t^{-1}\int_0^t W(u)du$ is a Brownian motion centered at its on-line average, and the process $\tilde{Y}(t) = B(t) - 12t(1-t)\int_0^1 B(u)du$ is a mean-centered Brownian bridge.

The spectra of the demeaned Brownian motion $\overline{W}$ and demeaned Brownian bridge $\overline{B}$ can be extracted from Karol et al. (2008), and are given by $\lambda_k^{(\overline{W})} = (k\pi)^{-2}, \lambda_{2k-1}^{(\overline{B})} = \lambda_{2k}^{(\overline{B})} = (2k\pi)^{-2}$ for $k \ge 1$. Consequently,

$$\int_0^1 \overline{B}^2(t)dt \overset{\text{law}}{=} \frac{1}{4}\int_0^1 \overline{W}^2(t)dt + \frac{1}{4}\int_0^1 \overline{W^*}^2(t)dt,$$

where $\overline{W}(t)$ and $\overline{W^*}(t)$ are two independent demeaned Brownian motions.

From statistical application point of view, similar results are studied as the limiting distributions for various generalized KPSS-tests of stationarity of univariate time series under the null hypothesis; see Nyblom and Harvey (2000), Hobijn et al. (2004), Taylor (2003) and Ahlgren and Nyblom (2008) for additional references and related work. In particular, the L_2 norm of the detrended Brownian motion that we defined below and studied in this paper is one of the asymptotic distribution used by Hobijn et al. (2004). The KL expansion for the generalized Brownian bridge is given by MacNeill (1978) and provides us the second distribution identity in law in Theorem 1.

To motive the definition of the detrended process, it is natural to view the demeaned process $\overline{X}$ as the orthogonal component of projection of $X(t)$ into a constant function subspace in $L^2([0, 1])$. That is

$$\int_0^1 \overline{X}(t)^2 dt = \min_{a\in\mathbb{R}} \int_0^1 (X(t) - a)^2 dt. \tag{1.3}$$

To generalize the projection idea into the linear detrended process, we consider

$$\min_{a,b\in\mathbb{R}} \int_0^1 (X(t) - a - bt)^2 dt, \tag{1.4}$$

where the optimal constant a and b satisfy

$$\frac{\partial}{\partial a}\int_0^1 (X(t) - a - bt)^2 dt = 0, \qquad \frac{\partial}{\partial b}\int_0^1 (X(t) - a - bt)^2 dt = 0.$$

It follows that

$$a = 4\int_0^1 X(s)ds - 6\int_0^1 sX(s)ds, \qquad b = 12\int_0^1 sX(s)ds - 6\int_0^1 X(s)ds.$$

Then we can define the detrended Gaussian process, the orthogonal component of the projection,

$$\widehat{X}(t) = X(t) - a - bt = X(t) + (6t-4)\int_0^1 X(s)ds + (6-12t)\int_0^1 sX(s)ds, \tag{1.5}$$

with covariance functions $\widehat{K}_X(t,s) = \mathbb{E}\widehat{X}(t)\widehat{X}(s), 0 \le s, t \le 1$.

In this paper, we will concentrate on the KL expansion of the detrended Brownian motion

$$\widehat{W}(t) = W(t) + (6t-4)\int_0^1 W(s)ds + (6-12t)\int_0^1 sW(s)ds. \tag{1.6}$$

One can also consider the detrended Brownian bridge

$$\widehat{B}(t) = B(t) + (6t-4)\int_0^1 B(s)ds + (6-12t)\int_0^1 sB(s)ds.$$

However, a simple covariance computation given in Lemma 2.1 in the next section shows that $\widehat{W}(t)$ and $\widehat{B}(t)$ are the same process on $C[0, 1]$.

Before stating Theorem 1, we need some notations and facts. For $v > -1$, let $J_v(\cdot)$ denote the Bessel function of the first kind with index v. The positive zeros of $J_v(\cdot)$ form an infinite sequence, denoted by $0 < z_{v,1} < z_{v,2} < \cdots$. These zeros are interlaced with zeros $0 < z_{\upsilon+1,1} < z_{\upsilon+1,2} < \cdots$ of $J_{\upsilon+1}(\cdot)$ (see, e.g., Watson, 1952, pp. 479) in such a way that $0 < z_{\upsilon,1} < z_{\upsilon+1,1} < z_{\upsilon,2} < z_{\upsilon+1,2} < \cdots$. Considering the special cases $\upsilon = 1/2$ and $\upsilon = 3/2$, for all $x > 0$,

$$J_{1/2}(x) = (2/(\pi x))^{1/2} \sin(x), \tag{1.7}$$

$$J_{3/2}(x) = (2/(\pi x))^{1/2}(\sin(x)/x - \cos(x)). \tag{1.8}$$

Since the positive zeros of $J_{1/2}(\cdot)$ are given by $z_{1/2,k} = k\pi$, $k = 1, 2, \ldots$, thus

$$0 < z_{1/2,1} = \pi < z_{3/2,1} < z_{1/2,2} = 2\pi < z_{3/2,2} < \cdots, \tag{1.9}$$

where $\{z_{3/2,k}, k \geq 1\}$ are the ordered positive zeros of $J_{3/2}(x)$.

Now we can state one of the main results of this paper.

Theorem 1. *The spectrum of the KL expansion for the detrended Brownian motion* $\{\widehat{W}(t), t \in [0, 1]\}$ *is given by* (2.27) *and* (2.28)*. In particular, we have the distribution identities*

$$\int_0^1 \widehat{W}(t)^2 dt \overset{\text{law}}{=} \int_0^1 B_2(t)^2 dt \overset{\text{law}}{=} \sum_{k\geq 1} \frac{\eta_k^2}{4\pi^2 k^2} + \sum_{k\geq 1} \frac{\eta_k^{*2}}{4 z_{3/2,k}^2}, \tag{1.10}$$

where $\{\eta_k, k \geq 1\}$ *and* $\{\eta_k^*, k \geq 1\}$ *denote two independent sequences of independently and identically distributed* $N(0, 1)$ *random variables, and the process*

$$\begin{aligned} B_2(t) &= W(t) - tW(1) + 3t(1-t)\left(W(1) - 2\int_0^1 W(s)ds\right) \\ &= B(t) - 6t(1-t)\int_0^1 B(u)du, \quad 0 \leq t \leq 1 \end{aligned} \tag{1.11}$$

is the second level (order) Brownian bridge.

Note that the process $B_2(t)$ given in (1.11) has properties of $B_2(0) = B_2(1) = 0$, $\int_0^1 B_2(t)dt = 0$ and the covariance function

$$\mathbb{E}B_2(t)B_2(s) = t \wedge s - st - 3s(1-s)t(1-t), \quad s, t \in [0, 1]. \tag{1.12}$$

Actually, the process $B_2(t)$ is a Brownian bridge $B(t)$ conditioned on $\int_0^1 B(t)dt = 0$, i.e. $B_2(t) = B(t)|_{\int_0^1 B(t)dt=0}$. Its KL expansion is given by MacNeill (1978) as a special case (second order) of a family of generalized Brownian bridges. More detailed study about $B_2(t)$ can be found from Deheuvels (2007) in connection with a family of mean-centered Brownian bridges, where the second identity in law in (1.10) is also presented.

The rest of the paper is organized as follows. In Section 2, we give the proof of Theorem 1. As applications, in Section 3, we study the Laplace transforms, large deviations and small deviations of the detrended Brownian motion $\widehat{W}(t)$.

2. The KL expansion for the detrended BM

We start with the following lemma that provides the explicit covariance function.

Lemma 2.1. *The covariance function of the detrended Brownian motion* $\widehat{W}(t)$ *on* $[0, 1]$ *is given by*

$$\begin{aligned} \widehat{K}_W(s,t) &= \mathbb{E}\widehat{W}(t)\widehat{W}(s) = t \wedge s - \frac{11}{10}t - \frac{11}{10}s + 2t^2 + 2s^2 - t^3 - s^3 \\ &\quad -3st^2 - 3ts^2 + 2st^3 + 2ts^3 + \frac{6}{5}st + \frac{2}{15}. \end{aligned} \tag{2.13}$$

Proof. For $0 \leq s, t \leq 1$, we have

$$\mathbb{E}\left(W(s)\int_0^1 W(u)du\right) = \int_0^1 \mathbb{E}(W(s)W(u))du = \int_0^1 (s \wedge u)du = s - \frac{s^2}{2},$$

$$\mathbb{E}\left(W(t)\int_0^1 vW(v)dv\right) = \int_0^1 v(v \wedge t)dv = \frac{t}{2} - \frac{t^3}{6}.$$

Similarly, and using the above computation, we obtain

$$\begin{aligned}\mathbb{E}\left(\int_0^1 W(u)du \int_0^1 W(v)dv\right) &= \int_0^1 \mathbb{E}\left(W(u)\int_0^1 W(v)dv\right) du \\ &= \int_0^1 \left(u - \frac{u^2}{2}\right) du = \frac{1}{3}, \\ \mathbb{E}\left(\int_0^1 W(u)du \int_0^1 vW(v)dv\right) &= \int_0^1 \mathbb{E}\left(W(u)\int_0^1 vW(v)dv\right) du \\ &= \int_0^1 \left(\frac{u}{2} - \frac{u^3}{6}\right) du = \frac{5}{24}, \\ \mathbb{E}\left(\int_0^1 uW(u)du \int_0^1 vW(v)dv\right) &= \int_0^1 u\mathbb{E}\left(W(u)\int_0^1 vW(v)dv\right) du \\ &= \int_0^1 u\left(\frac{u}{2} - \frac{u^3}{6}\right) du = \frac{2}{15}.\end{aligned}$$

Substituting the above equations into the product expansion

$$\begin{aligned}\mathbb{E}\widehat{W}(t)\widehat{W}(s) = \mathbb{E}\Bigg(&\left(W(t) + (6t-4)\int_0^1 W(u)du + (6-12t)\int_0^1 uW(u)du\right) \\ &\times \left(W(s) + (6s-4)\int_0^1 W(v)dv + (6-12s)\int_0^1 vW(v)dv\right)\Bigg),\end{aligned}$$

we obtain (2.13) after simplification. □

Proof of Theorem 1. We first compute the eigenvalues of $\widehat{W}(t)$ by substituting $\widehat{K}_W(s,t)$ of (2.13) into

$$T_{\widehat{W}}f(t) = \int_0^1 \widehat{K}_W(s,t)f(s)ds = \lambda f(t). \tag{2.14}$$

In order to handle the $t \wedge s = \min(t,s)$ term, we split the integration range and obtain

$$\begin{aligned}&\int_0^t sf(s)ds + t\int_t^1 f(s)ds + \int_0^1 \bigg(-\frac{11}{10}t - \frac{11}{10}s + 2t^2 + 2s^2 - t^3 - s^3 - 3st^2 - 3ts^2 + 2st^3 + 2ts^3 \\ &+ \frac{6}{5}st + \frac{2}{15}\bigg)f(s)ds = \lambda f(t)\end{aligned} \tag{2.15}$$

with the boundary conditions

$$\lambda f(0) = \int_0^1 \left(-\frac{11}{10}s + 2s^2 - s^3 + \frac{2}{15}\right) f(s)ds \tag{2.16}$$

and

$$\lambda f(1) = \int_0^1 \left(\frac{1}{10}s - s^2 + s^3 + \frac{1}{30}\right) f(s)ds. \tag{2.17}$$

By differentiating both sides of (2.15) with respect to t, we obtain

$$\int_t^1 f(s)ds + \int_0^1 \left(-\frac{11}{10} + 4t - 3t^2 - 6st - 3s^2 + 6st^2 + 2s^3 + \frac{6}{5}s\right) f(s)ds = \lambda f'(t) \tag{2.18}$$

with the boundary conditions

$$\lambda f'(0) = \int_0^1 \left(-\frac{1}{10} - 3s^2 + 2s^3 + \frac{6}{5}s\right) f(s)ds, \tag{2.19}$$

$$\lambda f'(1) = \int_0^1 \left(-\frac{1}{10} - 3s^2 + 2s^3 + \frac{6}{5}s\right) f(s)ds. \tag{2.20}$$

By differentiating again both sides of (2.18), we obtain the equation

$$\lambda f''(t) + f(t) + a + bt = 0, \tag{2.21}$$

where the constants $a = \int_0^1 (6s-4)f(s)ds$ and $b = \int_0^1 (6-12s)f(s)ds$.

The general solution of the inhomogeneous second order differential equation (2.21) is given by

$$f(t) = C_2 \sin(\lambda^{-1/2}t) + C_1 \cos(\lambda^{-1/2}t) - a - bt, \tag{2.22}$$

where C_1 and C_2 are constants.

The boundary conditions (2.19) and (2.20) give $f'(0) = f'(1)$ which reduces to

$$C_2(\cos(\lambda^{-1/2}) - 1) - C_1 \sin(\lambda^{-1/2}) = 0. \tag{2.23}$$

The boundary conditions (2.16), (2.17) and (2.19) imply $f'(0) = f(1) - f(0)$ which simplifies to

$$C_2(\sin(\lambda^{-1/2}) - \lambda^{-1/2}) + C_1(\cos(\lambda^{-1/2}) - 1) = 0. \tag{2.24}$$

In order to have constants C_1 and C_2 such that $C_1^2 + C_2^2 \neq 0$, Eqs. (2.23) and (2.24) imply

$$2(1 - \cos(\lambda^{-1/2})) - \lambda^{-1/2}\sin(\lambda^{-1/2}) = 0, \tag{2.25}$$

which can be rewritten as

$$2^{-1}\pi\lambda^{-1}J_{1/2}(2^{-1}\lambda^{-1/2})J_{3/2}(2^{-1}\lambda^{-1/2}) = 0, \tag{2.26}$$

where $J_{1/2}(x)$ and $J_{3/2}(x)$ are given in (1.7) and (1.8) respectively. Thus the solutions of (2.26) are

$$\lambda_{2k-1} = (2k\pi)^{-2}, \quad k = 1, 2, \ldots, \tag{2.27}$$

$$\lambda_{2k} = (2z_{3/2,k})^{-2}, \quad k = 1, 2, \ldots, \tag{2.28}$$

where $z_{3/2,k}$ are the ordered positive zeros of Bessel function and $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > 0$.

The eigenfunctions are $\sqrt{2}\cos(2k\pi t), k = 1, 2, \ldots$ associated with the eigenvalues $\lambda_{2k-1}, k = 1, 2, \ldots$ and the other eigenfunctions associated with the eigenvalues $\lambda_{2k}, k = 1, 2, \ldots$ can be found by using two boundary conditions (2.16) and (2.19) with normalizing condition $\int_0^1 (f(t))^2 dt = 1$. Furthermore, $\lambda_{2k-1} = (2k\pi)^{-2}, k = 1, 2, \ldots$ and $\lambda_{2k} = (2z_{3/2,k})^{-2}, k = 1, 2, \ldots$ give

$$\int_0^1 \widehat{W}(t)^2 dt \overset{\text{law}}{=} \sum_{k=1}^{\infty} \lambda_k \eta_k^2 = \sum_{k\geq 1} \eta_k^2 k^{-2}\pi^{-2}/4 + \sum_{k\geq 1} \eta_k^{*2} z_{3/2,k}^{-2}/4, \tag{2.29}$$

where $\{\eta_k, k \geq 1\}$ and $\{\eta^*{}_k, k \geq 1\}$ denote two independent sequences of independently and identically distributed $N(0, 1)$ random variables. From (1.20) of Deheuvels (2007), the mean-centered Gaussian process or the second order Brownian bridge $B_2(t) = B(t) - 6t(1-t)\int_0^1 B(u)du$ has the same norm in $L^2[0, 1]$ as $\widehat{W}(t)$. The proof is completed. □

3. Applications

There are very few Gaussian processes where the KL expansions are known through the explicit eigenvalues of $\{\lambda_k, k \geq 1\}$ and with simple forms of the eigenfunctions. However, like the KL expansion given in Theorem 1, the Laplace transform for $\int_0^1 \widehat{W}^2(t)dt$ can be more explicit by using the associated Fredholm determinant.

Proposition 3.1. *For each $\theta \in \mathbb{R}$,*

$$\mathbb{E}\exp\left(-\frac{\theta^2}{2}\int_0^1 \widehat{W}(t)^2 dt\right) = (12\theta^{-4}(2 + \theta\sinh(\theta) - 2\cosh(\theta)))^{-1/2}.$$

Proof. This follows from the general fact that

$$\mathbb{E}\exp\left(-\frac{\theta^2}{2}\int_0^1 \widehat{W}(t)^2 dt\right) = \mathbb{E}\exp\left(-\frac{\theta^2}{2}\sum_{k=1}^{\infty}\lambda_k \xi_k^2\right) = \prod_{k=1}^{\infty}(1 + \lambda_k\theta^2)^{-1/2} = (D(-\theta^2))^{-1/2}$$

with $\lambda_1 > \lambda_2 > \cdots > 0$ and $\sum_{k=1}^{\infty}\lambda_k < \infty$.

Here Fredholm determinant $D(\lambda)$ of the covariance $\widehat{K}_W(s,t)$ of the detrended Brownian motion $\widehat{W}(t)$ can be written as

$$D(\lambda) = 12\lambda^{-2}(2 - \lambda^{1/2}\sin(\lambda^{1/2}) - 2\cos(\lambda^{1/2})), \tag{3.30}$$

with $D(0) = 1$; see page 151–154 in Tanaka (1996) for a similar argument. In particular, one needs the Taylor expansion of the sin and cos functions to ensure $D(0) = 1$. □

One additional consequence of the associated Fredholm determinant $D(\lambda)$ is Smirnov formula (see Martynov, 1977), which gives for all $x > 0$,

$$\mathbb{P}\left(\int_0^1 \widehat{W}(t)^2 dt > x\right) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\int_{\gamma_{2k-1}}^{\gamma_{2k}} \frac{e^{-\frac{u}{2}x}}{u\sqrt{-D(u)}}\, du, \tag{3.31}$$

where $\gamma_k = \lambda_k^{-1}$, $k = 1, 2, \dots$.

Next we give the large deviation probability for L_2-norm of the detrended Brownian motion.

Proposition 3.2. *Let $\widehat{W}(t)$ be a detrended Brownian motion, then as $x \to \infty$,*

$$\mathbb{P}\left(\int_0^1 \widehat{W}(t)^2 dt > x\right) = (c + o(1))x^{-1/2}\exp(-2\pi^2 x), \tag{3.32}$$

where $c = \pi^{-1/2}(4\pi^2)^{-1}(3(32\pi^2 - 1)\cos(2\pi)^{-1} + 30\pi\sin(2\pi)^{-1} - 96\pi^2)^{-1/2}$.

Proof. By Lemma 1.1 and Remark 1.2 in Deheuvels and Martynov (2003), we have for all $x > 0$,

$$\mathbb{P}\left(\int_0^1 \widehat{W}(t)^2 dt > x\right) = (1 + o(1))(2/\pi)^{1/2}\gamma_1^{-1}(-D'(\gamma_1))^{-1/2}x^{-1/2}\exp(-\gamma_1 x/2),$$

where $\gamma_1 = (2\pi)^2$, $D(\gamma_1)$ is from Eq. (3.30) and $D'(\gamma_1)$ is the derivative of $D(\gamma_1)$, and the proof is completed. □

Finally, we briefly describe the small deviation probability for L_2-norm of the detrended Brownian motion. The argument is well developed and applied to many similar problems. We also choose to provide less precise description of several constants involved since they do not play significant role in applications.

Proposition 3.3. *There exists some constant $c > 0$ such that as $\varepsilon \to 0$,*

$$\mathbb{P}\left(\int_0^1 \widehat{W}(t)^2 dt \le \varepsilon\right) = (c + o(1))\varepsilon^{-1}\exp\left(-\frac{1}{8\varepsilon}\right).$$

Proof. The starting point is Theorem 2 of Li (1992a) that we recall here. Given any two sequences $a_k > 0$ and $b_k > 0$ with

$$\sum_{k\ge1} a_k < \infty, \qquad \sum_{k\ge1} b_k < \infty, \qquad \sum_{k\ge1} |1 - a_k/b_k| < \infty, \tag{3.33}$$

we have, as $\varepsilon \to 0$,

$$\mathbb{P}\left(\sum_{k\ge1} a_k\xi_k^2 \le \varepsilon\right) = (1 + o(1))\left(\prod_{k\ge1} b_k/a_k\right)^{1/2}\mathbb{P}\left(\sum_{k\ge1} b_k\xi_k^2 \le \varepsilon\right). \tag{3.34}$$

So for our setting, by the asymptotic formula for zeros of the Bessel function, we have $z_{3/2,k} = (k + 1/2)\pi + O(k^{-1})$, as $k \to \infty$ (see Korenev, 2002; Barczy and Igloi, 2011). We set $a_k = \lambda_k$, $b_{2k-1} = (2k\pi)^{-2}$, $b_{2k} = ((2k+1)\pi)^{-2}$ and they satisfy (3.33). Then by the distribution identity $\int_0^1 \widehat{W}(t)^2 dt \overset{\text{law}}{=} \sum_{k\ge1}\lambda_{2k-1}\eta_k^2 + \sum_{k\ge1}\lambda_{2k}\eta_k^{*2}$ and (3.34), there exists a constant c_1 such that as $\varepsilon \to 0$,

$$\begin{aligned}
\mathbb{P}\left(\int_0^1 \widehat{W}(t)^2 dt \le \varepsilon\right) &= \mathbb{P}\left(\sum_{k\ge1}\lambda_{2k-1}\eta_k^2 + \sum_{k\ge1}\lambda_{2k}\eta_k^{*2} \le \varepsilon\right)\\
&= (1 + o(1))\prod_{k\ge1}(b_k/a_k)^{1/2}\mathbb{P}\left(\sum_{k\ge1} b_k\xi_k^2 \le \varepsilon\right)\\
&= (1 + o(1))c_1\mathbb{P}\left(\sum_{k\ge1}\frac{\xi_{2k-1}^2}{(2k\pi)^2} + \sum_{k\ge1}\frac{\xi_{2k}^2}{((2k+1)\pi)^2} \le \varepsilon\right)\\
&= (1 + o(1))c_1\mathbb{P}\left(\sum_{k\ge1}(k+1)^{-2}\xi_k^2 \le \varepsilon\pi^2\right).
\end{aligned} \tag{3.35}$$

Now from Lemma 1 of Li (1992b), $\forall d > -1$, there exists a constant $c_2 > 0$, such that as $\varepsilon \to 0$,

$$\mathbb{P}\left(\sum_{k\geq 1}(k+d)^{-2}\xi_k^2 \leq \varepsilon\pi^2\right) = (1+o(1))c_2\varepsilon^{-d}\exp\left(-\frac{1}{8\varepsilon}\right). \tag{3.36}$$

Combining together (3.35) and (3.36), we complete the proof. □

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References

Ahlgren, N., Nyblom, J., 2008. Tests against stationary and explosive alternatives in vector autoregressive models. Journal of Time Series Analysis 29 (3), 421–443.

Barczy, M., Igloi, E., 2011. Karhunen–Loeve expansions of α-Wiener bridges. Central European Journal of Mathematics 9, 65–84.

Beghin, L., Nikitin, Ya., Orsingher, E., 2005. Exact small ball constants for some Gaussian processes under the L_2-norm. Journal of Mathematical Sciences 128, 2493–2502.

Deheuvels, P., 2007. Karhunen–Loeve expansion for a mean-centered Brownian bridge. Statistics & Probability Letters 77, 1190–1200.

Deheuvels, P., Martynov, G.V., 2003. Karhunen–Loeve expansions for weighted Wiener processes and Brownian bridges via Bessel functions. Progress in Probability 55, 57–93.

Hobijn, B., Franses, P.H., Ooms, M., 2004. Generalization of the KPSS-test for stationarity. Statistica Neerlandica 58 (4), 483–502.

Karol, A., Nazarov, A., Nikitin, Y., 2008. Small ball probabilities for Gaussian random fields and tensor products of compact operators. Transactions of the American Mathematical Society 360, 1443–1474.

Korenev, B.G., 2002. Bessel Functions and their Applications. Taylor & Francis, London.

Li, W.V., 1992a. Comparison results for the lower tail of Gaussian seminorms. Journal of Theoretical Probability 5, 1–31.

Li, W.V., 1992b. Limit theorems for the square integral of Brownian motion and its increments. Stochastic Processes and their Applications 41, 223–239.

MacNeill, Ian B., 1978. Properties of sequences of partial sums of polynomial regression residuals with applications to tests for change of regression at unknown times. The Annals of Statistics 6 (2), 422–433.

Martynov, G.V., 1977. Generalization of the Smirnov formula for the quadratic forms distributions. Theory of Probability and its Applications 22, 614–620.

Nyblom, J., Harvey, A.C., 2000. Tests of common stochastic trends. Econometric Theory 16, 176–199.

Tanaka, K., 1996. Time Series Analysis: Nonstationary and Noninvertible Distribution Theory. Wiley Series in Probability and Statistics.

Taylor, A.M.R., 2003. Locally optimal tests against unit roots in seasonal time series processes. Journal of Time Series Analysis 24, 591–612.

Watson, G.N., 1952. A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge.