Probability of all real zeros for random polynomial with the exponential ensemble

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December 27, 2011

Abstract

Abstract: The probability that a random polynomial of degree n with i.i.d exponentially distributed coefficients has all real zeros is

$$\mathbb{P}(\text{All zeros are real}) = \mathbb{E} \prod_{1 \le j < k \le n} |U_j - U_k| = \left(\prod_{k=1}^{n-1} \binom{2k+1}{k} \right)^{-1}$$

where U_i are i.i.d uniform on the interval $[0, 1]$. The second identity is a form of Selberg integral with simplification. Our evaluation of the probability starts with a formula of Zaporozhets (2004) which is based on an integral geometry representation developed by Edelman and Kostlan (1995) and tools from differential geometry.

*Supported in part by NSF grant DMS-1106938, NSFC-A011003, CAS-2008DP173182.

1 Introduction

There is a long history on the study of zeros of a random polynomial whose coefficients are independent, identically distributed, non-degenerate random variables. Various questions regarding the distribution of the number of real zeros were studied in late 30's by Littlewood and Offord and then significant progress was made by Kac in the 40's, where the main result is that the expected number of zeros of a degree n polynomial with independent standard Gaussian coefficients is found explicitly as a single integral and asymptotically equivalent to $2\log n$ for large n. There has since been a huge amount of work on various aspects of the distribution of the zeros of random polynomials and systems of random polynomials for a wide range of models with coefficients that are possibly dependent and have distributions other than Gaussian. As pointed out by Evans [E06], it is impossible to survey this work adequately, but some of the more commonly cited early papers, reviews of the literature and the level of sophistication achieved in terms of results and methodology are provided in [EK95], [DPSZ02], [E06] and [LW09]. Here we only briefly overview results of probability estimates of rare events associated with the number of real zeros.

Consider the random polynomial of degree n

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{1.1}$$

with independent and identically distributed coefficients, and denote by N_n the number of real zeros of $P_n(x)$. In a series of papers [LO38],[LO39], [LO43], Littlewood and Offord proved that if the a_i are all uniform on $(-1,1)$, or all normal $N(0,1)$, or all Rademacher/uniform on $\{-1,1\}$, i.e. $\mathbb{P}(a_i = \pm 1) = 1/2$, then

$$\mathbb{P}\left(N_n > 25(\log n)^2\right) \le \frac{12\log n}{n} \quad \text{and} \quad \mathbb{P}\left(N_n < \frac{c\log n}{(\log\log n)^2}\right) \le \frac{C}{\log n}$$

for some positive constants c and C. Years later, see e.g. [IM71] and [M74], it was shown that $\mathbb{E}\,N_n \sim (2/\pi)\log n$ and $\mathrm{Var}(N_n) \sim (4/\pi)(1-2/\pi)\log n$. They together imply, as pointed out in [DPSZ02], $\limsup_{n\to\infty} \log n \cdot \mathbb{P}(N_n \le 1) \le \pi - 2$. The bound has the same form as the above mentioned work of Littlewood and Offord, but the constant has been made explicit. The small value probability of $\mathbb{P}(N_n \le (1-\delta)\mathbb{E}\,N_n)$ for fixed $0 < \delta < 1$ is still an open question. However, a significant breakthrough was made in [DPSZ02] for nondegenerate i.i.d coefficients having zero mean and finite moments of all orders. They showed that the probability of exactly k real zeros is $n^{-b+o(1)}$ as $n \to \infty$ through integers of the same parity as the fixed integer $k \ge 0$. In particular, the probability that a random polynomial of large even degree n has no real zeros is $n^{-b+o(1)}$. The finite, positive constant b depends neither on k nor upon the specific law of the coefficients, and is characterized via the centered, stationary Gaussian process of correlation function $\mathrm{sech}(t/2)$. Their numerical simulations for degree $n \le 2^{10}$ suggest $b \approx 0.76 \pm 0.03$. In recent updated work of Aurzada, Li and Shao [ALS11+], a new representation for b in terms of the limiting one-sided exit exponent of iterated integrated Brownian motion is given and in particular the best theoretical bounds for the so called positivity exponent b is $0.5 < b < 1$.

In this note, the main goal is to consider the probability of the largest possible value of N_n or smallest possible number of complex zeros. Our main result is the following explicit evaluation.

Theorem: Assume the coefficients of a random polynomial of degree n are independent and exponentially distributed with rate one. Then the probability of all real zeros or no complex roots is given by

$$p_n^e = \mathbb{P}(\text{All zeros are real}) = \mathbb{E} \prod_{1\le j<k\le n} |U_j - U_k| = \left(\prod_{k=1}^{n-1} \binom{2k+1}{k}\right)^{-1} \tag{1.2}$$

where U_i are i.i.d uniform on the interval $[0,1]$. In particular, we have

$$p_1^e = 1, \quad p_2^e = \frac{1}{3}, \quad p_3^e = \frac{1}{30}, \quad p_4^e = \frac{1}{1050} \quad p_5^e = \frac{1}{132300}, \quad p_6^e = \frac{1}{488980800}$$

and asymptotically as $n \to \infty$,

$$\begin{aligned} \log \mathbb{P}(N_n = n) &= -\log 2 \cdot n^2 + \frac{1}{2} n \log n + \frac{1}{2}(\log \pi - 1)\, n + \frac{3}{8} \log n \\ &\quad + \frac{1}{24}(7\log 2 + 12 \log \pi) + \frac{3}{2}\left(\frac{1}{12} - \log A\right) + o(1) \end{aligned} \tag{1.3}$$

where A is the Glaisher-Kinkelin constant in the asymptotic formula for the Barnes G-function.

We need several remarks. Our evaluation of the probability starts with a formula of Zaporozhets [Z04] which is based on an integral geometry representation developed by Edelman and Kostlan (1995) and tools from differential geometry; see Section 2 for precise statements. The most important observation is that one can compute everything explicitly in the exponential setting. While the Gaussian case seems difficult and remains unresolved, our efforts on that version of the problem led to the realization that the technical details would be relatively simple for the exponential case. The second identity in (1.2) is a form of Selberg integral with simplification, see again Section 2 for more details. As far as this author knows, no explicit computation for i.i.d coefficient random variables with unbounded support is known before this note. Of course, one can also consider the natural generalization of the exponential distribution to Gamma distributions. This does encounter some minor difficulties and we will not pursue that direction in this paper. A more natural question under consideration is the computation of the explicit distribution of N_n under an i.i.d exponential ensemble. In particular, we hope to find the positivity exponent b via expression for $\mathbb{P}(N_n = 0)$ for n even. Another interesting question under consideration is the limiting measure of the empirical distribution of all real zeros, given that they are all real. Note also that in the case of so called reciprocal polynomials, a result similar to that of [Z04] is given by Farmer, Mezzadri and Snaith [FMS06] related to L-functions.

When the coefficients have a bounded support away from the origin, Devlin, Li, Xu and Yao [DLXY11+] proved that the probability of all real zeros is *zero* for all $n \ge n_0 + 1$, where n_0 is given explicitly. In particular, for all coefficients with support $\{1, 2, \cdots m\}$,

$$n_0 = \max\{k : P_k \neq 0\} = \max\{l : \binom{l}{[(l+1)/2]} \le m\}.$$

This bound can be achieved by $(x+1)^l$. For the support set $\{-1,+1\}$, one can see $n_0 = 3$ which turns out to be the Putnam competition problem A6 from 1968. The author found this out well after detailed studies in the project [DLXY11+]. Note that all these are basically deterministic results. When the coefficients are i.i.d geometric distributed, it is easy to provide non trivial bounds for the probability of all real zeros but we could not find a nice general formula.

Finally we mention the fact that the probability that a random real Gaussian matrix has k real eigenvalues is studied by Edelman [E97]. In particular, the probability that all eigenvalues are real is $2^{-n(n-1)/4}$. Interesting enough, the probability for exactly k real eigenvalues is not as explicit as one would like to see, even though certain improvement is possible and under current investigation. Similar phenomenons also occurred in the very recent work of Bergqvist and Forrester [BF11] on rank probabilities for real random $N\times N\times 2$ tensors. As an application, they provided the exact evaluation of the probability that all eigenvalues are real for random matrices $A^{-1}B$ where A and B are real Gaussian matrices.

The rest of the paper is organized in the following way. Section 2 provides notations and statements of two important results that we need. One is the formula of Zaporozhets [Z04] and the other is the well-known formula of Selberg's integral. In Section 3, we compute the probability and prove the theorem.

Acknowledgement: The author would like to thank Dr. Dmitry Zaporozhets for useful discussions and the referee for valuable suggestions.

2 Notations and Preliminaries

Before we sate the main result of D. Zaporozhets [Z04], let us introduce the following notation for the elementary symmetric polynomials:

$$\begin{aligned}
\sigma_0(y_1,\cdots,y_n) &= 1,\\
\sigma_1(y_1,\cdots,y_n) &= y_1+\cdots+y_n,\\
\sigma_2(y_1,\cdots,y_n) &= y_1y_2+\cdots+y_{n-1}y_n,\\
\sigma_3(y_1,\cdots,y_n) &= y_1y_2y_3+\cdots+y_{n-2}y_{n-1}y_n,\\
&\vdots\\
\sigma_{n-1}(y_1,\cdots,y_n) &= y_1y_2\cdots y_{n-1}+\cdots+y_2y_3\cdots y_n,\\
\sigma_n(y_1,\cdots,y_n) &= y_1y_2\cdots y_n;
\end{aligned} \tag{2.4}$$

and denote the Vandermonde determinant by

$$\Delta(y_1,\cdots,y_n) = \prod_{1\le i<j\le n} |y_i-y_j|. \tag{2.5}$$

Theorem [Z04]: Assume that the coefficients in (1.1) have a joint density $p(a_0, \cdots, a_n)$. Then the distribution of the number N_n of real zeros is given by

$$\begin{aligned} & \mathbb{P}(N_n = n-2k) \\ = & \frac{2^k}{k!(n-2k)!}\int_{\mathbb{R}^{n-2k}} dx_1\cdots dx_{n-2k}\int_{\mathbb{R}_+^k} dr_1\cdots dr_k\int_{[0,\pi]^k} d\alpha_1\cdots d\alpha_k\int_{\mathbb{R}} da\phi(a;x,\alpha,r) \end{aligned}$$

for $0 \le k \le n/2$, where

$$\begin{aligned} x &= (x_1, \cdots, x_{n-2k}), \\ r &= (r_1, \cdots, r_k), \\ \alpha &= (\alpha_1, \cdots, \alpha_k), \\ \phi(a;x,\alpha,r) &= r_1\cdots r_k p(a\sigma_0, a\sigma_1, \cdots, a\sigma_n)|a^n\Delta|, \end{aligned} \tag{2.6}$$

and

$$\sigma_j = \sigma_j(x_1, \cdots, x_{n-2k}, r_1e^{\mathbf{i}\alpha_1}, r_1e^{-\mathbf{i}\sigma_1}, \cdots, r_ke^{\mathbf{i}\sigma_k}, r_ke^{-\mathbf{i}\alpha_k}), \tag{2.7}$$

$$\Delta = \Delta(x_1, \cdots, x_{n-2k}, r_1e^{\mathbf{i}\alpha_1}, r_1e^{-\mathbf{i}\sigma_1}, \cdots, r_ke^{\mathbf{i}\sigma_k}, r_ke^{-\mathbf{i}\alpha_k}). \tag{2.8}$$

The basic approach of the proof in [Z04] starts with the integral geometry representation of Kac's formula and its various generalizations that is developed by Edelman and Kostlan [EK95]. Their remarkable interrelation between the theory of random polynomials and integral geometry is then extended in [Z04] to obtain the probability distribution of the number of real zeros of a random polynomial, by using tools from differential geometry. A similar argument is also used to derive a formula for the expected number of complex zeros lying in a given domain of the complex plane. However, no explicit example with absolute continuous density for i.i.d coefficient random variables is known after the work of [Z04] until this note.

Next we state the Selberg Integral, an n-dimensional generalization of the Euler beta integral,

$$\begin{aligned} S_n(\alpha,\beta,\gamma) &= \int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{\alpha-1}(1-t_i)^{\beta-1}\prod_{1\le i<j\le n}|t_j-t_i|^{2\gamma}\,dt_1\cdots dt_n \\ &= \prod_{j=0}^{n-1}\frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)} \end{aligned}$$

For interesting history of its sudden rise to prominence and fundamental role in various parts of mathematics, such as random matrix theory, Forrester and Warnaar [FW08] provided an excellent overview. For several proofs and far reaching generalizations, see also [AAR99] and [F10] for details. We will only use the case $\alpha = \beta = 1$ and $\gamma = 1/2$ in Section 3.

3 Proof of the Theorem

Let the coefficients a_i, $0 \le i \le n$ be i.i.d exponential random variables with mean one. Their joint density is given by

$$p(x_0, x_1, \cdots, x_n) = \exp\left(-\sum_{j=0}^{n} x_j\right), \quad x = (x_0, \cdots, x_n) \in \mathbb{R}_+^{n+1} \tag{3.9}$$

where $\mathbb{R}_+ = \{x \in \mathbb{R} : x \ge 0\}$. Based on the general formula given in the theorem of [Z04] in section 2, we have by taking $k = 0$,

$$\mathbb{P}(N_n = n) = \frac{1}{n!}\int_{\mathbb{D}} \exp\left(-\lambda\sum_{j=0}^{n}\sigma_j\right) \cdot |\lambda|^n \prod_{1\le j<k\le n} |x_j - x_k| dx_1 \cdots dx_n d\lambda \tag{3.10}$$

where the integration domain

$$\mathbb{D} = \{(x_1, \cdots, x_n, \lambda) \in \mathbb{R}^{n+1} : \lambda\sigma_j \ge 0, j = 0, 1, \cdots, n\}$$

and

$$\sigma_j = \sigma_j(x_1, \cdots, x_n)$$

as given in (2.4). The first critical step is the fact that the exponent in the integrand can be written in an alternative form:

$$\sum_{j=0}^{n}\sigma_j = \prod_{j=1}^{n}(1 + x_j). \tag{3.11}$$

This alternative form, while not helpful for the Gaussian ensemble, suggested that the problem would be easier to analyze for the exponential ensemble.

The second step is the realization that the integration region $\mathbb{D}$ is the first quadrant $\mathbb{R}_+^{n+1}$. Indeed, it is clear $\lambda = \lambda\sigma_0 \ge 0$ and we only need to show

$$\sigma_j(x_1, \cdots, x_n) \ge 0 \quad \text{for} \quad j = 1, 2, \cdots, n$$

if and only if

$$x_j \ge 0 \quad \text{for} \quad j = 1, 2, \cdots, n.$$

This follows from the relation

$$\prod_{k=1}^{n}(x + x_k) = \sum_{j=0}^{n}\sigma_j x^{n-j} > 0 \quad \text{for all} \quad x > 0.$$

In fact, if $x_k < 0$ for some k, then there is a contradiction by taking $x = -x_k > 0$.

Thus we can carry out the integration

$$\int_{\mathbb{R}_+} \exp\left(-\lambda\prod_{j=1}^{n}(1 + x_j)\right) \cdot \lambda^n d\lambda = \prod_{j=1}^{n}(1 + x_j)^{-n-1}\int_0^\infty t^n e^{-t} dt = n! \cdot \prod_{j=1}^{n}(1 + x_j)^{-n-1}$$

in (3.10) with respect to λ, and we obtain

$$\mathbb{P}(N_n = n) = \int_{\mathbb{R}_+^n} \prod_{j=1}^{n} (1+x_j)^{-n-1} \cdot \prod_{1\le j<k\le n} |x_j - x_k| dx_1 \cdots dx_n. \tag{3.12}$$

Now using the transformation $t_j = (1+x_j)^{-1}$, we have

$$\begin{aligned}
\prod_{1\le j<k\le n} |x_j - x_k| &= \prod_{1\le j<k\le n} |t_j^{-1} - t_k^{-1}| \\
&= \prod_{1\le j<k\le n} (t_j t_k)^{-1} \cdot \prod_{1\le j<k\le n} |t_j - t_k| \\
&= \prod_{j=1} t_j^{-n+1} \cdot \prod_{1\le j<k\le n} |t_j - t_k|, \\
dx_1 \cdots dx_n &= \prod_{j=1}^{n} (-t_j^{-2}).
\end{aligned}$$

Thus we obtain

$$\mathbb{P}(N_n = n) = \int_0^1 \cdots \int_0^1 \prod_{1\le j<k\le n} |t_j - t_k| dt_1 \cdots dt_n \tag{3.13}$$

which is a form of Selberg integral. Of course, we can also rewrite in terms of expectation involving i.i.d uniform random variables U_i on the interval $[0, 1]$,

$$\mathbb{P}(N_n = n) = \mathbb{E} \prod_{1\le j<k\le n} |U_j - U_k|. \tag{3.14}$$

The explicit evaluation, as a very special case of Selberg's integral discussed in section 2 with $\alpha = \beta = 1$ and $\gamma = 1/2$, is given by

$$\mathbb{P}(N_n = n) = \prod_{j=1}^{n} \frac{\Gamma^2((j+1)/2) \cdot \Gamma((j+2)/2)}{\Gamma((n+j+2)/2) \cdot \Gamma(3/2)} \tag{3.15}$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-x} dx$ is the standard Gamma function.

Next we simplify the gamma functions in the exact expression above and separate out even and odd terms, based on even or odd n, to cancel terms like $\Gamma(1/2) = \sqrt{\pi}$. To be more precise, we start by writing the Gamma functions involved in terms of factorials, that is

$$\Gamma(j/2) = \begin{cases} (j/2)! & \text{if } j \text{ is even,} \\ \frac{(j-1)!}{2^{j-1}((j-1)/2)!}\Gamma(1/2) & \text{if } j \text{ is odd.} \end{cases}$$

Thus we can rewrite (3.15) as

$$\begin{aligned}
\mathbb{P}(N_n = n) &= \prod_{2k-1\le n} \frac{\Gamma^2(k) \cdot \Gamma(k+1/2)}{\Gamma((n+2k+1)/2) \cdot \Gamma(1/2)} \cdot \prod_{2k\le n} \frac{\Gamma^2(k+1/2) \cdot \Gamma(k+1)}{\Gamma((n+2k+2)/2) \cdot \Gamma(1/2)} \\
&= \prod_{2k-1\le n} \frac{(k-1)!(2k-1)!}{2^{2k-2}\Gamma((n+2k+1)/2)} \cdot \prod_{2k\le n} \frac{((2k)!)^2 \cdot \Gamma(1/2)}{2^{4k-1} k! \Gamma((n+2k+2)/2)}.
\end{aligned}$$

When $n = 2m$, we have

$$\begin{aligned}
\mathbb{P}(N_{2m} = 2m) &= \prod_{k=1}^{m} \frac{(k-1)!(2k-1)!}{2^{2k-2}\Gamma((2m+2k+1)/2)} \cdot \prod_{k=1}^{m} \frac{((2k)!)^2 \cdot \Gamma(1/2)}{2^{4k-1}k!\Gamma((2m+2k+2)/2)} \\
&= \prod_{k=1}^{m} \frac{2^{2m+2}(k-1)!(2k-1)!(m+k)!}{(2m+2k)! \cdot \Gamma(1/2)} \cdot \prod_{k=1}^{m} \frac{((2k)!)^2 \cdot \Gamma(1/2)}{2^{4k-1}k!(m+k)!} \\
&= \frac{1}{(m!)^2} \prod_{k=1}^{m} \frac{((2k)!)^3}{(2m+2k)!} = \left(\prod_{k=1}^{m} \frac{1}{2^2} \binom{4k-2}{2k-1} \binom{4k}{2k} \right)^{-1} \\
&= \left(\prod_{k=1}^{2m} \frac{1}{2} \binom{2k}{k} \right)^{-1} = \left(\prod_{k=1}^{2m-1} \binom{2k+1}{k} \right)^{-1}.
\end{aligned}$$

When $n = 2m-1$, we have

$$\begin{aligned}
\mathbb{P}(N_{2m-1} = 2m-1) &= \prod_{k=1}^{m} \frac{(k-1)!(2k-1)!}{2^{2k-2}\Gamma((2m+2k)/2)} \cdot \prod_{k=1}^{m-1} \frac{((2k)!)^2 \cdot \Gamma(1/2)}{2^{4k-1}k!\Gamma((2m+2k+1)/2)} \\
&= \frac{(m-1)!}{2^{2m-2}} \prod_{k=1}^{m-1} \frac{(k-1)!(2k-1)!}{2^{2k-2}(m+k-1)!)} \cdot \prod_{k=1}^{m-1} \frac{((2k)!)^2(m+k)!}{2^{2k-1-2m}k!(2m+2k)!} \\
&= \frac{1}{2^{m-1}(m-1)!} \prod_{k=1}^{m-1} \frac{((2k)!)^3}{(2m-1+2k)!} = \left(\prod_{k=1}^{m-1} \frac{1}{2^2} \binom{4k}{2k} \binom{4k+2}{2k+1} \right)^{-1} \\
&= \left(\prod_{k=1}^{2m-1} \frac{1}{2} \binom{2k}{k} \right)^{-1} = \left(\prod_{k=1}^{2m-2} \binom{2k+1}{k} \right)^{-1}.
\end{aligned}$$

Combining the above formulas, we finish the explicit evaluation of the probability.

To find the large n asymptotic behavior of the probability, we use the well-known results on factorial function $\Gamma(n+1) = n!$ and super-factorial function

$$G(n+1) = \prod_{k=1}^{n-1} k!$$

and their extensions, the Gamma function $\Gamma(z)$ and the Barnes G-function $G(z)$. By using Euler-MacLaurin Summation, it is classical fact that, as $n \to \infty$,

$$\begin{aligned}
\log \Gamma(n+1) &= \log n! = n \log n - n + \frac{1}{2} \log n + \frac{\log(2\pi)}{2} + o(1); \\
\log G(n+1) &= \sum_{k=1}^{n-1} \log k! = \frac{1}{2} n^2 \log n - \frac{3}{4} n^2 + \frac{\log(2\pi)}{2} n - \frac{1}{12} \log n + \frac{1}{12} - \log A + o(1).
\end{aligned}$$

where A is the Glaisher-Kinkelin constant. Observe also that

$$G(2n+1) = \prod_{k=1}^{2n-1} k! = \left(\prod_{k=1}^{n-1} (2k)! \right) \cdot \left(\prod_{k=1}^{n-1} (2k+1)! \right) = \left(\prod_{k=1}^{n-1} (2k+1)! \right)^2 \cdot \left(\prod_{k=1}^{n-1} (2k+1) \right)^{-1}$$

and

$$\prod_{k=1}^{n-1}(2k+1) = \left(\prod_{k=1}^{n-1}(2k+2)(2k+1)\right) \cdot \left(\prod_{k=1}^{n-1} 2(k+1)\right)^{-1} = (2n)! \cdot (2^n \cdot n!)^{-1}.$$

Thus

$$\left(\prod_{k=1}^{n-1}(2k+1)!\right)^2 = \frac{(2n-1)!}{2^{n-1}(n-1)!}G(2n+1).$$

Now we are ready to obtain asymptotically the probability of all real zeros. We have as $n \to \infty$,

$$\begin{aligned}
\log \mathbb{P}(N_n = n) &= -\log \prod_{k=1}^{n-1}(2k+1)! + \log \prod_{k=1}^{n-1}(k+1)! + \log \prod_{k=1}^{n-1} k! \\
&= -\frac{1}{2}\log\left(\frac{(2n)!}{2^n n!}G(2n+1)\right) + \log G(n+2) + \log G(n+1) \\
&= -\frac{1}{2}\left(\log(2n)! - n\log 2 - \log n!\right) - \frac{1}{2}G(2n+1) + \log G(n+2) + \log G(n+1) \\
&= -\frac{1}{2}\left(n\log n - (1-\log 2)n + \frac{\log 2}{2} + o(1)\right) \\
&\quad -\frac{1}{2}\left(2n^2\log(2n) - 3n^2 + n\log(2\pi) - \frac{1}{12}\log(2n) + \frac{1}{12} - \log A + o(1)\right) \\
&\quad +\left(\frac{1}{2}(n+1)^2\log(n+1) - \frac{3}{4}(n+1)^2 + \frac{\log(2\pi)}{2}(n+1) - \frac{1}{12}\log(n+1)\right. \\
&\qquad \left. +\frac{1}{12} - \log A + o(1)\right) \\
&\quad +\left(\frac{1}{2}n^2\log n - \frac{3}{4}n^2 + \frac{\log(2\pi)}{2}n - \frac{1}{12}\log n + \frac{1}{12} - \log A + o(1)\right) \\
&= -\log 2 \cdot n^2 + \frac{1}{2}n\log n + \frac{1}{2}\left(\log\pi - 1\right)n + \frac{3}{8}\log n \\
&\quad +\frac{1}{24}\left(7\log 2 + 12\log\pi\right) + \frac{3}{2}\left(\frac{1}{12} - \log A\right) + o(1)
\end{aligned}$$

by using the approximation for $\log\Gamma(n+1)$ and $\log G(n+1)$.

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