Small value probabilities for supercritical branching processes with Immigration

Weijuan Chu* Wenbo V. Li† Yan-Xia Ren ‡

March 17, 2011

Abstract

We consider a supercritical Galton-Watson branching process with immigration. It is well known that under suitable conditions on the offspring and immigration distributions, there is a finite, strictly positive and non-degenerate limit for the normalized population size, denoted as $\mathcal{W}$. The main purpose of this paper is to investigate the small value probabilities of $\mathcal{W}$, that is to estimate $\mathbb{P}(\mathcal{W} \le \varepsilon)$ for $\varepsilon > 0$ small. In comparison with the well-studied results for supercritical Galton-Watson branching process without immigration, precise effects of the balance between offspring and immigration distributions on small value probability of $\mathcal{W}$, are obtained. Several illustrative examples are analyzed carefully. They demonstrate the sharpness of our results and the significant effect of the immigration which can cause the near-constancy phenomena even when there is no oscillation in the setting without immigration.

Key words and phrases: Supercritical Galton-Watson branching process, small value property, immigration

1 Introduction

Small value probability for a positive random variable V studies the rate of decay of the so called left tail probability $\mathbb{P}(V \le \varepsilon)$ as $\varepsilon \to 0^+$. When V is the norm of a random element in a Banach space, one is dealing with small ball probability, see [LS01] for a survey of Gaussian

*LMAM School of Mathematical Sciences, Peking University, Beijing 100871, P.R.China, E-mail: chuwj@pku.edu.cn;

†Department of Mathematical Sciences, University of Delaware, wli@math.udel.edu Supported in part by NSF grant DMS–0805929, NSFC-6398100, CAS-2008DP173182.

‡Corresponding author; LMAM School of Mathematical Sciences, Center for Statistical Science, Peking University, Beijing 100871, P.R.China, E-mail: yxren@math.pku.edu.cn; Supported in part by NSFC (Grant No. 10871103 and 10971003) and Specialized Research Fund for the Doctoral Program of Higher Education of China.

measure. When V is the maximum of a continuous random process starting at zero, one is estimating lower tail probability which is closely related to studies of boundary crossing probabilities or the first exit time associated with a general domain, see [L03] and [LS04] for Gaussian processes. A comprehensive study of small value probability is emerging and available in various talks and lecture notes formats in [L03], [L11+], see also the literature compilation [Lif10].

In this paper, we further develop the most natural aspect of the branching tree approach originated in [MO08] on the martingale limit of a supercritical Galton-Watson process. The problem has been solved initially in [D71a], [D71b] in the Schröder case and, up to a Tauberian theorem of Bingham [B88], also in the Böttcher case, see later part of this section for more details. The approach uses an integral transform together with some nontrivial complex analysis, which is powerful but inflexible and un-intuitive. On the other hand, the "branching tree heuristic" method developed in [MO08] for the martingale limit of a supercritical Galton-Watson process is very simple and based on an easy intuition. The main goal of this paper is developing additional tools to treat small value probabilities for the martingale limit of a supercritical Galton-Watson process with immigration. The interplay between the offspring and the immigration distribution can be seen clearly from our main result Theorem 2. We next provide a more detailed and precise discussion by introducing additional notations, surveying relevant results and stating our results.

Let $(Z_n, n \geq 0)$ be a supercritical Galton-Watson branching process with Z_0=1, offspring distribution $p_k = \mathbb{P}(X = k), k \geq 0$, and mean $m = \mathbb{E}\,X \in (1, \infty)$. To avoid non-branching case, we suppose $p_k < 1$ for all k throughout this paper. Under the natural condition $\mathbb{E}\,[X \log^+ X] < \infty$, the positive martingale $Z_n m^{-n}$ converges to a nontrivial random variable $W < \infty$ in the sense (see Kesten and Stigum [KS66])

$$Z_n m^{-n} \longrightarrow W \qquad a.s. \ \& \ L^1 \text{ as } n \to \infty.$$

Here and throughout this paper, $\log^+ x = \log \max(x, 1) \geq 0$. The distribution of the limit W is of great interests in various applications. However, except for some very special cases, the explicit distribution of W is not available, see, for example, Harris [H48], Williams [W08] Section 0.9. In general, it is known that W has a continuous positive density on $(0, \infty)$ satisfying a Lipschitz condition, see [AN72], Ch. II, p.84 Lemma 2. However it is not clear what type of densities can arise in this way. This lack of complete information on the distribution of W prompts a search for asymptotic information such as the behavior of the left tail, or the small value probabilities of W and its density. General estimates, near-constancy phenomena, specific examples, and various implications have been studied to various degree of accuracy in Harris [H48], Karlin and McGregor [KM68a] [KM68b], Dubuc [D71a] , [D71b] and [D82], Barlow and Perkins [BP88], Goldstein [G87], Kusuoka [K87], Bingham [B88], Biggins and Bingham [BB91] and [BB93], Biggins and Nadarajah [BN93], Fleischman and Wachtel [FW07] and [FW09].

For instance, in [D71b], the following results were given with assumption $p_0 = 0$ which holds without the loss of generality after the standard Harris-Sevastyanov transformation, see [H48], p.478 Theorem 3.2 or [B88] p.216) Here and throughout this paper we use $f(x) \asymp$

$g(x)$ *as* $x \to 0^+$ (∞) to represent $c \le f(x)/g(x) \le C$ *as* $x \to 0^+$ (∞) for two constants $C > c > 0$ and $f(x) \sim g(x)$ *as* $x \to 0^+$ (∞) to represent $f(x)/g(x) \to 1$ *as* $x \to 0^+$ (∞).

Theorem 1 *(Dubuc 1971(b))*
(a) If $p_1 > 0$*, then*

$$\mathbb{P}(W \le \varepsilon) \asymp \varepsilon^{|\log p_1|/\log m}.$$

(b) If $p_1 = 0$*, then*

$$-\log \mathbb{P}(W \le \varepsilon) \asymp \varepsilon^{-\beta/(1-\beta)},$$

with $\beta := \log \gamma / \log m$ *and* $\gamma := \inf\{n : p_n > 0\} \ge 2$*.*

Note that $\mathbb{P}(W \le \varepsilon) = \mathbb{P}(W < \varepsilon)$ since W has a continuous density, see, e.g. Athreya and Ney [AN72], Ch. I, Section 10, Corollary 4. Also, the so called near-constancy phenomena refers to the fact that the rough asymptotic $\asymp$ in Theorem 1 can not be improved into more precise asymptotic $\sim$ and the oscillation is very small, see [B88] for more details. In fact, it is still an open conjecture that the Laplace transform of W being non-oscillating near ∞ (and hence the small value probability of W being non-oscillating near 0) is only specific to the case $p_1 > 0$ in [KM68a] p.127.

In the present paper, we consider the supercritical branching process with immigration denoted by $(\mathcal{Z}_n, n \ge 0)$, and follow the definition in [AN72], Ch. VI, Section 7.1, p.263. To be more precise, we have

$$\mathcal{Z}_0 = Y_0, \quad \mathcal{Z}_{n+1} = X_1^n + X_2^n + \cdots + X_{\mathcal{Z}_n}^n + Y_{n+1}, \quad n \ge 0,$$

where $X_1^n, X_2^n, \cdots$ are independent and identically distributed with the same offspring distribution as X, the $Y_0, Y_1, \cdots$ are i.i.d. with the same immigration distribution $\{q_k, k \ge 0\}$ and the $X's$ and $Y's$ are independent. It is classic result, see [S70] for example, that

$$\lim_{n\to\infty} \mathcal{Z}_n / m^n = \mathcal{W} \tag{1.1}$$

exists and is finite a.s. if and only if

$$\mathbb{E} \log^+ Y < \infty \quad \text{and} \quad \mathbb{E}\,(X \log^+ X) < \infty. \tag{1.2}$$

Our main result of this paper is the following small value probabilities for $\mathcal{W}$.

Theorem 2 *Assume the condition (1.2) holds.*
(a) If $p_0 = 0$ *and* $0 < q_0 < 1$*, then*

$$\mathbb{P}(\mathcal{W} \le \varepsilon) \asymp \varepsilon^{|\log q_0|/\log m} \qquad as\ \varepsilon \to 0^+. \tag{1.3}$$

(b) If $p_0 = 0$*,* $q_0 = 0$ *and* $p_1 > 0$*, then*

$$\log \mathbb{P}(\mathcal{W} \le \varepsilon) \sim -\frac{K|\log p_1|}{2(\log m)^2} \cdot |\log \varepsilon|^2, \qquad as\ \varepsilon \to 0^+, \tag{1.4}$$

with $K=\inf\{n: q_n>0\}$.

(c) If $p_0=0$, $q_0=0$ *and* $p_1=0$, *then*

$$\log \mathbb{P}(\mathcal{W}\le\varepsilon)\asymp -\varepsilon^{-\beta/(1-\beta)}, \qquad \text{as } \varepsilon\to 0^+,$$

with β *being defined as in Theorem 1(b).*

(d) If $p_0>0$, *then*

$$\mathbb{P}(\mathcal{W}\le\varepsilon)\asymp \varepsilon^{|\log h(\rho)|/\log m}, \qquad \text{as } \varepsilon\to 0^+, \tag{1.5}$$

where ρ *is the solution of* $f(s)=s$ *between* $(0,1)$, *and* h *is the generating function of immigration.*

Similar to the case of Theorem 1, as it can be seen from Proposition 1 in Section 6, there is also the near-constancy phenomena here and thus the rough asymptotic $\asymp$ in our Theorem 2 can not be improved into more precise asymptotic $\sim$.

Our proof of Theorem 2, which appears in sections 3, 4 and 5, is based on Dubuc's result. Note that it seems impossible to extend the involved analytic method used in [D71b] to the branching process with immigration. However, Mörters and Ortgiese [MO08] provided a very useful probabilistic approach for Theorem 1. Our approach is built on top of their powerful arguments, and overcome additional difficulties of immigration effects. More specifically, we start with a fundamental decomposition for $\mathcal{W}$ given in (2.2). A suitable truncation is needed in order to handle the infinite series. To estimate the lower bound of $\mathbb{P}(\mathcal{W}\le\varepsilon)$, we investigate when the least population size happens. For the upper bound, we use the exponential Chebyshev's inequality and estimate the Laplace transform of $\mathcal{W}$. The property of $\mathbb{P}(\mathcal{W}\le\varepsilon)$ is then obtained through Tauberian type Theorems.

We put the proof of $p_0>0$ case in Section 5. In this case, the extinction probability is strictly positive, and plays the most important role in the small value probability of $\mathcal{W}$. Moreover, the immigration makes effect also.

Next we turn to consider a slightly different type of supercritical branching process with immigration, which is denoted by $(\widetilde{\mathcal{Z}}_n, n\ge 0)$. The only difference is to assume $\widetilde{\mathcal{Z}}_0=1$. The corresponding limit of $\widetilde{\mathcal{Z}}_n/m^n$ is denoted by $\widetilde{\mathcal{W}}$. Then by simple computation we get that

$$\widetilde{\mathcal{W}} =^d W+\frac{\mathcal{W}}{m} \tag{1.6}$$

in distribution, as denoted by $=^d$ throughout this paper. Due to (1.6) and the fact that

$$\begin{aligned}\mathbb{P}(W+\mathcal{W}/m\le\varepsilon) &\ge \mathbb{P}(W\le\varepsilon/2)\cdot\mathbb{P}(\mathcal{W}/m\le\varepsilon/2),\\ \mathbb{P}(W+\mathcal{W}/m\le\varepsilon) &\le \mathbb{P}(W\le\varepsilon)\cdot\mathbb{P}(\mathcal{W}/m\le\varepsilon),\end{aligned} \tag{1.7}$$

we can obtain the following result as a consequence of combining Theorem 1 and Theorem 2.

Theorem 3 *Assume the condition (1.2) holds.*

(a) If $p_0 = 0$, $p_1 > 0$ and $q_0 > 0$, then

$$\mathbb{P}(\widetilde{\mathcal{W}} \le \varepsilon) \asymp \varepsilon^{|\log(p_1 q_0)|/\log m} \qquad as\ \varepsilon \to 0^+.$$

(b) If $p_0 = 0$, $p_1 > 0$ and $q_0 = 0$, then

$$\log \mathbb{P}(\widetilde{\mathcal{W}} \le \varepsilon) \sim -\frac{K|\log p_1|}{2(\log m)^2}|\log \varepsilon|^2 \qquad as\ \varepsilon \to 0^+,$$

with K being defined as in Theorem 2(b).

(c) If $p_0 = 0$ and $p_1 = 0$, then

$$\log \mathbb{P}(\widetilde{\mathcal{W}} \le \varepsilon) \asymp -\varepsilon^{-\beta/(1-\beta)} \qquad as\ \varepsilon \to 0^+,$$

with β being defined as in Theorem 1(b).

(d) If $p_0 > 0$, then

$$\mathbb{P}(\widetilde{\mathcal{W}} \le \varepsilon) \asymp \varepsilon^{|\log h(\rho)|/\log m}, \qquad as\ \varepsilon \to 0^+.$$

Taking $q_0 = 1$ in Theorem 3, then it degenerates into Theorem 1.

In Section 6, we give some examples, where the offspring and immigration are both shifted geometric distributed (for definition see Section 6). Then we can calculate the Laplace transform of $\mathcal{W}$ directly. As enumerated in Proposition 1, we obtain more precise results than Theorem 2. For those results, we use the conclusions of [W08] Section 0.9 and Proposition 3.1 of Barlow and Perkins [BP88]. Especially in [BP88], they showed us a case when the Laplace transform of W indeed has an oscillation. Based on this, we can also construct an example when the Laplace transform of $\mathcal{W}$ is oscillating. As we can see in Section 2, the small value probabilities of $\mathcal{W}$ can be drawn from its Laplace transform directly. Additionally, we show that there are also oscillations in other case with immigration.

2 Basic relations and estimates

The following two Tauberian type theorems are useful tools in our investigation. The asymptotic equivalent type can be found in Bingham, Goldie and Teugels [BGT87], Theorem 1.7.1 on p.37 and Theorem 4.12.9 on p.254. The one-sided equivalent type is given in Li [L11+].

Lemma 1 *Assume V is a positive random variable and $\alpha > 0$ is a constant.*

(i) For constant $C > 0$,

$$\mathbb{E}\, e^{-\lambda V} \sim C\lambda^{-\alpha} \qquad as\ \lambda \to \infty,$$

if and only if

$$\mathbb{P}(V \le t) \sim \frac{C}{\Gamma(1+\alpha)} t^{\alpha} \qquad as\ t \to 0^+.$$

(ii) The one-sided relation

$$\mathbb{P}(V \le t) \le C_1 t^{\alpha} \quad \text{for some constant } C_1 > 0 \text{ and all } t > 0$$

is equivalent to

$$\mathbb{E}\, e^{-\lambda V} \le C_2 \lambda^{-\alpha} \quad \text{for some constant } C_2 > 0 \text{ and all } \lambda > 0.$$

Lemma 2 *Assume V is a positive random variable and $\alpha > 0, \theta \in \mathbb{R}$, or $\alpha = 0, \theta > 0$ are constants.*

(i) For constant $C > 0$,

$$\log \mathbb{P}(V \le t) \sim -Ct^{-\alpha} |\log t|^{\theta} \qquad \text{as } t \to 0^+,$$

if and only if

$$\log \mathbb{E}\, e^{-\lambda V} \sim -(1+\alpha)^{1-\theta/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} C^{1/(1+\alpha)} \lambda^{\alpha/(1+\alpha)} (\log \lambda)^{\theta/(1+\alpha)} \qquad \text{as } \lambda \to \infty.$$

When $\alpha = 0$ and $\theta > 0$, the product of constants above-mentioned is just C.

(ii) The one-sided relation

$$\log \mathbb{P}(V \le t) \le -C_1 t^{-\alpha} |\log t|^{\theta} \quad \text{for some constant } C_1 > 0 \text{ and all } t > 0$$

is equivalent to

$$\log \mathbb{E}\, e^{-\lambda V} \le -C_2 \lambda^{\alpha/(1+\alpha)} (\log \lambda)^{\theta/(1+\alpha)} \quad \text{for some constant } C_2 > 0 \text{ and all } \lambda > 0.$$

Now we consider the supercritical branching process with immigration $(\mathcal{Z}_n, n \ge 0)$ and $\mathcal{Z}_0 = Y_0$. For fixed integer $r \ge 0$ and $l \ge 1$, let $\xi_r(1), \cdots, \xi_r(\mathcal{Z}_r)$ be the individuals in generation r, and $\eta_l(j), j = 1, \cdots, Y_l$ be the individuals of immigration in generation l. Then for any $r \ge 0$ and $n \ge r+1$,

$$\mathcal{Z}_n = \sum_{i=1}^{\mathcal{Z}_r} Z_{n-r}(\xi_r(i)) + \sum_{l=r+1}^{n} \sum_{j=1}^{Y_l} Z_{n-l}(\eta_l(j)).$$

Here $(Z_n(v), n \ge 0)$ is a supercritical G-W branching process initiated with one individual v and $W(v)$ is the limit of positive martingale $m^{-n} Z_n(v)$.

Divided by m^n on both sides, then let $n \to \infty$, we get

$$\mathcal{W} = m^{-r} \sum_{i=1}^{\mathcal{Z}_r} W(\xi_r(i)) + \sum_{l=r+1}^{\infty} m^{-l} \sum_{j=1}^{Y_l} W(\eta_l(j)). \tag{2.1}$$

For simplicity, we rewrite (2.1) as

$$\mathcal{W} = m^{-r} \sum_{i=1}^{\mathcal{Z}_r} W_i + \sum_{l=r+1}^{\infty} m^{-l} \sum_{j=1}^{Y_l} W_l^j. \tag{2.2}$$

Here all the $W_i, W_l^j, i = 1, \cdots, \mathcal{Z}_r, l = r+1, \cdots, n, j = 1, \cdots, Y_l$ are independent and identically distributed as W. The relation (2.2) is the fundamental distribution identity of $\mathcal{W}$ and it is used repeatedly in our approach.

3 Proof of Theorem 2: Lower bound

We start with a simple but crucial probability estimates that is a consequence of the condition $\mathbb{E}\,\log^+ Y < \infty$ in (1.2).

Lemma 3 *Under condition $\mathbb{E}\,\log^+ Y < \infty$ in (1.2), for any fixed constant $\delta > 0$, there exists integer l such that*

$$\mathbb{P}(\max_{i\geq l+1} Y_i e^{-\delta i} \leq 1) \quad \geq \quad e^{-1}. \tag{3.1}$$

Proof. For any given $\delta > 0$, we have

$$\begin{aligned} \sum_{i=1}^{\infty} \mathbb{P}(\log^+ Y \geq \delta i) &= \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} \mathbb{P}(k \leq \delta^{-1}\log^+ Y < k+1) \\ &= \sum_{k=1}^{\infty} k\mathbb{E}\,\mathbb{I}(k \leq \delta^{-1}\log^+ Y < k+1) \\ &\leq \delta^{-1}\mathbb{E}\,\log^+ Y < \infty. \end{aligned}$$

Let Y_i and Y be our independent and identically distributed immigration random variables. Then for any large integer l such that

$$\sum_{i=l+1}^{\infty} \mathbb{P}(\log^+ Y \geq \delta i) \leq 1/2 \tag{3.2}$$

we have

$$\begin{aligned} \mathbb{P}(\max_{i\geq l+1} Y_i e^{-\delta i} \leq 1) &\geq \prod_{i=l+1}^{\infty} \big(1 - \mathbb{P}(\log^+ Y \geq \delta i)\big) \\ &\geq \exp\left(-2\sum_{i=l+1}^{\infty} \mathbb{P}(\log^+ Y \geq \delta i)\right) \\ &\geq e^{-1} \end{aligned}$$

where we used the fact that $(1-x)e^{2x}$ is increasing for $0 \leq x < 1/2$. This finishes our proof of the lemma.

Proof of (a) and (b). For any $\varepsilon > 0$, let $k = k_\varepsilon$ be the integer such that

$$m^{-k} \leq \varepsilon < m^{-k+1}, \tag{3.3}$$

which is equivalent to say that

$$k - 1 < |\log\varepsilon|/\log m \leq k, \quad \text{or} \quad k = \lceil |\log\varepsilon|/\log m \rceil. \tag{3.4}$$

Using the fundamental distribution identity (2.2) with $r = 0$, we have for a fixed integer l to be chosen later,

$$\begin{aligned} \mathbb{P}(\mathcal{W} \le \varepsilon) &= \mathbb{P}\Big(\sum_{i=0}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \varepsilon\Big) \\ &\ge \mathbb{P}\Big(\sum_{i=0}^{k+l} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{\varepsilon}{2}\Big) \cdot \mathbb{P}\Big(\sum_{i=k+l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{\varepsilon}{2}\Big). \end{aligned} \tag{3.5}$$

For the second term in (3.5), we have by using $\varepsilon \ge m^{-k}$ in (3.3),

$$\begin{aligned} \mathbb{P}\Big(\sum_{i=k+l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{\varepsilon}{2}\Big) &\ge \mathbb{P}\Big(\sum_{i=k+l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{m^{-k}}{2}\Big) \\ &= \mathbb{P}\Big(\sum_{i=l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{1}{2}\Big). \end{aligned} \tag{3.6}$$

Note that the last equality follows from the independence and identical distribution of all W_i^j's and Y_i's.

Next we have by controlling the size of Y_i, $i \ge l+1$, given in the Lemma 3,

$$\begin{aligned} \mathbb{P}\Big(\sum_{i=l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{1}{2}\Big) &\ge \mathbb{P}\Big(\sum_{i=l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{1}{2}, \max_{i \ge l+1} Y_i e^{-\delta i} \le 1\Big) \\ &\ge \mathbb{P}\Big(\sum_{i=l+1}^{\infty} m^{-i} \sum_{j=1}^{e^{\delta i}} W_i^j \le \frac{1}{2}\Big) \cdot \mathbb{P}\Big(\max_{i \ge l+1} Y_i e^{-\delta i} \le 1\Big). \end{aligned} \tag{3.7}$$

Using Chebyshev's inequality for the first part of (3.7), we get

$$\mathbb{P}\Big(\sum_{i=l+1}^{\infty} m^{-i} \sum_{j=1}^{e^{\delta i}} W_i^j \le \frac{1}{2}\Big) \ge 1 - 2\mathbb{E} \sum_{i=l+1}^{\infty} m^{-i} \sum_{j=1}^{e^{\delta i}} W_i^j \tag{3.8}$$

$$= 1 - \frac{2e^{\delta(l+1)}}{(m - e^{\delta})m^l}. \tag{3.9}$$

We can now choose δ such that $e^{\delta} < m$, and then find large enough integer l so that

$$\frac{2e^{\delta(l+1)}}{(m - e^{\delta})m^l} < \frac{1}{2}. \tag{3.10}$$

Combining (3.6)–(3.10) and Lemma 3, we obtain that

$$\mathbb{P}\Big(\sum_{i=k+l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{\varepsilon}{2}\Big) \ge \mathbb{P}\Big(\sum_{i=l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{1}{2}\Big) \ge \frac{1}{2e}. \tag{3.11}$$

Now back to the first part of (3.5), we have to handle it under conditions (a) and (b) separately. In the case (a) with $q_0 > 0$, we have the simple estimate

$$\mathbb{P}\Big(\sum_{i=0}^{k+l} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{\varepsilon}{2}\Big) \ge \mathbb{P}\big(Y_0 = \cdots = Y_{k+l} = 0\big) = q_0^{k+l+1}. \tag{3.12}$$

Using $k - 1 < |\log \varepsilon| / \log m$ in (3.4), it's easy to deduce that

$$q_0^k \;\; \ge \;\; q_0 \cdot q_0^{|\log \varepsilon|/\log m} = q_0 \varepsilon^{|\log q_0|/\log m}. \tag{3.13}$$

Combining (3.5) and (3.11)–(3.13) we have shown the lower bound in Theorem 2(a).

For the case (b) with $q_0 = 0$, we have, recalling the definition of $K = \inf\{n : q_n > 0\}$,

$$\begin{aligned} \mathbb{P}\Big(\sum_{i=0}^{k+l} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{\varepsilon}{2}\Big) \;\; &\ge \;\; \mathbb{P}\Big(\sum_{i=0}^{k+l} m^{-i} \sum_{j=1}^{Y_i} W_i^j \le \frac{\varepsilon}{2}, Y_0 = \cdots = Y_{k+l} = K\Big) \\ &= \;\; \mathbb{P}\Big(\sum_{i=0}^{k+l} m^{-i} \sum_{j=1}^{K} W_i^j \le \frac{\varepsilon}{2}\Big) \cdot q_K^{k+l+1}. \end{aligned} \tag{3.14}$$

The above probability of sums can be bounded termwise, and thus

$$\begin{aligned} \mathbb{P}\Big(\sum_{i=0}^{k+l} m^{-i} \sum_{j=1}^{K} W_i^j \le \frac{\varepsilon}{2}\Big) \;\; &\ge \;\; \mathbb{P}\Big(\max_{0\le i \le k+l} \max_{1 \le j \le K} m^{-i} W_i^j \le \frac{\varepsilon/2}{K(k+l+1)}\Big) \\ &= \;\; \prod_{i=0}^{k+l} \mathbb{P}^K\Big(m^{-i} W \le \frac{\varepsilon/2}{K(k+l+1)}\Big) \\ &\ge \;\; \prod_{i=0}^{k+l} \mathbb{P}^K\Big(W \le \frac{m^{i-k}/2}{K(k+l+1)}\Big). \end{aligned} \tag{3.15}$$

where we used the independence of all W_i^j's in the last equality and $\varepsilon \ge m^{-k}$ from (3.3) in the last inequality.

From Theorem 1(a) there exists a constant $c > 0$ such that, for $i = 0, 1, \cdots, k+l$,

$$\mathbb{P}\Big(W \le \frac{m^{i-k}/2}{K(k+l+1)}\Big) \;\; \ge \;\; c\Big(\frac{m^{i-k}/2}{K(k+l+1)}\Big)^{|\log p_1|/\log m}. \tag{3.16}$$

Combining (3.5), (3.11) and (3.14)–(3.16) together and take care of summation over $0 \le i \le k+l$ after taking logarithm, we have

$$\begin{aligned} \log \mathbb{P}(\mathcal{W} \le \varepsilon) \;\; &\ge \;\; -\frac{K|\log p_1|}{2} k^2 - O(k \log k) \\ &\ge \;\; -\frac{K|\log p_1|}{2(\log m)^2} |\log \varepsilon|^2 - O(\log \varepsilon^{-1} \log\log \varepsilon^{-1}) \end{aligned}$$

where we used $k < 1 + |\log \varepsilon|/\log m$ from (3.4).

Proof of (c). First observe that, in this case with $\gamma = \inf\{n : p_n > 0\} \geq 2$, $K = \inf\{n : q_n > 0\} \geq 1$, the smallest number of particles in generation n ($n \geq 1$) is

$$b(n) := K(\gamma^n + \gamma^{n-1} + \cdots + 1) = K(\gamma^{n+1} - 1)/(\gamma - 1). \tag{3.17}$$

It is also easy to see that the chance this occurs is

$$\mathbb{P}(\mathcal{Z}_n = b(n)) = p_\gamma^{b(n-1)+\cdots+b(0)} q_K^{n+1} := p_\gamma^{B(n)} q_K^{n+1}, \tag{3.18}$$

where

$$B(0) = 0, \qquad B(n) = b(n-1) + \cdots + b(0) = \frac{K\left(\gamma^{n+1} - (n+1)\gamma + n\right)}{(\gamma - 1)^2}, \; n \geq 1. \tag{3.19}$$

Given $\varepsilon > 0$, we can choose $k = k_\varepsilon$ such that

$$\frac{\gamma^k}{m^k} \leq \varepsilon < \frac{\gamma^{k-1}}{m^{k-1}}, \tag{3.20}$$

which is equivalent to say that

$$k - 1 < |\log \varepsilon|/\log(m/\gamma) \leq k, \quad \text{or} \quad k = \lceil |\log \varepsilon|/\log(m/\gamma) \rceil. \tag{3.21}$$

Next let l be an integer that will be determined later. Using the fundamental distribution identity (2.2) with $r = k + l$ and (3.18), we have that

$$\begin{aligned}
\mathbb{P}(\mathcal{W} \leq \varepsilon) &\geq \mathbb{P}\big(\mathcal{W} \leq (\gamma/m)^k | \mathcal{Z}_{k+l} = b(k+l)\big)\mathbb{P}(\mathcal{Z}_{k+l} = b(k+l)) \\
&= \mathbb{P}\Bigg(m^{-k-l} \sum_{i=1}^{b(k+l)} W_i + \sum_{i=k+l+1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \leq (\gamma/m)^k\Bigg) p_\gamma^{B(k+l)} q_K^{k+l+1} \\
&\geq \mathbb{P}\Bigg(\sum_{i=1}^{b(k+l)} W_i \leq \frac{m^l \gamma^k}{2}\Bigg)\mathbb{P}\Bigg(\sum_{i=1}^{\infty} m^{-i} \sum_{j=1}^{Y_i} W_i^j \leq \frac{m^l \gamma^k}{2}\Bigg) p_\gamma^{B(k+l)} q_K^{k+l+1}. \qquad (3.22)
\end{aligned}$$

For the first term in (3.22) we have by Chebyshev's inequality and choosing suitable l

$$\begin{aligned}
\mathbb{P}\Bigg(\sum_{i=1}^{b(k+l)} W_i \leq m^l \gamma^k/2\Bigg) &\geq 1 - \frac{2}{m^l \gamma^k} \mathbb{E} \sum_{i=1}^{b(k+l)} W_i = 1 - \frac{2b(k+l)}{m^l \gamma^k} \\
&\geq 1 - \frac{2K\gamma}{\gamma - 1}(\gamma/m)^l \geq 1/2 \qquad (3.23)
\end{aligned}$$

where we used the fact that $\mathbb{E}\, W = 1$ and $b(n) \leq K(\gamma - 1)^{-1}\gamma^{n+1}$ from (3.17).

For the second part of (3.22), we have

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{\infty} m^{-i}\sum_{j=1}^{Y_i} W_i^j \le \frac{m^l\gamma^k}{2}\Big) &= \mathbb{P}\Big(\sum_{i=l+1}^{\infty} m^{-i}\sum_{j=1}^{Y_i} W_i^j \le \frac{\gamma^k}{2}\Big)\\ &\ge \mathbb{P}\Big(\sum_{i=l+1}^{\infty} m^{-i}\sum_{j=1}^{Y_i} W_i^j \le \frac{1}{2}\Big) \ge e^{-1}/2 \qquad (3.24)\end{aligned}$$

where the last inequality follows from (3.11).

Combing (3.22)–(3.24), we get

$$\mathbb{P}(\mathcal{W} \le \varepsilon) \ge p_\gamma^{B(k+l)} q_K^{k+l+1} e^{-1}/4. \qquad (3.25)$$

Recalling the definition of $B(k+l)$ in (3.19) and $k-1 < |\log\varepsilon|/\log(m/\gamma)$ in (3.21), we see

$$B(k+l) \le \frac{K}{(\gamma-1)^2}\gamma^{k+l+1} \le C\gamma^{|\log\varepsilon|/\log(m/\gamma)} = C\varepsilon^{-\beta/(1-\beta)},$$

where β is defined as in Theorem 1(b) and C is a fixed constant. Therefore from (3.25) we obtain

$$\log\mathbb{P}(\mathcal{W} \le \varepsilon) \ge -C\varepsilon^{-\beta/(1-\beta)}$$

for some constant $C > 0$.

4 Proof of Theorem 2: Upper bound

As we can see from the arguments in section 3, only the finite terms in (2.2) are contributing to the small value probabilities of $\mathcal{W}$. Hence we take only $r = 0$ in (2.2), choose suitable cut off k, and focus on properties of $\sum_{l=0}^{k} m^{-l}\sum_{j=1}^{Y_l} W_l^j$.

Proof of (a). Let $k = k_\varepsilon$ be the integer defined as in (3.3). Using the fundamental distribution identity (2.2) with $r = 0$ and exponential Chebyshev's inequality, we have for any $\lambda > 0$,

$$\begin{aligned}\mathbb{P}(\mathcal{W} \le \varepsilon) &\le \mathbb{P}\Big(\sum_{i=0}^{k} m^{-i}\sum_{j=1}^{Y_i} W_i^j \le \varepsilon\Big)\\ &\le e^{\lambda\varepsilon}\cdot\mathbb{E}\exp\Big(-\lambda\sum_{i=0}^{k} m^{-i}\sum_{j=1}^{Y_i} W_i^j\Big). \qquad (4.1)\end{aligned}$$

Notice that all the $(W_i^j, i = 0, \cdots, k, j = 1, \cdots, Y_i)$ are independent, we have

$$\mathbb{E}\exp\Big(-\lambda\sum_{i=0}^{k} m^{-i}\sum_{j=1}^{Y_i} W_i^j\Big) = \prod_{i=0}^{k}\mathbb{E}\exp\Big(-\lambda m^{-i}\sum_{j=1}^{Y_i} W_i^j\Big). \qquad (4.2)$$

Conditioning on $Y_i = 0$ or $Y_i \geq 1$, we have

$$\mathbb{E}\,\exp\Big(-\lambda m^{-i}\sum_{j=1}^{Y_i} W_i^j\Big) \leq q_0 + (1-q_0)\mathbb{E}\,\exp\big(-\lambda m^{-i}W_i^1\big) \leq q_0(1+\delta_i), \tag{4.3}$$

where

$$\delta_i = q_0^{-1}\mathbb{E}\,\exp\big(-\lambda m^{-i}W_i^1\big) = q_0^{-1}\mathbb{E}\,\exp\big(-\lambda m^{-i}W\big),\ i = 0,\cdots,k. \tag{4.4}$$

Substituting (4.3) into (4.1) and letting $\lambda = \varepsilon^{-1}$, we obtain

$$\mathbb{P}(\mathcal{W} \leq \varepsilon) \leq e q_0^{k+1}\prod_{i=0}^{k}(1+\delta_i).$$

Since $k \geq |\log\varepsilon|/\log m$ in (3.4), we have

$$q_0^k \leq \varepsilon^{|\log q_0|/\log m}.$$

So we finish the proof by showing

$$\sum_{i=0}^{k}\log(1+\delta_i) \leq \sum_{i=0}^{k}\delta_i \leq M \tag{4.5}$$

where $M > 0$ is a constant independent of ε (noticing that the k depends on ε). To show (4.5), we have to argue separately according to $p_1 > 0$ or $p_1 = 0$.

When $p_1 > 0$, by Theorem 1(a) and Lemma 1(ii), there exists a constant $C > 0$ satisfying that

$$\mathbb{E}\,e^{-\lambda W} \leq C\lambda^{-|\log p_1|/\log m}, \qquad \lambda > 0. \tag{4.6}$$

Combining (4.4) with $\lambda = \varepsilon^{-1}$, and then using (4.6), we have

$$\begin{aligned}\sum_{i=0}^{k}\delta_i &= q_0^{-1}\sum_{i=0}^{k}\mathbb{E}\,\exp(-\varepsilon^{-1}m^{-i}W)\\ &\leq q_0^{-1}C\sum_{i=0}^{k}(\varepsilon m^i)^{|\log p_1|/\log m}\\ &= Cq_0^{-1}\varepsilon^{|\log p_1|/\log m}\sum_{i=0}^{k}p_1^{-i}\\ &\leq C'\varepsilon^{|\log p_1|/\log m}\cdot p_1^{-k} \leq C'p_1^{-1}\end{aligned}$$

where C' is a constant and the last inequality follows from (3.4).

When $p_1 = 0$, using Theorem 1(b) and Lemma 2(ii) with $\alpha = \beta/(1-\beta)$ and $\theta = 0$, we have for some constant $b > 0$,

$$\log \mathbb{E}\, e^{-\lambda W} \le -b\lambda^\beta, \qquad \lambda > 0, \tag{4.7}$$

from which it's similar to show that (4.5) holds. Indeed, setting $\lambda = \varepsilon^{-1}$ in (4.4), and then using (4.7) and (3.3), we obtain

$$\begin{aligned}
\sum_{i=0}^{k} \delta_i &= q_0^{-1} \sum_{i=0}^{k} \mathbb{E}\, \exp(-\varepsilon^{-1} m^{-i} W) \\
&\le q_0^{-1} \sum_{i=0}^{k} \exp(-b\varepsilon^{-\beta} m^{-i\beta}) \\
&\le q_0^{-1} \sum_{i=0}^{k} \exp(-b m^{(k-i-1)\beta}) \\
&\le q_0^{-1} \sum_{i=0}^{\infty} \exp(-b m^{(i-1)\beta}) < \infty.
\end{aligned}$$

Proof of (b). Let k be defined as in (3.3). Using (4.1) and the fact that $Y_i \ge K$ for any $i \ge 0$,

$$\mathbb{P}(\mathcal{W} \le \varepsilon) \le e^{\lambda\varepsilon} \prod_{i=0}^{k} \prod_{j=1}^{K} \mathbb{E}\, \exp\left(-\lambda m^{-i} W_i^j\right), \qquad \lambda > 0. \tag{4.8}$$

In the case (b) with $p_1 > 0$, substituting (4.6) into (4.8) with $\lambda = \varepsilon^{-1}$, we obtain

$$\mathbb{P}(\mathcal{W} \le \varepsilon) \;\le\; e \prod_{i=0}^{k} \prod_{j=1}^{K} C(\varepsilon m^i)^{|\log p_1|/\log m}.$$

Taking the logarithm we obtain

$$\begin{aligned}
\log \mathbb{P}(\mathcal{W} \le \varepsilon) &\le 1 + K(k+1)(\log C - |\log \varepsilon| \cdot |\log p_1|/\log m) + k(k+1) \cdot K|\log p_1|/2 \\
&= -k \cdot |\log \varepsilon| \cdot K|\log p_1|/\log m + (k-1)^2 \cdot K|\log p_1|/2 + O(k) \\
&\le -\frac{K|\log p_1|}{2(\log m)^2} |\log \varepsilon|^2 + O(|\log \varepsilon|)
\end{aligned}$$

where we used in the last inequality the fact that $k - 1 < |\log \varepsilon|/\log m \le k$ in (3.4).

Proof of (c). It is clear that

$$\mathbb{P}(\mathcal{W} \le \varepsilon) \le \mathbb{P}(W \le \varepsilon), \tag{4.9}$$

and therefore we finish the proof of (c) by using estimate in Theorem 1(b).

5 Proof of Theorem 2(d)

If $p_0 > 0$, then $f(s) = s$ has a unique solution $\rho \in (0, 1)$ and $\mathbb{P}(W = 0) = \rho$. By means of the Harris-Sevastyanov transformation

$$\widetilde{f}(s) := \frac{f((1-\rho)s + \rho) - \rho}{(1-\rho)},$$

$\widetilde{f}$ defines a new branching mechanism with $\widetilde{p}_0 = 0$ and $\widetilde{f}'(1) = m$. Denote $(\widetilde{Z}_n, n \geq 0)$ as the corresponding branching process and $\widetilde{W}$ as the limit of $m^{-n}\widetilde{Z}_n$. By Theorem 3.2 in [H48],

$$W =^d W_0 \cdot \widetilde{W}, \tag{5.1}$$

where W_0 is independent of $\widetilde{W}$ and takes the values 0 and $1/(1-\rho)$ with probabilities ρ and $1-\rho$ respectively. Notice that the small value probability of $\widetilde{W}$ has the asymptotic behavior described in Theorem 1(a) with $\widetilde{p}_1 = \widetilde{f}'(0) = f'(\rho) > 0$, and $\tau = |\log \widetilde{p}_1|/\log m$, that is

$$\mathbb{P}(\widetilde{W} \leq \varepsilon) \asymp \varepsilon^\tau. \tag{5.2}$$

Now we start to prove Theorem 2(d).

Proof of Lower Bound For any $\varepsilon > 0$, let $k = k_\varepsilon$ be the integer defined in (3.3), then using (3.5) and (3.11), we only need to estimate the first part of (3.5).

$$\begin{aligned}
\mathbb{P}\Big(\sum_{i=0}^{k+l} m^{-i} \sum_{j=1}^{Y_i} W_i^j \leq \frac{\varepsilon}{2}\Big) \quad &\geq \quad \prod_{i=0}^{k+l} \mathbb{P}\big(\sum_{j=1}^{Y_i} W_i^j = 0\big) \\
&= \quad \prod_{i=0}^{k+l} \Big(\sum_{k=0}^{\infty} q_k \rho^k\Big) = h(\rho)^{k+l+1},
\end{aligned} \tag{5.3}$$

where h is the generating function of immigration Y. Using $k - 1 < |\log \varepsilon|/\log m$ in (3.4), it's easy to deduce that

$$h(\rho)^k \quad \geq \quad h(\rho) \cdot h(\rho)^{|\log \varepsilon|/\log m} = h(\rho) \cdot \varepsilon^{|\log h(\rho)|/\log m}. \tag{5.4}$$

Combining (3.5), (3.11), (5.3) and (5.4) we obtain the lower bound of (d).

Proof of Upper Bound Using (5.1), we have, for any $\lambda > 0$,

$$\mathbb{E}\, e^{-\lambda W} = \rho + \mathbb{E}\, e^{-\lambda W} I_{\{W>0\}} := \rho + \delta(\lambda). \tag{5.5}$$

Using (4.1), (4.2) and the independence of all the $(W_i^j, i = 0, \cdots, k, j = 1, \cdots, Y_i)$, we have

$$\begin{aligned}\mathbb{P}(\mathcal{W} \le \varepsilon) &\le e^{\lambda\varepsilon}\mathbb{E}\exp\Big(-\lambda\sum_{i=0}^{k} m^{-i}\sum_{j=1}^{Y_i} W_i^j\Big)\\ &= e^{\lambda\varepsilon}\prod_{i=0}^{k} h\left(\rho + \delta(\lambda m^{-i})\right)\\ &= (h(\rho))^{k+1}\exp\left(\lambda\varepsilon + \sum_{i=0}^{k}\log\left(h\left(\rho+\delta(\lambda m^{-i})\right)/h(\rho)\right)\right), \end{aligned} \tag{5.6}$$

where $\lambda = \lambda_k$ depends on $k (= k_\varepsilon)$ and will be given later. Since $k \ge |\log\varepsilon|/\log m$ in (3.4), we have

$$(h(\rho))^k \le \varepsilon^{|\log h(\rho)|/\log m}. \tag{5.7}$$

So we finish the proof by showing that there is a constant $M > 0$, which does not depend on ε, such that

$$\begin{aligned}&\lambda\varepsilon + \sum_{i=0}^{k}\log\left(h\left(\rho+\delta(\lambda m^{-i})\right)/h(\rho)\right)\\ \le\ & \lambda m^{-k+1} + h(\rho)^{-1}\sum_{i=0}^{k}\left(h\left(\rho+\delta(\lambda m^{-i})\right) - h(\rho)\right) \le M. \end{aligned} \tag{5.8}$$

Since $\delta(\lambda m^{-x})$ is increasing in x, we have

$$\sum_{i=0}^{k}\left(h\left(\rho+\delta(\lambda m^{-i})\right) - h(\rho)\right) \le \int_0^{k+1}\left(h\left(\rho+\delta(\lambda m^{-x})\right) - h(\rho)\right)\mathrm{d}x. \tag{5.9}$$

Note that $\delta(\lambda) = (1-\rho)\mathbb{E}\, e^{-(\lambda/(1-\rho))\widetilde{W}}$. By (5.2) and Lemma 1(ii), there exists $C > 0$ such that

$$\delta(\lambda m^{-x}) \le C(\lambda m^{-x})^{-\tau}, \tag{5.10}$$

with $\tau = |\log f'(\rho)|/\log m$, thus

$$\begin{aligned}&\sum_{i=0}^{k}\left(h\left(\rho+\delta(\lambda m^{-i})\right) - h(\rho)\right)\\ \le\ & \int_0^{k+1}\left(h\left(\rho + C(\lambda m^{-x})^{-\tau}\right) - h(\rho)\right)\mathrm{d}x\\ =\ & 1/(\tau\log m)\cdot\int_{\lambda^{-\tau}}^{\lambda^{-\tau}m^{(k+1)\tau}} 1/y\cdot\left(h\left(\rho+Cy\right) - h(\rho)\right)\mathrm{d}y\\ \le\ & 1/(\tau\log m)\cdot\int_{0}^{\lambda^{-\tau}m^{(k+1)\tau}} 1/y\cdot\left(h\left(\rho+Cy\right) - h(\rho)\right)\mathrm{d}y. \end{aligned} \tag{5.11}$$

Since $\rho < 1$, we may choose $\delta_0 > 0$ such that $\rho + \delta_0 < 1$. We choose $\lambda = (C/\delta_0)^{1/\tau} m^{(k+1)}$. Then

$$\lambda m^{-k+1} = m^2 (C/\delta_0)^{1/\tau} := M_1, \tag{5.12}$$

and

$$\rho + Cy \le \rho + C\lambda^{-\tau} m^{(k+1)\tau} = \rho + \delta_0 < 1, \quad \forall\, y \le \lambda^{-\tau} m^{(k+1)\tau}.$$

Then we continue (5.11) to get

$$\begin{aligned} & \sum_{i=0}^{k} \left(h\left(\rho + \delta(\lambda m^{-i})\right) - h(\rho)\right) \\ \le\ & 1/(\tau \log m) \cdot \int_0^{\delta_0/C} 1/y \cdot \left(h\left(\rho + Cy\right) - h(\rho)\right) \mathrm{d}y \\ :=\ & M_2 < \infty, \end{aligned} \tag{5.13}$$

where we used the fact that

$$\lim_{y \to 0} 1/y \cdot \left(h\left(\rho + Cy\right) - h(\rho)\right) = Ch'(\rho) < \infty.$$

From (5.8), (5.12) and (5.13) we obtain that (5.8) holds with $M = M_1 + M_2$. We finished the proof of Theorem 2(d).

6 Examples

We start with some well known facts on the generating function approach to branching processes. Suppose that f and h are generating functions of the offspring number X and immigration number Y respectively, i.e.

$$f(s) = \mathbb{E}\, s^X \quad \text{and} \quad h(s) = \mathbb{E}\, s^Y, \qquad 0 < s < 1.$$

For fixed integer $k \ge 1$, let $f_k = f(f_{k-1})$ be the k-fold composition of f with $f_0(s) = s$. Set

$$\phi(\lambda) = \mathbb{E}\, e^{-\lambda W} \quad \text{and} \quad \Phi(\lambda) = \mathbb{E}\, e^{-\lambda \mathcal{W}}, \qquad 0 < s < 1. \tag{6.1}$$

Using (2.2) with $r = 0$, we get

$$\Phi(\lambda) = \prod_{k=0}^{\infty} h\left(\phi\left(m^{-k}\lambda\right)\right). \tag{6.2}$$

For the remaining part of this section, we assume for $0 < p, q \le 1$, $X =^d a + Geo(p)$ and $Y =^d b + Geo(q)$ where a and b are integers, and $X =^d a + Geo(p)$ is called shift geometric random variable with $\mathbb{P}(X = a + k) = p(1-p)^{k-1}$, $k \geq 1$. Clearly, the mean of $X =^d a + Geo(p)$ is $m = \mathbb{E}\, X = a + 1/p$ and the generating function is

$$f(s) = \frac{s^{a+1}}{s + (1-s)/p}, \quad a \geq -1,\ 0 < p < 1. \tag{6.3}$$

The generating function for $Y =^d b + Geo(q)$ is

$$h(s) = s^{b+1}/(s + (1-s)/q), \quad b \geq -1,\ 0 < q \leq 1. \tag{6.4}$$

When $a = -1$, it is known that (see [W08] Section 0.9) the generating function of the offspring is

$$f(s) = p/(1 - (1-p)s) \qquad 0 < p < 1/2,$$

The mean of the offspring and extinction probability are

$$m = (1-p)/p > 1, \qquad \rho = p/(1-p) = 1/m \tag{6.5}$$

respectively, and the Laplace transform of limit W of the martingale $m^{-n} Z_n$ is given by

$$\mathbb{E}\, e^{-\lambda W} = (p\lambda + 1 - 2p)/((1-p)\lambda + 1 - 2p). \tag{6.6}$$

When $a = 0$ and $X =^d Geo(p)$ is standard geometric, it is well known that (see Example 3 of [FW07], p.237 for instance)

$$f_k(s) = \frac{s}{s + (1-s)/p^k}, \qquad k = 1, 2, \cdots, \tag{6.7}$$

thus we have for any $\lambda > 0$

$$\mathbb{E}\, e^{-\lambda W} = \lim_{n\to\infty} f_n\big(\exp(-\lambda m^{-n})\big) = 1/(\lambda + 1). \tag{6.8}$$

When $a \geq 1$, then $\phi(\lambda)$ in (6.1) satisfies

$$\phi(m\lambda) = f(\phi(\lambda)) = \frac{\phi(\lambda)^{a+1}}{\phi(\lambda) + (1 - \phi(\lambda))/p}. \tag{6.9}$$

For $\lambda > 0$, let

$$g(\lambda) = -\lambda^{-\beta} \log \phi(\lambda) \tag{6.10}$$

with $\beta = \log(a+1)/\log m$ as defined in Theorem 1(b). Combining (6.9) and (6.10), we obtain

$$g(m\lambda) = g(\lambda) + \lambda^{-\beta} |\log p|/(a+1) + \lambda^{-\beta} \log\big(1 - (1-p)\phi(\lambda)\big)/(a+1).$$

The special case of $a = 1$ and $p = 1/4$ in the above equality is studied in detail by Barlow and Perkins [BP88], in connection with Brownian motion on the sierpinski gasket. We follow their approach to obtain (6.14) below. Using $0 < \phi(\lambda) < 1$, it is clear

$$g(\lambda) \le g(m\lambda) \le g(\lambda) + \lambda^{-\beta} |\log p|/(a+1),$$

which implies that $g(m^n\lambda)$ is increasing with respect to n and

$$g(\lambda) \le g(m^n\lambda) \le g(\lambda) + \lambda^{-\beta}\big(|\log p|/(a+1)\big)\big((1-m^{-\beta n})/(1-m^{-\beta})\big).$$

Thus there exists a positive function, denoted by $G(\lambda)$, such that

$$\lim_{n\to\infty} g(m^n\lambda) = G(\lambda), \qquad \lambda > 0, \tag{6.11}$$

and it is easy to check that

$$G(m^k\lambda) = G(\lambda), \qquad \text{for any integer } k. \tag{6.12}$$

When $a = 1$ and $p = 1/4$, from p.573-p.574 in [BP88], it is known that there is an oscillation for $G(x)$ near $0+$, that is

$$\liminf_{x\to 0+} G(x) < \limsup_{x\to 0+} G(x). \tag{6.13}$$

By the similar argument in the proof of Proposition 3.1(b) of [BP88], p.572, we can prove easily that there exist some strictly positive constants C_1, C_2 and C_3 such that

$$\exp(-C_1\lambda^\beta) \le \mathbb{E}\, e^{-\lambda W} \le C_2 \exp(-C_3\lambda^\beta), \qquad \lambda > 0. \tag{6.14}$$

This can be refined significantly, even with immigration, and here is our precise estimates.

Proposition 1 *Assume that the offspring* $X =^d a + Geo(p)$*, and the immigration* $Y =^d b + Geo(q)$ *with integers* $a \ge -1$*,* $b \ge -1$*, and* $m = \mathbb{E}\, X = a + 1/p$*.*

(i) When $a = 0$ *and* $b \ge 0$*, we have*

$$\log \mathbb{E}\, e^{-\lambda \mathcal{W}} \sim -\frac{b+1}{2\log m} (\log\lambda)^2 \qquad \text{as } \lambda \to \infty, \tag{6.15}$$

Equivalently, according to Lemma 2(i),

$$\log \mathbb{P}(\mathcal{W} \le \varepsilon) \sim -\frac{b+1}{2\log m} |\log\varepsilon|^2 \qquad \text{as } \varepsilon \to 0. \tag{6.16}$$

(ii) When $a = 0$ *and* $b = -1$*, we have*

$$\liminf_{\lambda\to\infty} \lambda^{|\log q|/\log m} \mathbb{E}\, e^{-\lambda\mathcal{W}} = C \inf_{0<x<1} e^{C(x)} \tag{6.17}$$

$$\limsup_{\lambda\to\infty} \lambda^{|\log q|/\log m} \mathbb{E}\, e^{-\lambda\mathcal{W}} = C \sup_{0<x<1} e^{C(x)} \tag{6.18}$$

where $C(x)$ is defined in (6.41) *and*

$$C = \exp\big(- |\log q|^2/(2\log m) - |\log q|/2\big). \tag{6.19}$$

Especially, when $m = 4$, and $q = 1/8$, we have

$$0 < \liminf_{\lambda\to\infty} \lambda^{|\log q|/\log m}\mathbb{E}\, e^{-\lambda\mathcal{W}} < \limsup_{\lambda\to\infty} \lambda^{|\log q|/\log m}\mathbb{E}\, e^{-\lambda\mathcal{W}} < \infty. \tag{6.20}$$

Equivalently, according to Lemma 1(i),

$$0 < \liminf_{\varepsilon\to 0} \varepsilon^{-|\log q|/\log m}\mathbb{P}(\mathcal{W} \le \varepsilon) < \limsup_{\varepsilon\to 0} \varepsilon^{-|\log q|/\log m}\mathbb{P}(\mathcal{W} \le \varepsilon) < \infty. \tag{6.21}$$

(iii) When $a \ge 1$ and $b \ge 0$, for $G(\lambda)$ defined as in (6.11), *we have*

$$\liminf_{\lambda\to\infty} \lambda^{-\beta}\log\mathbb{E}\, e^{-\lambda\mathcal{W}} = -\sup_{0<x<1} G(x)\cdot(b+1)(a+1)/a, \tag{6.22}$$

$$\limsup_{\lambda\to\infty} \lambda^{-\beta}\log\mathbb{E}\, e^{-\lambda\mathcal{W}} = -\inf_{0<x<1} G(x)\cdot(b+1)(a+1)/a \tag{6.23}$$

where $\beta = \log(a+1)/\log m$.

Especially when $a = 1$ and $p = 1/4$, from (6.13)*, then using Lemma 2(i), we have*

$$-\infty < \liminf_{\varepsilon\to 0} \varepsilon^{-\beta/(1-\beta)}\log\mathbb{P}(\mathcal{W} \le \varepsilon) < \limsup_{\varepsilon\to 0} \varepsilon^{-\beta/(1-\beta)}\log\mathbb{P}(\mathcal{W} \le \varepsilon) < 0. \tag{6.24}$$

(iv) When $a \ge 1$ and $b = -1$, we have

$$\liminf_{\lambda\to\infty} \lambda^{|\log q|/\log m}\mathbb{E}\, e^{-\lambda\mathcal{W}} = C\inf_{0<x<1} e^{-C(x,\phi)} \tag{6.25}$$

$$\limsup_{\lambda\to\infty} \lambda^{|\log q|/\log m}\mathbb{E}\, e^{-\lambda\mathcal{W}} = C\sup_{0<x<1} e^{-C(x,\phi)} \tag{6.26}$$

where $C(x,\phi)$ is defined in (6.62) *and $C = \exp(-C_1 - C_2 - |\log q|/2)$ with C_1 and C_2 being defined in* (6.58) *and* (6.60) *respectively.*

(v) When $a = -1$, we have

$$\liminf_{\lambda\to\infty} \lambda^{|\log h(\rho)|/\log m}\mathbb{E}\, e^{-\lambda\mathcal{W}} = C\inf_{0<x<1}\exp(-C_{-1}(x)), \tag{6.27}$$

$$\limsup_{\lambda\to\infty} \lambda^{|\log h(\rho)|/\log m}\mathbb{E}\, e^{-\lambda\mathcal{W}} = C\sup_{0<x<1}\exp(-C_{-1}(x)), \tag{6.28}$$

where $C = \exp(-C_1 - C_2 - |\log h(\rho)|/2)$ with C_1 and C_2 being defined in (6.66) *and* (6.67) *respectively, and $C_{-1}(\lambda)$ being defined in* (6.68).

Remark: In special cases, we can check that $C(x)$, $C(x,\phi)$ and $C_{-1}(x)$ are not constant but have near constancy phenomena using Matlab Calculus method. But for the general case, we can't prove this result. We only checked that $C(x)$ is oscillating near 0 for instance.

From Proposition 1(ii), the oscillation occurs with immigration even there is no oscillation without immigration. This is quite unexpected and demonstrates the significant effects of the immigration. Of course, Proposition 1 also shows that the asymptotic $\asymp$ in our main Theorem 2 is best possible in the sense that it can not be improved into the more precise asymptotic $\sim$.

Now we turn to consider $\widetilde{\mathcal{W}}$ defined in (1.6). Together with (6.6), (6.8), (6.11) and Proposition 1, it is easy to obtain the following results.

Corollary 1 *The same assumption as in Proposition 1.*

(i) When $a = 0$ and $b \geq 0$, we have

$$\log \mathbb{E}\, e^{-\lambda \widetilde{\mathcal{W}}} \sim -\frac{b+1}{2\log m}(\log \lambda)^2 \qquad \text{as } \lambda \to \infty.$$

Equivalently, according to Lemma 2(i),

$$\log \mathbb{P}(\widetilde{\mathcal{W}} \leq \varepsilon) \sim -\frac{b+1}{2\log m}|\log \varepsilon|^2 \qquad \text{as } \varepsilon \to 0.$$

(ii) When $a = 0$ and $b = -1$, we have

$$\begin{aligned} \liminf_{\lambda\to\infty} \lambda^{1+|\log q|/\log m}\mathbb{E}\, e^{-\lambda \widetilde{\mathcal{W}}} &= \inf_{0<x<1} C/q \cdot \exp(C(x)), \\ \limsup_{\lambda\to\infty} \lambda^{1+|\log q|/\log m}\mathbb{E}\, e^{-\lambda \widetilde{\mathcal{W}}} &= \sup_{0<x<1} C/q \cdot \exp(C(x)), \end{aligned}$$

where $C(\lambda)$ and C are defined in Proposition 1(ii).

(iii) When $a \geq 1$ and $b \geq 0$, for $G(\lambda)$ defined as in (6.11), we have

$$\begin{aligned} \liminf_{\lambda\to\infty} \lambda^{-\beta} \log \mathbb{E}\, e^{-\lambda \widetilde{\mathcal{W}}} &= -\sup_{0<x<1} G(x) \cdot (a+b+1)/a, \\ \limsup_{\lambda\to\infty} \lambda^{-\beta} \log \mathbb{E}\, e^{-\lambda \widetilde{\mathcal{W}}} &= -\inf_{0<x<1} G(x) \cdot (a+b+1)/a, \end{aligned}$$

where $\beta = \log(a+1)/\log m$.

(iv) When $a \geq 1$ and $b = -1$, we have

$$\liminf_{\lambda\to\infty} \lambda^{-\beta} \log \mathbb{E}\, e^{-\lambda \widetilde{\mathcal{W}}} = -\sup_{0<x<1} G(x),$$

$$\limsup_{\lambda\to\infty} \lambda^{-\beta} \log \mathbb{E}\, e^{-\lambda \widetilde{\mathcal{W}}} = -\inf_{0<x<1} G(x),$$

where $\beta = \log(a+1)/\log m$.

(v) When $a = -1$, we have

$$\liminf_{\lambda\to\infty} \lambda^{|\log h(\rho)|/\log m}\mathbb{E}\, e^{-\lambda \mathcal{W}} = C\rho/h(\rho) \cdot \inf_{0<x<1} \exp(-C_{-1}(x)),$$

$$\limsup_{\lambda\to\infty} \lambda^{|\log h(\rho)|/\log m}\mathbb{E}\, e^{-\lambda \mathcal{W}} = C\rho/h(\rho) \cdot \sup_{0<x<1} \exp(-C_{-1}(x)),$$

where C, $C(x)$ and ρ are defined in Proposition 1(v).

Remark: We should mention that when $q = 1$ and $b = -1$, we have $\widetilde{\mathcal{W}} = W$. Since (6.8) and (6.11), we have (ii) and (iv) holds obviously.

Proof of Proposition 1(i) Recall $m = 1/p$ in this case. Combining (6.2), (6.4) and (6.8), we obtain

$$I(\lambda) := -\log \mathbb{E}\, e^{-\lambda \mathcal{W}} = \sum_{k=0}^{\infty} \log\left((\lambda m^{-k}+1)^b (1+\lambda m^{-k}/q)\right). \tag{6.29}$$

Using the integral representation

$$\log(1+xm^{-k}) = \int_0^x (t+m^k)^{-1} \mathrm{d}t,$$

we can rewrite (6.29) via Fubini theorem to obtain

$$I(\lambda) = b\int_0^{\lambda} Q(t)\mathrm{d}t + \int_0^{\lambda q^{-1}} Q(t)\mathrm{d}t \tag{6.30}$$

where

$$Q(t) := \sum_{k=0}^{\infty} \frac{1}{t+m^k} = \int_0^{\infty} \frac{\mathrm{d}x}{t+m^x} + \sum_{k=0}^{\infty} \int_k^{k+1} \frac{m^x - m^k}{(t+m^k)(t+m^x)}\mathrm{d}x. \tag{6.31}$$

We can explicitly evaluate the integral term in (6.31),

$$\int_0^{\infty} \frac{\mathrm{d}x}{t+m^x} = (\log m)^{-1} \int_1^{\infty} \frac{\mathrm{d}y}{y(t+y)} = (\log m)^{-1} \frac{\log(t+1)}{t} \tag{6.32}$$

and also estimate the sum term

$$\begin{aligned} 0 \le \sum_{k=0}^{\infty} \int_k^{k+1} \frac{m^x - m^k}{(t+m^k)(t+m^x)}\mathrm{d}x &\le \int_0^{\infty} \frac{m^x}{(t+m^{x-1})(t+m^x)}\mathrm{d}x \\ &= \frac{C}{t}\log\left((1+mt)/(1+t)\right) \le mC(1\wedge t^{-1}) \end{aligned} \tag{6.33}$$

for some constant $C > 0$, by using the substitution $m^x = ty$ for the last integral. Combining (6.31), (6.32) and (6.33), we have as $\lambda \to \infty$

$$\begin{aligned} \int_0^{\lambda} Q(t)\mathrm{d}t &\sim \int_1^{\lambda} Q(t)\mathrm{d}t \\ &\sim (\log m)^{-1} \int_1^{\lambda} \frac{\log(t+1)}{t}\mathrm{d}t \\ &\sim (2\log m)^{-1} \cdot (\log \lambda)^2. \end{aligned}$$

Thus from (6.30), we have as $\lambda \to \infty$

$$I(\lambda) \sim (b+1)(2\log m)^{-1} \cdot (\log \lambda)^2$$

which finishes the proof of part (i), in the case $b \geq 0$.

Proof of Proposition 1(ii) Letting $b = -1$ in (6.29), then using

$$\log\Big(\big(1+\lambda m^{-k}/q\big)/\big(\lambda m^{-k}+1\big)\Big) = \int_\lambda^{\lambda/q} (t+m^k)^{-1}\mathrm{d}t,$$

if we define $f(x) = 1/(t+m^x)$, then we can write (6.29) as

$$I(\lambda) = \int_\lambda^{\lambda/q} \sum_{k=0}^{\infty} f(k)\mathrm{d}t \tag{6.34}$$

We first use Euler-Maclaurin formula to obtain

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\mathrm{d}x + (f(n)+f(0))/2 + R_n(t) \tag{6.35}$$

where the remainder

$$R_n(t) = \int_0^n f'(x)P_1(x)\mathrm{d}x, \tag{6.36}$$

and

$$P_1(x) = x - \lfloor x \rfloor - 1/2, \quad \text{and } \lfloor x \rfloor \text{ is the largest integer less than } x. \tag{6.37}$$

Substituting the definition of $f(x)$ and its derivative $f'(x)$ into (6.35), and letting $n \to \infty$, we have

$$\sum_{k=0}^{\infty} f(k) = \frac{\log(t+1)}{t\log m} + \frac{1}{2(t+1)} + R_\infty(t) \tag{6.38}$$

with

$$R_\infty(t) = -\log m \cdot \int_0^\infty P_1(x)(t+m^x)^{-2} m^x \mathrm{d}x. \tag{6.39}$$

Thus we obtain from (6.34) and (6.38), as $\lambda \to \infty$

$$\begin{aligned} I(\lambda) &= \int_\lambda^{\lambda/q} \frac{\log(t+1)}{t\log m}\mathrm{d}t + \int_\lambda^{\lambda/q} \frac{1}{2(t+1)}\mathrm{d}t + \int_\lambda^{\lambda/q} R_\infty(t)\mathrm{d}t \\ &= |\log q|/\log m \cdot \log\lambda + |\log q|^2/(2\log m) + |\log q|/2 - R(\lambda) + O(\lambda^{-1}) \end{aligned} \tag{6.40}$$

where

$$\begin{aligned}
R(\lambda) &= \log m \cdot \int_{\lambda}^{\lambda/q} \int_{0}^{\infty} P_1(x)(t+m^x)^{-2} m^x \mathrm{d}x\mathrm{d}t \\
&= \int_{1}^{1/q} \int_{1/\lambda}^{\infty} P_1\left(\log(\lambda y)/\log m\right)(s+y)^{-2}\mathrm{d}y\mathrm{d}s \\
&= \int_{1}^{1/q} \int_{0}^{\infty} P_1\left(\log(\lambda y)/\log m\right)(s+y)^{-2}\mathrm{d}y\mathrm{d}s + O(\lambda^{-1}) \\
&:= C(\lambda) + O(\lambda^{-1}), \qquad (6.41)
\end{aligned}$$

here we use variable substitutions $m^x = \lambda y$ and $t = \lambda s$. Clearly, the function $C(\lambda)$ is bounded since $|P_1(x)| \leq 1/2$, and in fact,

$$|C(\lambda)| \leq \int_{1}^{1/q} \int_{0}^{\infty} (s+y)^{-2}\mathrm{d}y\mathrm{d}s = |\log q| < \infty.$$

It is easy to check by the periodicity of $P_1(x)$ that for any integer $k \geq 1$ and positive real number x,

$$C(m^{-k}x) = C(x). \qquad (6.42)$$

Combining (6.40) and (6.42), we obtain (6.17) and (6.18). It is necessary here to show that the $C(x)$ is oscillating near $0+$, that is to say

$$\liminf_{x\to 0+} C(x) < \limsup_{x \to 0+} C(x). \qquad (6.43)$$

In fact, when $m = 4$ and $q = 1/8$, using Matlab calculus we found the value of $C(x)$ satisfies

$$\begin{aligned}
\sup_{0<x<1} C(x) &\geq C(25 \cdot 10^{-10}) = -1.568763331475900 \\
\inf_{0<x<1} C(x) &\leq C(11 \cdot 10^{-10}) = -1.609054498122461 \qquad (6.44)
\end{aligned}$$

when x takes values in $11 \cdot 10^{-10},\ 12 \cdot 10^{-10},\ 13 \cdot 10^{-10}, \cdots, 31 \cdot 10^{-10}$. Therefore we obtain (6.20).

Proof of Proposition 1(iii) From now on, let $a \geq 1$ be an integer. Recall the definition of Φ given by (6.2), and h given by (6.4). For any $\lambda > 0$ and integer $n > 0$, we have

$$\begin{aligned}
& (m^n\lambda)^{-\beta} \log \Phi(m^n\lambda) \\
=\ & (b+1)(m^n\lambda)^{-\beta} \sum_{k=0}^{\infty} \log \phi(m^{n-k}\lambda) - (m^n\lambda)^{-\beta} \sum_{k=0}^{\infty} \log\left(1/q - (1/q-1)\phi(m^{n-k}\lambda)\right) \\
:=\ & (b+1) II(n,\lambda) - III(n,\lambda), \qquad (6.45)
\end{aligned}$$

where $\beta = \log(a+1)/\log m$. Now we handle $II(n,\lambda)$ and $III(n,\lambda)$ separately. First observe that

$$\begin{aligned} II(n,\lambda) &= \sum_{k=0}^{n}(m^k)^{-\beta}(m^{n-k}\lambda)^{-\beta}\log\phi(m^{n-k}\lambda) + (m^n\lambda)^{-\beta}\sum_{j=1}^{\infty}\log\phi(m^{-j}\lambda) \\ &:= II^{(1)}(n,\lambda) + II^{(2)}(n,\lambda). \end{aligned} \tag{6.46}$$

By definition the of $G(\lambda)$, we know that for $\forall\, \varepsilon > 0$, there exists some integer n_0 such that

$$\left|(m^k\lambda)^{-\beta}\log\phi(m^k\lambda) + G(\lambda)\right| \le \varepsilon, \qquad \forall\, k \ge n_0. \tag{6.47}$$

For fixed integer n large enough, using (6.47), we have

$$\begin{aligned} &\left|II^{(1)}(n,\lambda) + \sum_{k=0}^{n} m^{-k\beta}G(\lambda)\right| \\ \le\;& \varepsilon\sum_{k=0}^{n-n_0} m^{-k\beta} + \sum_{k=n-n_0+1}^{n} m^{-k\beta}\Big(\left|(m^{n-k}\lambda)^{-\beta}\log\phi(m^{n-k}\lambda)\right| + G(\lambda)\Big). \end{aligned} \tag{6.48}$$

By (6.14), the second term of (6.48) is bounded by

$$\sum_{k=n-n_0+1}^{n} m^{-k\beta}\big(C_1 + G(\lambda)\big) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{6.49}$$

Combining (6.48) and (6.49), then taking $n \to \infty$ and $\varepsilon \to 0+$, we get

$$II^{(1)}(n,\lambda) \longrightarrow \frac{-m^\beta}{m^\beta - 1}G(\lambda) = -(a+1)/a \cdot G(\lambda) \qquad \text{as } n \to \infty,$$

for the last equality we used the definition of $\beta = \log(a+1)/\log m$. For $II^{(2)}(n,\lambda)$, by Jensen's inequality, we have

$$0 > \sum_{j=1}^{\infty}\log\phi(m^{-j}\lambda) \ge \sum_{j=1}^{\infty} -m^{-j}\lambda \ge -\lambda/(m-1),$$

and then $II^{(2)}(n,\lambda) \to 0$ as $n \to \infty$. Therefore we obtain

$$II(n,\lambda) \longrightarrow \frac{-m^\beta}{m^\beta - 1}G(\lambda) = -(a+1)/a \cdot G(\lambda) \qquad \text{as } n \to \infty. \tag{6.50}$$

Now, we only need to prove that $III(n,\lambda) \to 0$. Notice that

$$\begin{aligned} III(n,\lambda) &= (m^n\lambda)^{-\beta}\sum_{k=-n}^{\infty}\log\big(1 + (1/q - 1)(1 - \phi(m^{-k}\lambda)\big) \\ &\le (m^n\lambda)^{-\beta}\sum_{k=-n}^{\infty}(1/q - 1)\big(1 - \phi(m^{-k}\lambda)\big) \\ &\le (m^n\lambda)^{-\beta}(1/q - 1)\sum_{k=-n}^{\infty}\big(1 - \exp(-m^{-k}\lambda)\big), \end{aligned}$$

here we used $\log(1+x) \le x$ on $[0,\infty)$ and Jensen's inequality. Furthermore, since

$$\begin{aligned}
\sum_{k=-n}^{\infty} \big(1-\exp(-m^{-k}\lambda)\big) &\le \int_{-n-1}^{\infty} 1-\exp(-m^{-x}\lambda)\mathrm{d}x \\
&= \int_0^{\lambda m^{n+1}} \big(1-e^{-y}\big)/\big(y\log m\big)\mathrm{d}y \\
&\le \int_0^1 1/\log m \mathrm{d}y + \int_1^{\lambda m^{n+1}} 1/\big(y\log m\big)\mathrm{d}y \\
&= \big(1+\log\lambda\big)/\log m + (n+1),
\end{aligned}$$

we have

$$III(n,\lambda) \to 0 \qquad \text{as } n\to\infty. \tag{6.51}$$

Then combining (6.45), (6.50) and (6.51), we obtain

$$-(m^n\lambda)^{-\beta}\log\Phi(m^n\lambda) \longrightarrow (b+1)(a+1)/a\cdot G(\lambda) \qquad \text{as } n\to\infty. \tag{6.52}$$

From which we obtain (6.22) and (6.23).

Proof of Proposition 1(iv) By the definition of Φ given by (6.2) and h being defined in (6.4) with $b=-1$, we have for any real number $\lambda>0$,

$$-\log\Phi(\lambda) = \sum_{k=0}^{\infty}\log\big(1/q-(1/q-1)\phi(\lambda m^{-k})\big) := \sum_{k=0}^{\infty} f(k). \tag{6.53}$$

Using Euler-Maclaurin formula, we obtain

$$\sum_{k=0}^{\infty} f(k) = \int_0^{\infty} f(x)\mathrm{d}x + f(0)/2 + R(\lambda,\phi) \tag{6.54}$$

with

$$R(\lambda,\phi) = (1/q-1)\lambda\log m\cdot\int_0^{\infty}\frac{P_1(x)\phi'(\lambda m^{-x})m^{-x}}{1/q-(1/q-1)\phi(\lambda m^{-x})}\mathrm{d}x. \tag{6.55}$$

It is obvious from (6.14) that

$$f(0) = \log\left(1/q-(1/q-1)\phi(\lambda)\right) \sim |\log q| + o(\lambda^{-1}) \qquad \text{as } \lambda\to\infty. \tag{6.56}$$

For the first term in (6.54), by variable substitution $y=\lambda m^{-x}$, we have

$$\int_0^{\infty} f(x)\mathrm{d}x = \int_0^{\lambda}\log\big(1/q-(1/q-1)\phi(y)\big)/(y\log m)\mathrm{d}y. \tag{6.57}$$

It is obvious from the definition of ϕ that

$$\int_0^1 \log\big(1/q-(1/q-1)\phi(y)\big)/(y\log m)\mathrm{d}y := C_1 \tag{6.58}$$

is a finite positive constant. Thus we only need to estimate

$$\begin{aligned}
&\int_1^\lambda \log\big(1/q-(1/q-1)\phi(y)\big)/(y\log m)\mathrm{d}y \\
= \quad & |\log q|/\log m\cdot\log\lambda+\int_1^\lambda \log\big(1-(1-q)\phi(y)\big)/(y\log m)\mathrm{d}y. \qquad (6.59)
\end{aligned}$$

Using $\log(1-x)\sim -x$ as $x\to 0$ and (6.14), we obtain that

$$\begin{aligned}
&\int_\lambda^\infty \log\big(1-(1-q)\phi(y)\big)/(y\log m)\mathrm{d}y \\
\sim \quad & -(1-q)/\log m\cdot\int_\lambda^\infty \phi(y)/y\mathrm{d}y = -o(\lambda^{-1}) \qquad \text{as } \lambda\to\infty.
\end{aligned}$$

Thus we obtain that as $\lambda\to\infty$,

$$\begin{aligned}
&\int_1^\lambda \log\big(1-(1-q)\phi(y)\big)/(y\log m)\mathrm{d}y \\
= \quad & \int_1^\infty \log\big(1-(1-q)\phi(y)\big)/(y\log m)\mathrm{d}y + o(\lambda^{-1}) := C_2+o(\lambda^{-1}). \qquad (6.60)
\end{aligned}$$

Combining (6.57)–(6.60), we obtain that

$$\int_0^\infty f(x)\mathrm{d}x \sim |\log q|/\log m\cdot\log\lambda + C_1 + C_2 \qquad \text{as } \lambda\to\infty. \tag{6.61}$$

Now we only need to estimate $R(\lambda,\phi)$ in (6.55). Using variable substitution $\lambda m^{-x}=y$, we obtain

$$\begin{aligned}
R(\lambda,\phi) \quad &= \quad (1-q)\cdot\int_0^\lambda \frac{P_1\big(\log(\lambda/y)/\log m\big)\phi'(y)}{1-(1-q)\phi(y)}\mathrm{d}y \\
&= \quad (1-q)\cdot\int_0^\infty \frac{P_1\big(\log(\lambda/y)/\log m\big)\phi'(y)}{1-(1-q)\phi(y)}\mathrm{d}y + o(\lambda^{-1}) \\
&:= \quad C(\lambda,\phi)+o(\lambda^{-1}) \qquad \text{as } \lambda\to\infty. \qquad (6.62)
\end{aligned}$$

Here we use (6.14) to obtain the second equality. It is easy to check that $C(\lambda,\phi)$ is bounded by $|\log q|$. Then using the periodicity of P_1, we have for any integer $k>0$,

$$C(m^k\lambda,\phi)=C(\lambda,\phi). \tag{6.63}$$

Combining (6.53)–(6.56) and (6.61), (6.62), we obtain

$$\mathbb{E}\, e^{-\lambda\mathcal{W}} = \lambda^{-|\log q|/\log m} \exp\left(-C_1 - C_2 - |\log q|/2\right) \exp(-C(\lambda,\phi)),$$

then together with (6.63), we obtain (6.25) and (6.26).

Proof of Proposition 1(v) Now we assume $a=-1$. Using (6.2) and (6.6), we have when $h(s)$ satisfies (6.4),

$$\mathbb{E}\, e^{-\lambda\mathcal{W}} \;=\; \prod_{k=0}^{\infty} \frac{q(p\lambda m^{-k}+1-2p)^{b+1}((1-p)\lambda m^{-k}+1-2p)^{-b}}{(1-2p+pq)\lambda m^{-k}+(1-2p)q}.$$

Define

$$I(\lambda) = -\log \mathbb{E}\, \exp(-\lambda\mathcal{W}).$$

When $b=-1$, we have

$$I(\lambda) = \sum_{k=0}^{\infty} \log \frac{(1-2p+qp)\lambda m^{-k}+(1-2p)q}{(1-p)q\lambda m^{-k}+(1-2p)q}.$$

By Euler-Maclaurin formula, if we define

$$f(x) = \log \frac{(1-2p+qp)\lambda m^{-x}+(1-2p)q}{(1-p)q\lambda m^{-x}+(1-2p)q},$$

then we have

$$I(\lambda) = \int_0^\infty f(x)\mathrm{d}x + \frac{f(0)}{2} + \int_0^\infty P_1(x) f'(x)\mathrm{d}x \tag{6.64}$$

with

$$\begin{aligned}
\int_0^\infty f(x)\mathrm{d}x &= 1/\log m \cdot \int_0^\lambda 1/y \cdot \log \frac{(1-2p+qp)y+(1-2p)q}{(1-p)qy+(1-2p)q}\mathrm{d}y \\
&= C_1 + 1/\log m \cdot \int_1^\lambda 1/y \cdot \left(\log \frac{1-2p+qp}{(1-p)q} + \log \frac{y+q(1-2p)/(1-2p+qp)}{y+(1-2p)/(1-p)} \right) \mathrm{d}y \\
&= C_1 + |\log h(\rho)|/\log m \cdot \log\lambda + C_2 - O(\lambda^{-1}) \quad \text{as } \lambda\to\infty
\end{aligned} \tag{6.65}$$

where we used the definition of ρ in (6.5) and h in (6.4) with $b=-1$ for the last equality, and

$$C_1 = 1/\log m \cdot \int_0^1 1/y \cdot \log \frac{(1-2p+qp)y+(1-2p)q}{(1-p)qy+(1-2p)q}\mathrm{d}y \tag{6.66}$$

and

$$C_2 = 1/\log m \cdot \int_1^\infty 1/y \cdot \log \frac{y+q(1-2p)/(1-2p+qp)}{y+(1-2p)/(1-p)}\mathrm{d}y. \tag{6.67}$$

By the definition of $f'(x)$ and $P_1(x)$ in (6.37), we obtain that the third term of (6.64) is

$$\begin{aligned}
& -\int_0^\infty \frac{(1-2p)^2(1-q)\cdot\lambda m^{-x}\cdot\log m}{(1-p)\cdot\lambda m^{-x}+(1-2p)}\frac{P_1(x)}{(1-2p+qp)\lambda m^{-x}+(1-2p)q}\mathrm{d}x \\
= & -\int_0^\lambda \frac{(1-2p)^2(1-q)}{(1-p)\cdot y+(1-2p)}\frac{P_1\left(\log(\lambda/y)/\log m\right)}{(1-2p+qp)\cdot y+(1-2p)q}\mathrm{d}y \\
= & -\int_0^\infty \frac{(1-2p)^2(1-q)}{(1-p)\cdot y+(1-2p)}\frac{P_1\left(\log(\lambda/y)/\log m\right)}{(1-2p+qp)\cdot y+(1-2p)q}\mathrm{d}y + O(\lambda^{-1}) \\
:= & \; C_{-1}(\lambda)+O(\lambda^{-1}), \qquad \text{as } \lambda\to\infty
\end{aligned} \tag{6.68}$$

and it is easy to check that $C_{-1}(\lambda)$ is a bounded periodic function. Notice that

$$f(0) = |\log h(\rho)|, \tag{6.69}$$

then together with (6.64), (6.65), (6.68) and (6.69), we obtain

$$I(\lambda) = |\log h(\rho)|/\log m\cdot\log\lambda + C_1 + C_2 + |\log h(\rho)|/2 + C_{-1}(\lambda) \quad \text{as } \lambda\to\infty, \tag{6.70}$$

which implies (6.27) and (6.28).

When $b\geq 0$, we can similarly obtain (6.27) and (6.28), and the details are omitted.

References

[AN72] Athreya, K. B. and Ney, P. E. (1972). *Branching Processes.* New York: Springer-Verlag.

[B88] Bingham, N. H. (1988). On the limit of a supercritical branching process. *J. Appl. Probab.* **25A** 215-228.

[BB91] Biggins, J. D. and Bingham, N. H. (1991). Near-constancy of phenomena in branching processes. *Math. Proc. Camb. Phil. Soc.* **110** 545-558.

[BB93] Biggins, J. D. and Bingham, N. H. (1993). Large deviation in the supercritical branching process. *Adv. Appl. Prob.* **25** 757-772.

[BN93] Biggins, J. D. and Nadarajah, S. (1993). Near-constancy of the Harris function in the simple branching process. *Comm. Statist. Stochastic Models* **n.3** 435-444.

[BGT87] Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1987). *Regular Variation.* Cambridge: Cambridge Univ. Press.

[BP88] Barlow, M. T. and Perkins, E. A. (1988). Brownian Motion on the sierpinski gasket. *Probab. Theory Related Fields* **79** 543-623.

[D71a] Dubuc, S. (1971a). La Densité de la loi-limite *d'*un processus en cascade expansif. *Z. Wahrsch. Verw. Gebiete* **19** 281-290.

[D71b] Dubuc, S. (1971b). Problèmes relatifs *à l'*itération de fonctions suggér*é*s par les processus en cascade. *Ann. Inst. Fourier(Grenoble)* **21** 171-251.

[D82] Dubuc, S. (1982). *É*tude th*é*orique et num*é*rique de la fonction de Karlin-McGregor. *J. Analyse Math.* **42** 15-37.

[FW07] Fleischmann, K. and Wachtel, V. (2007). Lower deviation probabilities for supercritical Galton-Watson processes. *Ann.Inst. H. Poincaré Probab. Statist.* **43** 233-255.

[FW09] Fleischmann, K. and Wachtel, V. (2009). On the left tail asymtotics for the limit law of supercritical Galton-Watson processes in the *Böttcher* case. *Ann. Inst. H. Poincaré Probab. Statist.* **45** 201-225.

[G87] Goldstein, S. (1987). Random walks and diffusions on fractals. *Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984C1985), 121C129, IMA Vol. Math. Appl., 8, Springer, New York.*

[H95] Hambly, B. M. (1995). On constant tail behavior for the limiting random variable in a supercritical branching process. *J. Appl. Probab. Trust* **32** 267-273.

[H48] Harris, T. E. (1948). Branching processes. *Ann. Math. Statist.* **19** 474-494.

[KM68a] Karlin, S. and McGregor, J. (1968a). Embeddability of discrete time simple branching processes into continuous time branching processes. *Trans. Amer. Math. Soc.* **132** 115-136.

[KM68b] Karlin, S. and McGregor, J. (1968b). Embedding iterates of analytic functions with two fixed points into continuous groups. *Trans. Amer. Math. Soc.* **132** 137-145.

[KS66] Kesten, H. and Stigum, B. (1966). A limit theorem for multidimensional Galton-Watson processes. *Ann. Math. Statist.* **37** 1211-1223.

[K87] Kusuoka, S. (1987). A diffusion process on a fractal. *Probabilistic methods in mathematical physics (Katata/Kyoto, 1985)* 251-274.

[L03] Li W. V. (2003). The first exit time of Brownian motion from unbounded convex domain, *Annals of Probability*, Vol. 31, 1078-1096.

[L11+] Li W. V. (2011+). Small Value Probabilities: Techniques and Applications. Lecture notes. `http://www.math.udel.edu/~wli/svp.html`

[Lif10] Lifshits. M. A. (2010). Bibliography compilation on small deviation probabilities. `http://www.proba.jussieu.fr/pageperso/smalldev/biblio.html`, 2010.

[LS01] Li W.V. and Shao Q.-M.(2001). Gaussian processes: inequalities, small ball probabilities and applications. *Stochastic processes: theory and methods, Handbook of Statist.* **19** 533–597.

[LS04] Li W.V. and Shao Q.-M. (2004). Lower tail probabilities for Gaussian processes. *Ann. Probab.* **32** 216-242.

[MO08] Mörters, P. and Ortgiese, M. (2008). Small value probabilities via the branching tree heuristic. *Bernoulli* **14(1)** 277-299.

[S70] Seneta, E. (1970). On the supercririical branching process with immigration. *Math. Biosci.* **7** 9-14.

[W08] Williams, D. (2008). *Probability with Martingales.* Cambridge: Cambridge Mathematical textbooks.