



On the supremum of certain families of stochastic processes

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ABSTRACT

We consider a family of stochastic processes $\{X_t^\epsilon, t \in T\}$ on a metric space T , with a parameter $\epsilon \downarrow 0$. We study the conditions under which

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left(\sup_{t \in T} |X_t^\epsilon| < \delta \right) = 1$$

when one has an *a priori* estimate on the modulus of continuity and the value at one point. We compare our problem to the celebrated Kolmogorov continuity criteria for stochastic processes, and finally give an application of our main result for stochastic integrals with respect to compound Poisson random measures with infinite intensity measures.

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1. Introduction

Let (T, d) be a metric space with finite diameter,

$$D(T) = \sup \{d(s, t) : s, t \in T\} < \infty.$$

Let $N(T, d, \delta)$ denote the covering number, *i.e.*, for every $\delta > 0$, let $N(T, d, \delta)$ denote the minimal number of closed d -balls of radius δ required to cover T . The supremum of a stochastic process X_t defined on T , $\sup_{t \in T} X_t$ can be quantified in terms of $N(T, d, \delta)$ (see Talagrand (2005, Chapter 1) for instance) under various assumptions on the process X_t .

In this article we consider a family of stochastic processes X_t^ϵ on T , with a parameter $\epsilon > 0$. In certain applications in nonparametric statistics (see Section 4) it is of interest to study the limiting behaviour of the supremum, $\lim_{\epsilon \rightarrow 0} \sup_{t \in T} X_t^\epsilon$ when one has an *a priori* estimate of the form

$$\mathbb{E}|X_t^\epsilon - X_s^\epsilon|^\beta \leq B_\epsilon d(s, t)^\gamma$$

for some $\beta, \gamma > 0$ and $B_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. In particular, we would like to identify conditions under which, for every $\delta > 0$,

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left(\sup_{t \in T} |X_t^\epsilon| < \delta \right) = 1. \tag{1.1}$$

In our main result in Section 2, we find conditions in terms of the covering number $N(T, d, \delta)$ that ensure (1.1) holds. Although our technique is based on well known chaining methods, our principal result appears to be new. In Section 3 we discuss briefly the optimality of our hypotheses and compare our theorem with the Kolmogorov criterion for continuity of stochastic processes. In Section 4 we present an application of our main theorem to random fields constructed from Lévy random measures.

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2. Main result

Let (T, d) be a complete separable metric space and $(X_t^\epsilon)_{t \in T}$ a family of real-valued, centered, L_2 stochastic processes on T , indexed by $\epsilon > 0$.

Let n_0 be the largest integer n such that $N(T, d, 2^{-n}) = 1$ (note $n_0 < 0$ is possible). For every $n \geq n_0$, fix a covering of T of cardinality $N_n = N(T, d, 2^{-n})$ by closed balls of radius 2^{-n} . From this we can construct a partition \mathcal{A}_n of T of cardinality $|\mathcal{A}_n| = N_n$ by Borel sets with diameter at most 2^{-n+1} . For each $n \geq n_0$, fix a designated point in each element A of the partition \mathcal{A}_n , and denote by T_n the collection of these points. Without loss of generality, let the designated point in the single element of partition $\mathcal{A}_{n_0} = \{T\}$ be $T_{n_0} = \{t_0\}$ for a point $t_0 \in T$ to be specified in the statement of Theorem 1 below. For $t \in T$ denote by $\mathcal{A}_n(t)$ the partition element $A \in \mathcal{A}_n$ that contains t . For every t and every n , let $s_n(t)$ be the element of T_n in t 's partition element, so that $t \in \mathcal{A}_n(s_n(t))$. It is clear that $d(t, s_n(t)) \leq 2^{-n+1}$ for every $t \in T$ and $n \geq n_0$. By the triangle inequality

$$d(s_n(t), s_{n-1}(t)) \leq d(s_n(t), t) + d(t, s_{n-1}(t)) \leq 2^{-n+1} + 2^{-n+2} = 6 \cdot 2^{-n}.$$

Define the set

$$H_n \equiv \{(u, v) \in T_n \times T_{n-1} : d(u, v) \leq 6 \cdot 2^{-n}\}. \tag{2.2}$$

The following is our main result:

Theorem 1. *Suppose that:*

1. *There exists a point $t_0 \in T$ such that*

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}(X_{t_0}^\epsilon)^2 = 0. \tag{2.3a}$$

2. *There exist $\alpha, \beta > 0$ and positive numbers $\{B_\epsilon\}$ with $\lim_{\epsilon \rightarrow 0} B_\epsilon = 0$ such that for any $s, t \in T$*

$$\mathbb{E}|X_t^\epsilon - X_s^\epsilon|^\beta \leq B_\epsilon d(s, t)^{1+\alpha}. \tag{2.3b}$$

3. *There exists a family of partitions \mathcal{A}_n of T of sets of diameter no more than 2^{1-n} and a constant $\gamma < \alpha$ such that*

$$\sum_{n=1}^{\infty} |H_n| 2^{-(1+\gamma)n} < \infty \tag{2.3c}$$

where H_n is as defined in (2.2).

Then for any $\delta > 0$,

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left(\sup_{t \in T} |X_t^\epsilon| < \delta \right) = 1. \tag{2.6}$$

Remark. For each fixed $\epsilon > 0$, Eq. (2.3b) guarantees the existence of a path-continuous version of (X_t^ϵ) (by Kolmogorov's continuity criterion; see Durrett, 1996, p. 375. For more on this connection see Section 3). Since H_n satisfies the bound

$$|H_n| \leq |T_n| \cdot |T_{n-1}| \leq N^2(T, d, 2^{-n}),$$

the monotonicity of $N(T, d, \delta)$ implies that the entropy condition (2.3c) holds whenever

$$\int_0^{D(T)} a^\gamma N^2(T, d, a) da < \infty.$$

Frequently in applications we have a bound of the form

$$|H_n| \leq C \cdot |T_n| \leq C \cdot N(T, d, 2^{-n}) \tag{2.7}$$

for a universal constant C and in this case (2.3c) holds if

$$\int_0^{D(T)} a^\gamma N(T, d, a) da < \infty. \tag{2.8}$$

For example it can be easily shown that, with $T = [0, 1]$ and $d(u, v) = |u - v|$, the dyadic partition

$$\mathcal{A}_n = \{[i2^{1-n}, (i + 1)2^{1-n}] : 0 \leq i < 2^{n-1}\}$$

of T into $N_n = 2^{n-1}$ d -balls of radius 2^{-n} for $n \geq n_0 = 1$ satisfies (2.3c) for $C = 5$.

Proof. Fix $\delta > 0$. First observe that

$$\begin{aligned} \mathbb{P}\left(\sup_{t \in T} |X_t^\epsilon| < \delta\right) &\geq \mathbb{P}\left(\sup_{t \in T} |X_t^\epsilon - X_{t_0}^\epsilon| < \delta/2, |X_{t_0}^\epsilon| < \delta/2\right) \\ &\geq \mathbb{P}\left(\sup_{t \in T} |X_t^\epsilon - X_{t_0}^\epsilon| < \delta/2\right) - \mathbb{P}\left(|X_{t_0}^\epsilon| \geq \delta/2\right) \end{aligned}$$

and

$$\mathbb{P}(|X_{t_0}^\epsilon| \geq \delta/2) \leq 4\delta^{-2} \mathbb{E}|X_{t_0}^\epsilon|^2 \rightarrow 0$$

as $\epsilon \rightarrow 0$ by Eq. (2.3a). Thus we only need to control $\sup_{t \in T} |X_t^\epsilon - X_{t_0}^\epsilon|$.

We employ the so-called generic chaining principle of Ledoux (1996) (see also Talagrand, 2005, or Xiao, 2010 for a refinement similar in spirit to our approach). The fundamental relation is the convergent telescoping sum

$$X_t - X_{t_0} = \sum_{n > n_0} (X_{s_n(t)} - X_{s_{n-1}(t)})$$

for every $t \in T$, where we note that $s_{n_0}(t) = t_0$ for every $t \in T$. Then,

$$\begin{aligned} \sup_{t \in T} |X_t - X_{t_0}| &\leq \sup_{t \in T} \sum_{n > n_0} |X_{s_n(t)} - X_{s_{n-1}(t)}| \\ &\leq \sum_{n > n_0} \max_{(u,v) \in H_n} |X_u - X_v|. \end{aligned}$$

For $(u, v) \in T \times T$, let $\{w_n(u, v)\}_{n \geq n_0}$ be a sequence of non-negative real numbers such that $\sum_{n \geq n_0} w_n(u, v) = 1$. For any $\delta > 0$, by the triangle inequality

$$\bigcap_{n > n_0} \bigcap_{(u,v) \in H_n} \{|X_u - X_v| \leq w_n(u, v) \delta/2\} \subset \left\{ \sup_{t \in T} |X_t - X_{t_0}| \leq \delta/2 \right\}.$$

Therefore,

$$\begin{aligned} \mathbb{P}\left(\sup_{t \in T} |X_t^\epsilon - X_{t_0}^\epsilon| > \delta/2\right) &\leq \mathbb{P}\left(\bigcup_{n > n_0} \bigcup_{(u,v) \in T_n} \{|X_u^\epsilon - X_v^\epsilon| > w_n(u, v) \delta/2\}\right) \\ &\leq \sum_{n > n_0} \sum_{(u,v) \in H_n} \mathbb{P}(|X_u^\epsilon - X_v^\epsilon| > w_n(u, v) \delta/2). \end{aligned}$$

Next use Eqs. (2.3b) and (2.3c) to find optimal choices for $w_n(v)$ (the so-called ‘‘majorizing measure’’, see Talagrand, 2005, Chapter 1). Set

$$w_n(u, v) \equiv w_n \equiv (1 - 2^{-h})2^{-h(n-n_0)}, \quad h = (\alpha - \gamma)/\beta, \quad v \in T.$$

Notice that $\sum_{n \geq n_0} w_n(u, v) = 1$. By Markov’s inequality and (2.3b), for $v \in T_n$,

$$\begin{aligned} \mathbb{P}(|X_u^\epsilon - X_v^\epsilon| \geq w_n \delta/2) &\leq (w_n \delta/2)^{-\beta} \mathbb{E}|X_u^\epsilon - X_v^\epsilon|^\beta \\ &\leq (\delta/2)^{-\beta} (1 - 2^{-h})^{-\beta} 2^{\beta h(n-n_0)} B_\epsilon d(u, v)^{1+\alpha} \\ &\leq (\delta/2)^{-\beta} (1 - 2^{-h})^{-\beta} 2^{-\beta h n_0} 2^{\beta h n} B_\epsilon (6 \cdot 2^{-n})^{1+\alpha}. \end{aligned}$$

Putting all the estimates together,

$$\begin{aligned} \mathbb{P}\left(\sup_{t \in T} |X_t^\epsilon - X_{t_0}^\epsilon| \geq \delta/2\right) &\leq (\delta/2)^{-\beta} (1 - 2^{-h})^{-\beta} 6^{1+\alpha} 2^{-\beta h n_0} B_\epsilon \left(\sum_{n > n_0} \sum_{(u,v) \in H_n} 2^{\beta h n} 2^{-(1+\alpha)n}\right) \\ &= C B_\epsilon \sum_{n > n_0} |H_n| 2^{-(1+\gamma)n} \end{aligned}$$

for a finite constant $C < \infty$. Since $B_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and the sum converges by (2.3c),

$$\lim_{\epsilon \rightarrow 0} \mathbb{P}\left(\sup_{t \in T} |X_t^\epsilon - X_{t_0}^\epsilon| \geq \delta/2\right) = 0$$

and the theorem is proved. \square

3. Near optimality of our hypothesis

Kolmogorov’s continuity criterion asserts the existence of a path continuous version of any stochastic process $X_t, t \in [0, 1]$ that satisfies

$$\mathbb{E}(|X_t - X_s|^\beta) \leq C|t - s|^{1+\alpha}$$

for some fixed $\alpha, \beta > 0, C < \infty$ and all $0 \leq s, t \leq 1$ (cf. (2.3b)). Strict inequality $\alpha > 0$ is necessary, as illustrated by the well known counter example

$$X_t = \mathbf{1}_{\{U \leq t\}}$$

for $U \sim \text{Un}[0, 1]$ which satisfies $\mathbb{E}|X_t - X_s|^\beta \leq C|t - s|$ for all $\beta > 0$ and $C \geq 1$ but is almost surely discontinuous. In the spirit of this example, here we construct a stochastic process which shows that our hypothesis (2) in Theorem 1 is “very close” to optimal.

Let $U \sim \text{Un}[0, 1], 0 < \epsilon < 1$ and $X_t^\epsilon = \mathbf{1}_{\{t < U \leq t + \epsilon\}}, 0 \leq t \leq 1$. Then for any fixed $t \in [0, 1]$,

$$\mathbb{E}(X_t^\epsilon)^2 = \mathbb{P}(t < U \leq t + \epsilon) = \min(\epsilon, 1 - t). \tag{3.9}$$

Since

$$\mathbb{E}X_t^\epsilon X_s^\epsilon = \begin{cases} 0 & \text{if } |t - s| > \epsilon \\ \epsilon - |t - s| & \text{if } |t - s| \leq \epsilon, \min(s, t) \leq 1 - \epsilon \\ 1 - \max(s, t) & \text{if } |t - s| \leq \epsilon, \min(s, t) \geq 1 - \epsilon, \end{cases}$$

it follows that

$$\mathbb{E}(X_t^\epsilon - X_s^\epsilon)^2 \leq 2 \min(\epsilon, |t - s|). \tag{3.10}$$

By (3.9) X_t^ϵ satisfies (2.3a) for any $t_0 \in [0, 1]$, and by (3.10) we have bounds on $\mathbb{E}[(X_t^\epsilon - X_s^\epsilon)^\beta]$ for $\beta = 2$ both of the form $B|t - s|$ (for fixed $B = 2$) and of the form $B_\epsilon \rightarrow 0$ (with $B_\epsilon = 2\epsilon$), but not quite a bound of the form required by (2.3b). The conclusion (2.6) of Theorem 1 fails for the process X_t^ϵ since, for any $\epsilon > 0, \sup_{0 \leq t \leq 1} X_t^\epsilon = 1$ almost surely.

Notice that the ongoing example does not show the optimality of hypothesis (2.3b) since the above calculations do not imply the existence of a sequence $B_\epsilon \rightarrow 0$ with

$$\mathbb{E}(X_t^\epsilon - X_s^\epsilon)^2 \leq B_\epsilon |t - s|.$$

However, we believe that the condition $\alpha > 0$ in Eq. (2.3b) cannot be relaxed and state this as a conjecture.

Conjecture 1. *Theorem 1 is not true if hypothesis (2) (Eq. (2.3b)) is replaced by*

$$\mathbb{E}|X_t^\epsilon - X_s^\epsilon|^\beta \leq B_\epsilon d(s, t).$$

4. Application: Compensated Poisson random measures

In this section we present an application of Theorem 1 to a stochastic process constructed from compensated Poisson random measures.

Let Ω be a Polish space and $\nu(du d\omega)$ be a positive sigma-finite measure on $(-1, 1) \times \Omega$ such that

$$\begin{aligned} \nu((-a, a) \times \Omega) &= \infty, \quad \forall a \in [0, 1] \\ \iint_{(-1, 1) \times \Omega} u^2 \nu(du d\omega) &< \infty. \end{aligned}$$

Let

$$N(du d\omega) \sim \text{Po}(\nu)$$

be a Poisson random measure on $(-1, 1) \times \Omega$ which assigns independent $\text{Po}(\nu(B_i))$ distributions to disjoint Borel sets $B_i \subset (-1, 1) \times \Omega$. Let

$$\tilde{N}(du d\omega) \equiv N(du d\omega) - \nu(du d\omega)$$

denote the compensated Poisson measure with mean 0, an isometry from $L_2((-1, 1) \times \Omega, \nu(du d\omega))$ to the square-integrable zero-mean random variables (Sato, 1999, p. 38).

Let $K(t, \omega) : [0, 1] \times \Omega \rightarrow \mathbb{R}$ be a Borel measurable function such that

$$\iint_{(-1, 1) \times \Omega} K^2(t, \omega) u^2 \nu(du d\omega) < \infty \tag{4.11}$$

for all $0 \leq t \leq 1$. For $0 < \epsilon \leq 1$ define a stochastic process X_t^ϵ by

$$X_t^\epsilon \equiv \iint_{\{0 < |u| < \epsilon\} \times \Omega} K(t, \omega) u \tilde{N}(du d\omega), \quad t \in [0, 1]. \tag{4.12}$$

For every $t \in [0, 1]$ the stochastic integral (4.12) is well defined by (4.11) (see Wolpert and Taqqu, 2005; Rajput and Rosiński, 1989). For $t \in [0, 1]$, we have:

$$\begin{aligned} \mathbb{E}[X_t^\epsilon] &= 0 \\ \mathbb{E}[(X_t^\epsilon)^2] &= \iint_{(-\epsilon, \epsilon) \times \Omega} K^2(t, \omega) u^2 v(du d\omega) < \infty \\ \mathbb{E}\left[e^{i\zeta X_t^\epsilon}\right] &= \exp\left\{\iint_{(-\epsilon, \epsilon) \times \Omega} \left[e^{i\zeta K(t, \omega)u} - 1 - i\zeta K(t, \omega)u\right] v(du d\omega)\right\}, \end{aligned}$$

the Lévy–Khintchine formula for the characteristic function of an infinitely divisible random variable.

The stochastic process $\{X^\epsilon \equiv X_t^\epsilon, t \in [0, 1]\}$ is the discretization error arising from the approximation of certain stochastic integrals by finite sums (see Pillai and Wolpert, 2008; Wolpert et al., 2006). The limiting behaviour of the process X^ϵ as ϵ goes to zero (see Pillai and Wolpert, 2008, Section 3) is of particular interest; we would like to identify the conditions on the function K under which

$$\lim_{\epsilon \rightarrow 0} \mathbb{P}\left(\sup_{t \in [0, 1]} |X_t^\epsilon| > \delta\right) = 0 \tag{4.13}$$

for all $\delta > 0$, so the approximation error vanishes in the limit. Concentration equalities similar to (4.13) were studied by Reynaud-Bouret (2006) for finite intensity measures (i.e., $\nu((-1, 1) \times \Omega) < \infty$) using methods that are not applicable to our infinite intensity case.

In the next proposition we apply Theorem 1 to identify conditions for the kernel $K(\cdot, \cdot)$ under which (4.13) holds. Although it is stated for the index set $T \equiv [0, 1]$, the conclusions hold for any T satisfying the hypothesis of Theorem 1.

Proposition 1. Let $K(t, \omega) : [0, 1] \times \Omega \rightarrow \mathbb{R}$ satisfy (4.11) and

$$|K(t, \omega) - K(s, \omega)|^2 \leq C(\omega) |t - s|^{1+\alpha}, \quad s, t \in [0, 1], \omega \in \Omega \tag{4.14}$$

for some $\alpha > 0$ and Borel measurable function $C : \Omega \rightarrow \mathbb{R}_+$ satisfying

$$\iint_{(-1, 1) \times \Omega} C(\omega) u^2 v(du d\omega) < \infty. \tag{4.15}$$

Let X^ϵ be the stochastic process on $[0, 1]$ given in (4.12). Then, for any $\delta > 0$,

$$\lim_{\epsilon \rightarrow 0} \mathbb{P}\left(\sup_{t \in [0, 1]} |X_t^\epsilon| > \delta\right) = 0.$$

Proof. For any $t_0 \in [0, 1]$, by (4.11) and the dominated convergence theorem

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}\left[(X_{t_0}^\epsilon)^2\right] = \lim_{\epsilon \rightarrow 0} \int_{\{|u| \leq \epsilon\} \times \Omega} K^2(t_0, \omega) u^2 v(du d\omega) = 0, \tag{4.16}$$

verifying hypothesis 1 (Eq. (2.3a)) of Theorem 1. For $t, s \in [0, 1]$, by (4.14) and by the isometric property of $\tilde{N}(du d\omega)$,

$$\begin{aligned} \mathbb{E}\left[(X_t^\epsilon - X_s^\epsilon)^2\right] &= \iint_{\{|u| \leq \epsilon\} \times \Omega} |K(t, \omega) - K(s, \omega)|^2 u^2 v(du d\omega) \\ &\leq B_\epsilon |t - s|^{1+\alpha}, \quad \text{where} \\ B_\epsilon &\equiv \iint_{\{|u| \leq \epsilon\} \times \Omega} C(\omega) u^2 v(du d\omega) \rightarrow 0 \end{aligned} \tag{4.17}$$

as $\epsilon \rightarrow 0$ by (4.15), so hypothesis 2 (Eq. (2.3b)) of Theorem 1 is satisfied with the Euclidean metric $d(t, s) \equiv |t - s|$. For dyadic partitions of $[0, 1]$, we have already shown that (2.7) holds. Since $N([0, 1], d, a) = \lceil \frac{1}{a} \rceil \leq \frac{2}{a}$ for all $0 < a < 1$, for any $\gamma > 0$ (say, $\gamma = \alpha/2$),

$$\int_0^1 a^\gamma N([0, 1], d, a) da \leq 2 \int_0^1 a^{\gamma-1} da = \frac{2}{\gamma} < \infty \tag{4.18}$$

verifying Eq. (2.8). Therefore by (4.16)–(4.18) and Theorem 1, it follows that for any $\delta > 0$,

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left(\sup_{t \in [0,1]} |X_t^\epsilon| > \delta \right) = 0$$

and we are done. \square

Remark. It is not known whether the conclusion of the above proposition still holds if (4.14) is weakened to

$$|K(t, \omega) - K(s, \omega)|^2 \leq C(\omega)|t - s|, \quad s, t \in [0, 1].$$

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References

- Durrett, Richard, 1996. Probability: Theory and Examples, second ed. Duxbury Press, Belmont, CA.
- Ledoux, Michel, 1996. Isoperimetry and Gaussian analysis. In: Lectures on Probability Theory and Statistics. Saint-Flour, 1994. In: Lecture Notes in Mathematics, vol. 1648. Springer-Verlag, Berlin, DE, pp. 165–294.
- Pillai, Natesh S., Wolpert, Robert L., 2008. Posterior consistency of Bayesian nonparametric models using Lévy random field priors. Discussion Paper 2008–08. Duke Univ. Dept. Statist. Science, USA.
- Rajput Balram S., Rosiński, 1989. Spectral representations of infinitely divisible processes. Probab. Theory Rel. 82 (3), 451–487.
- Reynaud-Bouret, Patricia, 2006. Compensator and exponential inequalities for some suprema of counting processes. Statist. Probab. Lett. 76 (14), 1514–1521.
- Sato, Ken-iti, 1999. Lévy Processes and Infinitely Divisible Distributions. In: Cambridge Studies in Advanced Mathematics, vol. 68. Cambridge Univ. Press, Cambridge, UK.
- Talagrand, Michel, 2005. The Generic Chaining. In: Springer Monographs in Mathematics, Springer-Verlag, Berlin, DE.
- Wolpert, Robert L., Clyde, Merlise A., Tu, Chong, 2006. Lévy Adaptive Regression Kernels. Discussion Paper 2006–08. Duke Univ. Dept. Statist. Science, USA. Revised, November 2009.
- Wolpert, Robert L., Taqqu, Murad S., 2005. Fractional Ornstein–Uhlenbeck Lévy processes and the telecom process: Upstairs and downstairs. Signal Process. 85 (8), 1523–1545.
- Xiao, Yimin, 2010. Uniform modulus of continuity of random fields. Monatsh. Math. 159 (1), 163–184. doi:10.1007/s00605-009-0133-z. <http://www.springerlink.com/content/w724160533801310>.