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A CLT FOR THE L^2 MODULUS OF CONTINUITY OF BROWNIAN LOCAL TIME[1]

BY XIA CHEN, WENBO V. LI, MICHAEL B. MARCUS AND JAY ROSEN

University of Tennessee, University of Delaware, City University of New York and City University of New York

Let $\{L_t^x; (x,t) \in R^1 \times R_+^1\}$ denote the local time of Brownian motion, and

$$\alpha_t := \int_{-\infty}^{\infty} (L_t^x)^2 \, dx.$$

Let $\eta = N(0,1)$ be independent of α_t. For each fixed t,

$$\frac{\int_{-\infty}^{\infty} (L_t^{x+h} - L_t^x)^2 \, dx - 4ht}{h^{3/2}} \xrightarrow{\mathcal{L}} \left(\frac{64}{3}\right)^{1/2} \sqrt{\alpha_t}\,\eta$$

as $h \to 0$. Equivalently,

$$\frac{\int_{-\infty}^{\infty} (L_t^{x+1} - L_t^x)^2 \, dx - 4t}{t^{3/4}} \xrightarrow{\mathcal{L}} \left(\frac{64}{3}\right)^{1/2} \sqrt{\alpha_1}\,\eta$$

as $t \to \infty$.

1. Introduction. In [10] almost sure limits are obtained for the L^p moduli of continuity of local times of a very wide class of symmetric Lévy processes. For Brownian motion the result is as follows: Let $\{L_t^x; (x,t) \in R^1 \times R_+^1\}$ denote the local time of Brownian motion. Then for all $p \geq 1$, and all $t \in R_+$,

$$\lim_{h \downarrow 0} \int_a^b \left| \frac{L_t^{x+h} - L_t^x}{\sqrt{h}} \right|^p dx = 2^p E(|\eta|^p) \int_a^b |L_t^x|^{p/2} \, dx \tag{1.1}$$

for all a, b in the extended real line almost surely, and also in L^m, $m \geq 1$. (Here η is a normal random variable with mean zero and variance one.) When $p = 2$ and $a = -\infty$, $b = \infty$ we can write (1.1) in the form,

$$\lim_{h \downarrow 0} \int_{-\infty}^{\infty} \frac{(L_t^{x+h} - L_t^x)^2}{h} \, dx = 4t \qquad \text{a.s.} \tag{1.2}$$

This result in (1.1) uses the Eisenbaum Isomorphism theorem (see, e.g., [9], Theorem 8.1.1), and is a consequence of a similar result for the Ornstein–Uhlenbeck

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process [the stationary Gaussian process $\{G(x), x \in R^1\}$, with $E(G(x) - G(y))^2 = 2(1 - e^{-|x-y|})$], which is that, for all $p \geq 1$,

$$\lim_{h\to 0} \int_a^b \left| \frac{G(x+h) - G(x)}{\sqrt{h}} \right|^p dx = E|\eta|^p (b-a) \qquad \forall a, b \in R^1 \text{ a.s.} \tag{1.3}$$

This is also obtained in [10], in which this question is considered for a very large class of Gaussian processes. The right-hand side of (1.3) is the expected value of the left-hand side. Consequently, (1.3) can be thought of as a law of large numbers. In [11] we consider the central limit theorem for the left-hand side of (1.3). For the Ornstein–Uhlenbeck, when $p = 2$, we get

$$\lim_{h\downarrow 0} \frac{\int_a^b (G(x+h) - G(x))^2 \, dx - 2h(b-a)}{h^{3/2}} \stackrel{\mathcal{L}}{=} (16/3)^{1/2}(b-a)\eta. \tag{1.4}$$

The argument involving the Eisenbaum Isomorphism theorem that is used in [10] to show that (1.3) implies (1.1) does not work to show that (1.4) implies a similar result for the local times of Brownian motion. In this paper we obtain a central limit theorem corresponding to (1.1) by considering moments of

$$\int (L_t^{x+1} - L_t^x)^2 \, dx. \tag{1.5}$$

(An integral sign without limits is to be read as $\int_{-\infty}^{\infty}$.)

Let

$$\alpha_t = \int (L_t^x)^2 \, dx, \tag{1.6}$$

and let $\eta = N(0,1)$ be independent of α_t. We have the following weak convergence results.

THEOREM 1.1. *For each fixed t,*

$$\frac{\int (L_t^{x+h} - L_t^x)^2 \, dx - 4ht}{h^{3/2}} \stackrel{\mathcal{L}}{\to} c\sqrt{\alpha_t}\,\eta \tag{1.7}$$

as $h \to 0$, where $c = (64/3)^{1/2}$.

Equivalently,

$$\frac{\int (L_t^{x+1} - L_t^x)^2 \, dx - 4t}{t^{3/4}} \stackrel{\mathcal{L}}{\to} c\sqrt{\alpha_1}\,\eta \tag{1.8}$$

as $t \to \infty$.

The equivalence of (1.7) and (1.8) follows from the scaling relationship

$$\{L_{h^{-2}t}^x; (x,t) \in R^1 \times R_+^1\} \stackrel{\mathcal{L}}{=} \{h^{-1} L_t^{hx}; (x,t) \in R^1 \times R_+^1\} \tag{1.9}$$

(see, e.g., [9], Lemma 10.5.2), which implies that

$$\int (L_t^{x+h} - L_t^x)^2\,dx \overset{\mathcal{L}}{=} h^3 \int (L_{t/h^2}^{x+1} - L_{t/h^2}^x)^2\,dx. \tag{1.10}$$

Using this, and (1.7) with $t = 1$, and the change of variables $h^2 = 1/t$ gives (1.8).

We show in Lemma 8.1 that

$$E\left(\int (L_t^{x+1} - L_t^x)^2\,dx\right) = 4t + O(t^{1/2}). \tag{1.11}$$

Consequently, (1.8) can be written as

$$\frac{\int (L_t^{x+1} - L_t^x)^2\,dx - E(\int (L_t^{x+1} - L_t^x)^2\,dx)}{t^{3/4}} \overset{\mathcal{L}}{\to} c\sqrt{\alpha_1}\eta. \tag{1.12}$$

The weak law (1.7) can be written similarly.

Consider the limit in (1.7), and note that

$$\sqrt{\alpha_t}\eta \overset{\mathcal{L}}{=} \int_{-\infty}^{\infty} L_t^x\,dW_x, \tag{1.13}$$

where $\{W_x, x \in R^1\}$ is a new two-sided Brownian motion independent of $\{L_t^x; (x,t) \in R^1 \times R_+^1\}$. The process $\{\int_{-\infty}^{\infty} L_t^x\,dW_x, t \in R_+\}$ is often referred to as Brownian motion in Brownian scenery. It appeared first in the work of Kesten and Spitzer [8]. Let $\{S_k\}_{k=0}^{\infty}$ be a simple symmetric random walk and let $\{\sigma(x)\}_{x\in Z}$ be independent identically distributed symmetric random variables, with variance one, that are independent of $\{S_k\}$. The process

$$K_n = \sum_{k=0}^{n} \sigma(S_k), \qquad n \ge 1, \tag{1.14}$$

was introduced in [8] as a model of a self-interacting process. K_n is called a random walk in random scenery. In [8] it is shown that under certain moment conditions

$$n^{-3/4} K_{[nt]} \overset{\mathcal{L}}{\to} \int_{-\infty}^{\infty} L_t^x\,dW_x, \qquad \text{as } n \to \infty, \tag{1.15}$$

in the space of bounded functions on [0, 1] with the uniform topology.

More recently, the random variable $\int_{-\infty}^{\infty} L_1^x\,dW_x$ has appeared as the limit in a model for charged polymers [2]. Let $\{\omega_k\}_{k=1}^{\infty}$ be independent identically distributed symmetric random variables, with variance one, that satisfy certain integrability conditions and let $\{S_k\}_{k=1}^{\infty}$ be a simple symmetric random walk independent of $\{\omega_k\}$. Consider the stochastic process

$$H_n = \sum_{1\le j<k\le n} \omega_j\omega_k 1_{\{S_j=S_k\}}, \qquad n \ge 1. \tag{1.16}$$

H_n is referred to as the polymer energy of $\{S_1, S_2, \dots, S_n\}$. To understand the physical intuition behind this, think of assigning an electrical charge ω_k to the random site S_k, for all $k = 1, 2, \dots$. Assume that whenever $S_j = S_k$, we have an electrical interaction $\omega_j \omega_k$. In this case H_n represents the total electrical interaction of the polymer $\{S_1, S_2, \dots, S_n\}$. In [2] it is shown that

$$n^{-3/4} H_n \xrightarrow{\mathcal{L}} 2^{-1/2} \int_{-\infty}^{\infty} L_1^x \, dW_x. \tag{1.17}$$

Furthermore, Chen and Khoshnevisan ([3], Theorem 1.2) show that K_n and H_n are close in distribution.

In the proof of Theorem 1.1 we use the following result which is of independent interest: let $\{L_t^x, \widetilde{L}_t^x; (x,t) \in R^1 \times R_+^1\}$ denote the local times of two independent Brownian motions and let

$$\beta_{s,t} = \int L_s^x \widetilde{L}_t^x \, dx \tag{1.18}$$

denote their intersection local time.

THEOREM 1.2. *For each fixed s, t,*

$$\frac{\int (L_s^{x+h} - L_s^x)(\widetilde{L}_t^{x+h} - \widetilde{L}_t^x)\, dx}{h^{3/2}} \xrightarrow{\mathcal{L}} \widetilde{C} \sqrt{\beta_{s,t}}\, \eta \tag{1.19}$$

as $h \to 0$ where $\widetilde{C} = (32/3)^{1/2}$. Consequently,

$$\frac{\int (L_t^{x+1} - L_t^x)(\widetilde{L}_t^{x+1} - \widetilde{L}_t^x)\, dx}{t^{3/4}} \xrightarrow{\mathcal{L}} \widetilde{C} \sqrt{\beta_{1,1}}\, \eta \tag{1.20}$$

as $t \to \infty$.

We were motivated to try to find a central limit theorem for $\int (L_t^{x+h} - L_t^x)^2 \, dx$ by our interest in the expression

$$H_n = \sum_{i,j=1, i \neq j}^{n} 1_{\{S_i = S_j\}} - \frac{1}{2} \sum_{i,j=1, i \neq j}^{n} 1_{\{|S_i - S_j| = 1\}} \tag{1.21}$$

which appears as the Hamiltonian in a model for a polymer in a repulsive medium [6]. Here $S := \{S_n; n = 0, 1, 2, \dots\}$ is a simple random walk on Z^1. Note that

$$H_n = \frac{1}{2} \sum_{x \in Z^1} (l_n^x - l_n^{x+1})^2, \tag{1.22}$$

where $l_n^x = \sum_{i=1}^n 1_{\{S_i = x\}}$ is the local time for S.

Theorems 1.1 and 1.2 are proved by the method of moments. In Section 2 we show that Theorem 1.2 follows immediately from moment estimates in Lemma 2.1. Lemma 2.1 itself follows from Lemma 2.2, which obtains the moments of an expression analogous to the one in Lemma 2.1, except that the fixed time t is replaced by independent exponential times. Lemma 2.2 is proved in Section 4. Lemma 2.1 also requires Lemma 2.3 which allows us to use Laplace transform methods. Lemma 2.3 is proved in Section 5. In Section 3 we derive some estimates on the potential densities of Brownian motion that are used throughout this paper. In Section 6 we show that Theorem 1.1 follows from Lemma 6.2, on the moments of an expression analogous to the left-hand side of (1.7), in which t is replaced by an independent exponential time. Lemma 6.2 is proved in Section 7. In Section 8 we obtain (1.11).

The basic tool we use for studying moments of local times is Kac's moment formula. We use exponential times to make Kac's moment formula manageable. Moments at exponential times correspond to the Laplace transforms of the moments at fixed times. Since the left-hand side of (1.7) has no obvious monotonicity properties, an important part of our proof involves showing how to derive limit results for the moments of (1.7) from limit results for their Laplace transforms.

An alternate approach to proving Theorems 1.1 and 1.2 would be to use Tanaka's formula and martingale methods (see [14, 15]). For the results in this paper this would involve establishing results about the differentiability of triple intersection local times, as is done in [12] for ordinary intersection local times. We plan to return to this at a later date.

2. Proof of Theorem 1.2.

We derive Theorem 1.2 from the next lemma which is proved in this section.

LEMMA 2.1. *For all $s,t \ge 0$ and all integers $m \ge 0$,*

$$\lim_{h\to 0} E\left(\left(\frac{\int (L_s^{x+h}-L_s^x)(\widetilde{L}_t^{x+h}-\widetilde{L}_t^x)\,dx}{h^{3/2}}\right)^m\right) = \begin{cases} \dfrac{(2n)!}{2^n n!}\left(\dfrac{32}{3}\right)^n E\left\{\left(\displaystyle\int L_s^x \widetilde{L}_t^x\,dx\right)^n\right\}, & \text{if } m=2n,\\ 0, & \text{otherwise.}\end{cases} \tag{2.1}$$

PROOF OF THEOREM 1.2. It follows from [4], formula (6.12), that

$$E\left\{\left(\int (L_s^x)^2\,dx\right)^n\right\} \le C_s^n((2n)!)^{1/4}. \tag{2.2}$$

Therefore, the right-hand side of (2.1), which is the $2n$th moment of $c\sqrt{\beta_{s,t}}\eta$ is less than or equal to $C_{s,t}^n((2n)!)^{3/4}$. This implies that $c\sqrt{\beta_{s,t}}\eta$ is determined by its moments (see [5], pages 227–228). Thus (1.19) follows from [1], Theorem 30.2, which is often referred to as the method of moments. We then get (1.20) by using the scaling relationship (1.9). □

The next two lemmas are used in the proof of Lemma 2.1. Lemma 2.2 is proved in Section 4 and Lemma 2.3 is proved in Section 5.

Let λ_ζ and $\widetilde{\lambda}_{\zeta'}$ be independent exponential times with means $1/\zeta$ and $1/\zeta'$, respectively.

LEMMA 2.2. *For each integer* $m \geq 0$, *and any* $\zeta, \zeta' > 0$,

$$\lim_{h\to 0} E\left(\left(\frac{\int (L^{x+h}_{\lambda_\zeta} - L^x_{\lambda_\zeta})(\widetilde{L}^{x+h}_{\widetilde{\lambda}_{\zeta'}} - \widetilde{L}^x_{\widetilde{\lambda}_{\zeta'}})\,dx}{h^{3/2}}\right)^m\right) = a_m, \tag{2.3}$$

where

$$a_m = \begin{cases} \dfrac{(2n)!}{2^n n!}\left(\dfrac{32}{3}\right)^n E\left\{\left(\displaystyle\int L^x_{\lambda_\zeta} \widetilde{L}^x_{\widetilde{\lambda}_{\zeta'}}\,dx\right)^n\right\}, & \text{if } m = 2n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

We write the statement of Lemma 2.2 in the form,

$$\begin{aligned} &\lim_{h\to 0}\int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t} E\left(\left(\frac{\int (L^{x+h}_s - L^x_s)(\widetilde{L}^{x+h}_t - \widetilde{L}^x_t)\,dx}{h^{3/2}}\right)^m\right) ds\,dt \\ &\qquad = \int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t} E\left\{\eta^m \left(\frac{32}{3}\int L^x_s \widetilde{L}^x_t\,dx\right)^{m/2}\right\} ds\,dt. \end{aligned} \tag{2.5}$$

For $h > 0$, let

$$F_h(s,t;m) := E\left(\left(\frac{\int (L^{x+h}_s - L^x_s)(\widetilde{L}^{x+h}_t - \widetilde{L}^x_t)\,dx}{h^{3/2}}\right)^m\right) \tag{2.6}$$

and

$$F_0(s,t;m) = E\left\{\eta^m \left(\frac{32}{3}\int L^x_s \widetilde{L}^x_t\,dx\right)^{m/2}\right\}.$$

In this notation Lemma 2.2 states that for any $\zeta, \zeta' > 0$,

$$\begin{aligned} &\lim_{h\to 0}\int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t} F_h(s,t;m)\,ds\,dt \\ &\qquad = \int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t} F_0(s,t;m)\,ds\,dt. \end{aligned} \tag{2.7}$$

[Note that $F_0(s,t;m) = 0$ when m is odd.]

LEMMA 2.3. *For all integers* $m \geq 0$, *and* $0 \leq h \leq 1$, $F_h(s,t;m)$ *is a non-negative, polynomially bounded, continuous increasing function of* (s,t).

PROOF OF LEMMA 2.1. It follows from Lemma 2.3 that $F_h(s,t;m)$ is the distribution function of a measure $\mu_{h,m}$ on R^2_+; that is,

$$F_h(s,t;m) = \int_0^s \int_0^t d\mu_{h,m}(u,v). \tag{2.8}$$

For any $0 \le h \le 1$, consider

$$\int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t} F_h(s,t;m)\,ds\,dt. \tag{2.9}$$

Since $F_h(0,t;m) = F_h(s,0;m) = 0$, it follows from integrating by parts (in which we use Lemma 2.3), that for all $\zeta, \zeta' > 0$,

$$\zeta\zeta' \int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t} F_h(s,t;m)\,ds\,dt = \int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t}\,d\mu_{h,m}(s,t). \tag{2.10}$$

We see from (2.7) and (2.10) that for any $\zeta, \zeta' > 0$,

$$\lim_{h\to 0} \int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t}\,d\mu_{h,m}(s,t) = \int_0^\infty \int_0^\infty e^{-\zeta s - \zeta' t}\,d\mu_{0,m}(s,t). \tag{2.11}$$

It then follows from (2.11) and the extended continuity theorem [7], Theorem 5.22, that $\mu_{h,m} \stackrel{w}{\to} \mu_{0,m}$. Using this and Lemma 2.3 we see that

$$\lim_{h\to 0} F_h(s,t) = F_0(s,t) \qquad \forall s,t \tag{2.12}$$

which gives (2.1). (Actually [7], Theorem 5.22, is stated for probability measures on R_+^d. The case of general measures on R_+^d can be derived as in the proof of [5], Chapter XIII, Section 1, Theorem 2a. Unfortunately [5], Chapter XIII, Section 1, Theorem 2a, only considers measures on R_+^1. Its extension to R_+^d is routine.) $\square$

3. Estimates for the potential density of Brownian motion. The α-potential density of Brownian motion,

$$u^\alpha(x) = \int_0^\infty e^{-\alpha t} p_t(x)\,dt = \frac{e^{-\sqrt{2\alpha}|x|}}{\sqrt{2\alpha}}. \tag{3.1}$$

Let λ_α be an independent exponential random variable with mean $1/\alpha$.

Kac's moment formula [9], Theorem 3.10.1, states that

$$E^{x_0}\left(\prod_{j=1}^n L_{\lambda_\alpha}^{x_j}\right) = \sum_\pi \prod_{j=1}^n u^\alpha\big(x_{\pi(j)} - x_{\pi(j-1)}\big), \tag{3.2}$$

where the sum runs over all permutations π of $\{1,\dots,n\}$ and $\pi(0) = 0$.

Let Δ_x^h denote the finite difference operator on the variable x, that is,

$$\Delta_x^h f(x) = f(x+h) - f(x). \tag{3.3}$$

We write Δ^h for Δ_x^h when the variable x is clear.

The next lemma collects some facts about $u^\alpha(x)$ that are used in this paper.

LEMMA 3.1. *Fix $\alpha, \beta > 0$. For $0 < h \leq 1$,*

$$\Delta_x^h \Delta_y^h u^\alpha(x-y)|_{y=x} = 2\left(\frac{1-e^{-\sqrt{2\alpha}h}}{\sqrt{2\alpha}}\right) = 2h + O(h^2), \tag{3.4}$$

$$|\Delta^h u^\alpha(x)| \leq Chu^\alpha(x), \tag{3.5}$$

$$|\Delta^h \Delta^{-h} u^\alpha(x)| \leq Chu^\alpha(x), \tag{3.6}$$

$$|\Delta^h \Delta^{-h} u^\alpha(x)| \leq Ch^2 u^\alpha(x) \qquad \forall |x| \geq h. \tag{3.7}$$

In addition,

$$\int (\Delta^h \Delta^{-h} u^\alpha(x))(\Delta^h \Delta^{-h} u^\beta(x))\,dx = \big(8/3 + O(h)\big)h^3, \tag{3.8}$$

$$\int_{|x|\geq h} (\Delta^h \Delta^{-h} u^\alpha(x))^2\,dx = O(h^4), \tag{3.9}$$

$$\int |\Delta^h \Delta^{-h} u^\alpha(x)|\,dx = O(h^2). \tag{3.10}$$

In all these statements the constants C and the terms $O(h)$ may depend on α and β.

PROOF. Since

$$\begin{aligned}\Delta_x^h \Delta_y^h u^\alpha(x-y) &= \{u^\alpha(x-y) - u^\alpha(x-y-h)\} \\ &\quad - \{u^\alpha(x-y+h) - u^\alpha(x-y)\},\end{aligned} \tag{3.11}$$

we have

$$\begin{aligned}\Delta_x^h \Delta_y^h u^\alpha(x-y)|_{y=x} &= \{u^\alpha(0) - u^\alpha(-h)\} - \{u^\alpha(h) - u^\alpha(0)\} \\ &= 2\big(u^\alpha(0) - u^\alpha(h)\big) = 2\left(\frac{1-e^{-\sqrt{2\alpha}h}}{\sqrt{2\alpha}}\right)\end{aligned} \tag{3.12}$$

which gives (3.4).

To obtain (3.5) we note that

$$\Delta_x^h u^\alpha(x) = \left(\frac{e^{-\sqrt{2\alpha}|x+h|} - e^{-\sqrt{2\alpha}|x|}}{\sqrt{2\alpha}}\right). \tag{3.13}$$

Therefore,

$$\begin{aligned}|\Delta_x^h u^\alpha(x)| &= \frac{e^{-\sqrt{2\alpha}|x|}}{\sqrt{2\alpha}}\big|e^{\sqrt{2\alpha}(|x|-|x+h|)} - 1\big| \\ &\leq e^{-\sqrt{2\alpha}|x|}\big(\big||x| - |x+h|\big| + O\big(\big||x| - |x+h|\big|^2\big)\big)\end{aligned} \tag{3.14}$$

which gives (3.5) (since we allow C to depend on α).

To obtain (3.6) we note that

$$|\Delta^h \Delta^{-h} u^\alpha(x)| = |2u^\alpha(x) - u^\alpha(x+h) - u^\alpha(x-h)| \\ \le |\Delta^h u^\alpha(x)| + |\Delta^h u^\alpha(x-h)| \tag{3.15}$$

and use (3.5).

To obtain (3.7) we simply note that when $|x| \ge h$,

$$\Delta^h \Delta^{-h} u^\alpha(x) = 2u^\alpha(x) - u^\alpha(x+h) - u^\alpha(x-h) \\ = u^\alpha(x)\bigl(2 - e^{-\sqrt{2\alpha}h} - e^{\sqrt{2\alpha}h}\bigr). \tag{3.16}$$

The statement in (3.9) follows trivially from (3.7).

For (3.8) we note that for $|x| \le h$,

$$\begin{aligned} &\Delta^h \Delta^{-h} u^\alpha(x) \\ &\quad = 2u^\alpha(x) - u^\alpha(x+h) - u^\alpha(x-h) \\ &\quad = \bigl(u^\alpha(0) - u^\alpha(x+h)\bigr) + \bigl(u^\alpha(0) - u^\alpha(x-h)\bigr) - 2\bigl(u^\alpha(0) - u^\alpha(x)\bigr) \\ &\quad = |x+h| + |x-h| - 2|x| + O(h^2). \end{aligned} \tag{3.17}$$

Therefore when $0 \le x \le h$, we have

$$\Delta^h \Delta^{-h} u^\alpha(x) = x + h + h - x - 2x + O(h^2) = \bigl(2 + O(h)\bigr)(h - x) \tag{3.18}$$

and similarly for $\Delta^h \Delta^{-h} u^\beta(x)$. Consequently,

$$\int_0^h (\Delta^h \Delta^{-h} u^\alpha(x))(\Delta^h \Delta^{-h} u^\beta(x))\,dx = \bigl(4 + O(h)\bigr) \int_0^h (h-x)^2\,dx \\ = \bigl(4/3 + O(h)\bigr)h^3. \tag{3.19}$$

Similarly, when $-h \le x \le 0$, it follows from (3.17) that

$$\Delta^h \Delta^{-h} u^\alpha(x) = h - x + x + h + 2x + O(h^2) = \bigl(2 + O(h)\bigr)(h + x) \tag{3.20}$$

and similarly for $\Delta^h \Delta^{-h} u^\beta(x)$. Consequently,

$$\int_{-h}^0 (\Delta^h \Delta^{-h} u^\alpha(x))(\Delta^h \Delta^{-h} u^\beta(x))\,dx = \bigl(4 + O(h)\bigr) \int_{-h}^0 (h+x)^2\,dx \\ = \bigl(4/3 + O(h)\bigr)h^3. \tag{3.21}$$

Using (3.19), (3.21) and (3.9) we get (3.8).

To obtain (3.10) we write

$$\int |\Delta^h \Delta^{-h} u^\alpha(y)|\,dy = \int_{|y|\le h} |\Delta^h \Delta^{-h} u^\alpha(y)|\,dy + \int_{|y|\ge h} |\Delta^h \Delta^{-h} u^\alpha(y)|\,dy \\ \le Ch \int_{|y|\le h} 1\,dy + Ch^2 \int_{|y|\ge h} u^\alpha(y)\,dy = O(h^2), \tag{3.22}$$

where for the last line we use (3.6) and (3.7). $\square$

4. Proof of Lemma 2.2. Let X_t, $\widetilde{X}_t$ be two independent Brownian motions in R^1. Let L_t^x, $\widetilde{L}_t^x$ denote their local times, and let λ_ζ, $\widetilde{\lambda}_{\zeta'}$ be independent exponential times of mean $1/\zeta$, $1/\zeta'$, respectively. Set

$$\beta_2 = \int L^x_{\lambda_\zeta} \widetilde{L}^x_{\widetilde{\lambda}_{\zeta'}}\, dx. \tag{4.1}$$

It follows from (3.2), the Kac moment formula, that

$$\begin{aligned}
E\left(\prod_{i=1}^m L^{x_i}_{\lambda_\zeta} \widetilde{L}^{y_i}_{\widetilde{\lambda}_{\zeta'}}\right) &= E\left(\prod_{i=1}^m L^{x_i}_{\lambda_\zeta}\right) E\left(\prod_{i=1}^m \widetilde{L}^{y_i}_{\widetilde{\zeta}}\right) \\
&= \sum_\pi \prod_{j=1}^m u^\zeta\big(x_{\pi(j)} - x_{\pi(j-1)}\big)\, dr_j \\
&\quad \times \sum_{\pi'} \prod_{j=1}^m u^{\zeta'}\big(y_{\pi'(j)} - y_{\pi'(j-1)}\big),
\end{aligned} \tag{4.2}$$

where the sums run over all permutations π and π' of $\{1,\dots,m\}$, $\pi(0) = \pi'(0) = 0$ and $x_0 = 0$. Consequently, by setting each y_i equal to x_i, we see that

$$\begin{aligned}
&E\left(\left(\int L^x_{\lambda_\zeta} \widetilde{L}^x_{\widetilde{\lambda}_{\zeta'}}\, dx\right)^m\right) \\
&\quad = E\left(\prod_{i=1}^m \int L^{x_i}_{\lambda_\zeta} \widetilde{L}^{x_i}_{\widetilde{\lambda}_{\zeta'}}\, dx_i\right) \\
&\quad = \sum_{\pi,\pi'} \int \left(\prod_{j=1}^m u^\zeta\big(x_{\pi(j)} - x_{\pi(j-1)}\big) \prod_{j=1}^m u^{\zeta'}\big(x_{\pi'(j)} - x_{\pi'(j-1)}\big)\right) \prod_{i=1}^m dx_i.
\end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned}
&E\left(\prod_{i=1}^m \big(L^{x_i+h}_{\lambda_\zeta} - L^{x_i}_{\lambda_\zeta}\big)\big(\widetilde{L}^{y_i+h}_{\widetilde{\lambda}_{\zeta'}} - \widetilde{L}^{y_i}_{\widetilde{\lambda}_{\zeta'}}\big)\right) \\
&\qquad = \left(\prod_{i=1}^m \Delta^h_{x_i} \Delta^h_{y_i}\right) E\left(\prod_{i=1}^m L^{x_i}_{\lambda_\zeta} \widetilde{L}^{y_i}_{\widetilde{\lambda}_{\zeta'}}\right) \\
&\qquad = \left(\prod_{i=1}^m \Delta^h_{x_i} \Delta^h_{y_i}\right) E\left(\prod_{i=1}^m L^{x_i}_{\lambda_\zeta}\right) E\left(\prod_{i=1}^m \widetilde{L}^{y_i}_{\widetilde{\lambda}_{\zeta'}}\right) \\
&\qquad = \left(\prod_{i=1}^m \Delta^h_{x_i}\right) \sum_\pi \prod_{j=1}^m u^\zeta\big(x_{\pi(j)} - x_{\pi(j-1)}\big) \\
&\qquad\quad \times \left(\prod_{i=1}^m \Delta^h_{y_i}\right) \sum_{\pi'} \prod_{j=1}^m u^{\zeta'}\big(y_{\pi'(j)} - y_{\pi'(j-1)}\big).
\end{aligned} \tag{4.4}$$

Using the product rule for finite differences,

$$\Delta^h(fg)(x) = (\Delta^h f(x))g(x+h) + f(x)\Delta^h g(x), \tag{4.5}$$

we can write

$$\begin{aligned}
&\left(\prod_{i=1}^m \Delta_{x_i}^h\right) \sum_\pi \prod_{j=1}^m u^\zeta\big(x_{\pi(j)} - x_{\pi(j-1)}\big) \\
&\quad = \sum_{\pi,a} \prod_{j=1}^m \big((\Delta_{x_{\pi(j)}}^h)^{a_1(j)} (\Delta_{x_{\pi(j-1)}}^h)^{a_2(j)} u^{\zeta,\sharp}\big(x_{\pi(j)} - x_{\pi(j-1)}\big)\big),
\end{aligned} \tag{4.6}$$

where the sum runs over π and all $a = (a_1, a_2) : [1, \dots, m] \mapsto \{0, 1\} \times \{0, 1\}$, with the restriction that for each i there is exactly one factor of the form $\Delta_{x_i}^h$. [Here we define $(\Delta_{x_i}^h)^0 = 1$ and $(\Delta_0^h) = 1$.] In this formula, $u^{\zeta,\sharp}(x)$ can take any of the values $u^\zeta(x)$, $u^\zeta(x+h)$ or $u^\zeta(x-h)$. (We consider all three possibilities in the subsequent proofs.) It is important to recognize that in (4.6) each of the difference operators is applied to only one of the terms $u^{\zeta,\sharp}(\cdot)$.

Using (4.6) we see that if we set $x_i = y_i$, $i = 0, \dots, m$, in (4.4) we get

$$\begin{aligned}
&E\left(\left(\int (L_{\lambda_\zeta}^{x+h} - L_{\lambda_\zeta}^x)(\widetilde{L}_{\widetilde{\lambda}_{\zeta'}}^{x+h} - \widetilde{L}_{\widetilde{\lambda}_{\zeta'}}^x)\,dx\right)^m\right) \\
&\quad = \sum_{\pi,\pi',a,a'} \int \mathcal{T}_h'(x; \pi, \pi', a, a')\,dx,
\end{aligned} \tag{4.7}$$

where $x = (x_1, \dots, x_m)$, and

$$\begin{aligned}
&\mathcal{T}_h'(x; \pi, \pi', a, a') \\
&\quad = \prod_{j=1}^m \big((\Delta_{x_{\pi(j)}}^h)^{a_1(j)} (\Delta_{x_{\pi(j-1)}}^h)^{a_2(j)} u^{\zeta,\sharp}\big(x_{\pi(j)} - x_{\pi(j-1)}\big)\big) \\
&\qquad \times \prod_{k=1}^m \big((\Delta_{x_{\pi'(k)}}^h)^{a_1'(k)} (\Delta_{x_{\pi'(k-1)}}^h)^{a_2'(k)} u^{\zeta',\sharp}\big(x_{\pi'(k)} - x_{\pi'(k-1)}\big)\big),
\end{aligned} \tag{4.8}$$

and where the sum runs over all permutations π and π' of $\{1, \dots, m\}$ and all $a = (a_1, a_2) : [1, \dots, m] \mapsto \{0, 1\} \times \{0, 1\}$ and $a' = (a_1', a_2') : [1, \dots, m] \mapsto \{0, 1\} \times \{0, 1\}$ with the restriction that, for each i, there is exactly one factor of the form $\Delta_{x_i}^h$ in the second line of (4.8), and, similarly, in the third line of (4.8).

Let

$$\begin{aligned}
&\mathcal{T}_h(x; \pi, \pi', a, a') \\
&\quad = \prod_{j=1}^m \big((\Delta_{x_{\pi(j)}}^h)^{a_1(j)} (\Delta_{x_{\pi(j-1)}}^h)^{a_2(j)} u^\zeta\big(x_{\pi(j)} - x_{\pi(j-1)}\big)\big) \\
&\qquad \times \prod_{k=1}^m \big((\Delta_{x_{\pi'(k)}}^h)^{a_1'(k)} (\Delta_{x_{\pi'(k-1)}}^h)^{a_2'(k)} u^{\zeta'}\big(x_{\pi'(k)} - x_{\pi'(k-1)}\big)\big).
\end{aligned} \tag{4.9}$$

The difference between (4.8) and (4.9) is that $u^{\zeta,\sharp}$ is replaced by u^{ζ} and similarly for $u^{\zeta',\sharp}$. To simplify the computations, we first obtain

$$\lim_{h\to 0}\frac{1}{h^{3m/2}}\sum_{\pi,\pi',a,a'}\int \mathcal{T}_h(x;\pi,\pi',a,a')\,dx \tag{4.10}$$

and then explain why (4.10) is unchanged when $\mathcal{T}_h$ is replaced by $\mathcal{T}_h'$.

Recall that $\Delta^h f(u) = f(u+h) - f(h)$ so that

$$\Delta^h\Delta^{-h} f(u-v) = 2f(u-v) - f(u-v-h) - f(u-v+h). \tag{4.11}$$

Consequently,

$$\Delta_u^h\Delta_v^h f(u-v) = \Delta^h\Delta^{-h} f(u-v). \tag{4.12}$$

We proceed to evaluate (4.10) based on the different ways the difference operators in (4.9) are distributed. We examine these in three subsections. The reader will see that the only limits in (4.10), that are not zero, come from the case considered in Section 4.1.

Let $e = (e(1),\dots,e(2n))$ where $e(2j) = (1,1), e(2j-1) = (0,0)$, $j = 1,\dots,n$.

4.1. *$a = a' = e$ and compatible permutations.* Let $m = 2n$ and let $\mathcal{P} = \{(l_{2i-1}, l_{2i}), 1 \le i \le n\}$ be a pairing of the integers $[1, 2n]$. Let π and π' be two permutations of $[1, 2n]$ such that for each $1 \le j \le n$, $\{\pi(2j-1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\}$ for some, necessarily unique, $1 \le i \le n$ and similarly for π', that is, for each $1 \le j \le n$, $\{\pi'(2j-1), \pi'(2j)\} = \{l_{2k-1}, l_{2k}\}$ for some, necessarily unique, $1 \le k \le n$. In this case we say that π and π' are compatible with the pairing $\mathcal{P}$ and write this as $(\pi,\pi') \sim \mathcal{P}$. [Note that $\{\pi(2j-1), \pi(2j)\}$ is not necessarily equal to $\{\pi'(2j-1), \pi'(2j)\}$. Furthermore, when we write $\{\pi(2j-1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\}$ we mean as two sets, so, according to what π is, we may have $\pi(2j-1) = l_{2i-1}, \pi(2j) = l_{2i}$ or $\pi(2j-1) = l_{2i}, \pi(2j) = l_{2i-1}$. A similar situation exist for π'.] We write $\pi \sim \pi'$ to mean that $(\pi,\pi') \sim \mathcal{P}$ for some pairing $\mathcal{P}$. In this subsection we show that

$$\begin{aligned}&\sum_{\pi\sim\pi'}\int \mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n} dx_j\\ &\qquad = \frac{(2n)!}{2^n n!}\left(\frac{32h^3}{3}\right)^n E\left\{\left(\int L^x_{\lambda_\zeta}\widetilde{L}^x_{\widetilde{\lambda}_{\zeta'}}\,dx\right)^n\right\} + O(h^{3n+1}).\end{aligned} \tag{4.13}$$

In Sections 4.2 and 4.3 we show that

$$\sum_{\pi\not\sim\pi' \text{ or } (a,a')\neq(e,e)}\left|\int \mathcal{T}_h(x;\pi,\pi',a,a')\prod_{j=1}^{2n} dx_j\right| = O(h^{3n+1}). \tag{4.14}$$

Together these estimates give (2.1).

When π and π' are compatible it follows from (4.9) and (4.12) that

$$\begin{aligned}
&\mathcal{T}_h(x;\pi,\pi',e,e)\\
&\quad=\prod_{j=1}^{n}\bigl(\Delta^h\Delta^{-h}u^{\zeta}\bigl(x_{\pi(2j)}-x_{\pi(2j-1)}\bigr)\bigr)\\
&\qquad\times\prod_{j=1}^{n}u^{\zeta}\bigl(x_{\pi(2j-1)}-x_{\pi(2j-2)}\bigr)\\
&\qquad\times\prod_{k=1}^{n}\bigl(\Delta^h\Delta^{-h}u^{\zeta'}\bigl(x_{\pi'(2k)}-x_{\pi'(2k-1)}\bigr)\bigr)\\
&\qquad\times\prod_{k=1}^{n}u^{\zeta'}\bigl(x_{\pi'(2k-1)}-x_{\pi'(2k-2)}\bigr).
\end{aligned}\tag{4.15}$$

We would like to integrate $T_h(x;\pi,\pi',e,e)$ with respect to x, but this is not easy because the variables,

$$\{x_{\pi(2j)}-x_{\pi(2j-1)},x_{\pi'(2j)}-x_{\pi'(2j-1)},j\in[1,n]\}$$

and

$$\{x_{\pi(2j-1)}-x_{\pi(2j-2)},x_{\pi'(2j-1)}-x_{\pi'(2j-2)},j\in[1,n]\},$$

are not independent. To get around this difficulty we first write

$$1=\prod_{i=1}^{n}\bigl(1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\le h\}}+1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\ge h\}}\bigr)=\sum_{A\subseteq[1,\dots,n]}1_{D_A},\tag{4.16}$$

where

$$D_A=\{|x_{l_{2i}}-x_{l_{2i-1}}|\le h,i\in A\}\cap\{|x_{l_{2i}}-x_{l_{2i-1}}|>h,i\in A^c\};\tag{4.17}$$

and use it to write

$$\begin{aligned}
&\int\mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n}dx_j\\
&\quad=\int\prod_{i=1}^{n}\bigl(1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\le\sqrt{h}\}}\bigr)\mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n}dx_j+E_{1,h},
\end{aligned}\tag{4.18}$$

where

$$E_{1,h}=\sum_{A^c\ne\phi}\int_{D_A}\mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n}dx_j.\tag{4.19}$$

Let

$$w^{\zeta}(x) = |\Delta^h \Delta^{-h} u^{\zeta}(x)|. \tag{4.20}$$

We have

$$\begin{aligned}
&\left| \int_{D_A} \mathcal{T}_h(x; \pi, \pi', e, e) \prod_{j=1}^{2n} dx_j \right| \\
&\quad \le \int_{D_A} \prod_{j=1}^{n} u^{\zeta}\big(x_{\pi(2j-1)} - x_{\pi(2j-2)}\big) w^{\zeta}\big(x_{\pi(2j)} - x_{\pi(2j-1)}\big) \\
&\qquad\qquad \times \prod_{k=1}^{n} u^{\zeta'}\big(x_{\pi'(2k-1)} - x_{\pi'(2k-2)}\big) w^{\zeta'}\big(x_{\pi'(2k)} - x_{\pi'(2k-1)}\big) \prod_{j=1}^{2n} dx_j.
\end{aligned} \tag{4.21}$$

Let

$$\widetilde{D}_A = \{|x_{2j-1} - x_{2j-2}| \le h,\, j \le |A|\} \cap \{|x_{2j-1} - x_{2j-2}| > h,\, j > |A|\}. \tag{4.22}$$

Applying the Cauchy–Schwarz inequality in (4.21) to separate the terms in π from the terms in π', and then relabeling, we get

$$\begin{aligned}
&\left| \int_{D_A} \mathcal{T}_h(x; \pi, \pi', e, e) \prod_{j=1}^{2n} dx_j \right|^2 \\
&\quad \le \int_{\widetilde{D}_A} \prod_{j=1}^{n} \big(u^{\zeta}(x_{2j-1} - x_{2j-2})\big)^2 \big(w^{\zeta}(x_{2j} - x_{2j-1})\big)^2 \prod_{j=1}^{2n} dx_j \\
&\qquad \times \int_{\widetilde{D}_A} \prod_{j=1}^{n} \big(u^{\zeta'}(x_{2j-1} - x_{2j-2})\big)^2 \big(w^{\zeta'}(x_{2j} - x_{2j-1})\big)^2 \prod_{j=1}^{2n} dx_j \\
&\quad \le \left(\prod_{j=1}^{|A|} \int (w^{\zeta}(x_{2j}))^2 \, dx_{2j} \right) \left(\prod_{j=|A|+1}^{n} \int 1_{\{|x_{2j}| \ge h\}} (w^{\zeta}(x_{2j}))^2 \, dx_{2j} \right) \\
&\qquad \times \left(\prod_{j=1}^{|A|} \int (w^{\zeta'}(x_{2j}))^2 \, dx_{2j} \right) \left(\prod_{j=|A|+1}^{n} \int 1_{\{|x_{2j}| \ge h\}} (w^{\zeta'}(x_{2j}))^2 \, dx_{2j} \right) \\
&\quad \le \big(C^n h^{3n+|A^c|}\big)^2,
\end{aligned} \tag{4.23}$$

where the last inequality comes from (3.8) and (3.9). Combining this with (4.19) we see that

$$|E_{1,h}| = O(h^{3n+1}). \tag{4.24}$$

We now study

$$\widetilde{B}_h(\pi,\pi',e,e) := \int \prod_{i=1}^{n} \bigl(1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\le h\}}\bigr) \mathcal{T}_h(x;\pi,\pi',e,e) \prod_{j=1}^{2n} dx_j. \tag{4.25}$$

Recall that for each $1 \le j \le n$, $\{\pi(2j-1),\pi(2j)\} = \{l_{2i-1}, l_{2i}\}$, for some $1 \le i \le n$. We identify these relationships by setting $i = \sigma(j)$ when $\{\pi(2j-1),\pi(2j)\} = \{l_{2i-1}, l_{2i}\}$. We write

$$\begin{aligned} &\prod_{j=1}^{n} u^{\zeta}\bigl(x_{\pi(2j-1)} - x_{\pi(2j-2)}\bigr) \\ &\qquad = \prod_{j=1}^{n} \bigl(u^{\zeta}\bigl(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}\bigr) + \Delta^{h_j} u^{\zeta}\bigl(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}\bigr)\bigr), \end{aligned} \tag{4.26}$$

where $h_j = (x_{\pi(2j-1)} - x_{l_{2\sigma(j)-1}}) + (x_{l_{2\sigma(j-1)-1}} - x_{\pi(2j-2)})$. Note that because of the presence of the term $\prod_{i=1}^{n}(1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\le h\}})$ in the integral in (4.25) we need only be concerned with values of $|h_j| \le 2h$, $1 \le j \le n$.

Similarly we set $i = \sigma'(j)$ when $\{\pi'(2j-1),\pi'(2j)\} = \{l_{2i-1}, l_{2i}\}$, and write

$$\begin{aligned} &\prod_{j=1}^{n} u^{\zeta'}\bigl(x_{\pi'(2j-1)} - x_{\pi'(2j-2)}\bigr) \\ &\qquad = \prod_{j=1}^{n} \bigl(u^{\zeta'}\bigl(x_{l_{2\sigma'(j)-1}} - x_{l_{2\sigma'(j-1)-1}}\bigr) + \Delta^{h'_j} u^{\zeta'}\bigl(x_{l_{2\sigma'(j)-1}} - x_{l_{2\sigma'(j-1)-1}}\bigr)\bigr), \end{aligned} \tag{4.27}$$

where $h'_j = (x_{\pi'(2j-1)} - x_{l_{2\sigma'(j)-1}}) + (x_{l_{2\sigma'(j-1)-1}} - x_{\pi'(2j-2)})$. As above we need only be concerned with values of $|h'_j| \le 2h$, $1 \le j \le n$.

We substitute (4.26) and (4.27) into the term $\mathcal{T}_h(x;\pi,\pi',e,e)$ in (4.25) and expand the products so that we can write $\widetilde{B}_h(\pi,\pi',e,e)$ as a sum of many terms to get

$$\begin{aligned} &\widetilde{B}_h(\pi,\pi',e,e) \\ &\qquad := \int \prod_{i=1}^{n} \bigl(1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\le h\}}\bigr) \widetilde{\mathcal{T}}_h(x;\pi,\pi',e,e) \prod_{j=1}^{2n} dx_j + E_{2,h}, \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} &\widetilde{\mathcal{T}}_h(x;\pi,\pi',e,e) \\ &\quad = \prod_{i=1}^{n} \Delta^h \Delta^{-h} u^{\zeta}(x_{l_{2i}} - x_{l_{2i-1}}) \times \prod_{j=1}^{n} u^{\zeta}\bigl(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}\bigr) \\ &\qquad \times \prod_{i=1}^{n} \Delta^h \Delta^{-h} u^{\zeta'}(x_{l_{2i}} - x_{l_{2i-1}}) \times \prod_{j=1}^{n} u^{\zeta'}\bigl(x_{l_{2\sigma'(j)-1}} - x_{l_{2\sigma'(j-1)-1}}\bigr), \end{aligned} \tag{4.29}$$

and

$$E_{2,h} = \sum_{A,A' \subseteq [1,\dots,n]} E_{2,h;A,A'}, \tag{4.30}$$

where

$$\begin{aligned}
E_{2,h;A,A'} := \int \prod_{i=1}^{n} \big(1_{\{|x_{l_{2i}} - x_{l_{2i-1}}| \le h\}}\big) \prod_{j=1}^{n} \Delta^h \Delta^{-h} u^\zeta \big(x_{\pi(2j)} - x_{\pi(2j-1)}\big) \\
\times \prod_{j \in A} u^\zeta \big(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}\big) \prod_{j \in A^c} \Delta^{h_j} u^\zeta \big(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}\big) \\
\times \prod_{k=1}^{n} \Delta^h \Delta^{-h} u^{\zeta'} \big(x_{\pi'(2k)} - x_{\pi'(2k-1)}\big) \\
\times \prod_{k \in A'} u^{\zeta'} \big(x_{l_{2\sigma'(k)-1}} - x_{l_{2\sigma'(k-1)-1}}\big) \\
\times \prod_{k \in A'^c} \Delta^{h'_k} u^{\zeta'} \big(x_{l_{2\sigma'(k)-1}} - x_{l_{2\sigma'(k-1)-1}}\big) \prod_{j=1}^{2n} dx_j,
\end{aligned} \tag{4.31}$$

and A^c and A'^c are not both empty.

Using (3.5) we see that

$$\begin{aligned}
|E_{2,h;A,A'}| \le C^n h^{|A^c| + |A'^c|} \\
\times \int \prod_{i=1}^{n} \big(1_{\{|x_{l_{2i}} - x_{l_{2i-1}}| \le h\}}\big) \prod_{j=1}^{n} w^\zeta \big(x_{\pi(2j)} - x_{\pi(2j-1)}\big) \\
\times \prod_{j=1}^{n} u^\zeta \big(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}\big) \prod_{k=1}^{n} w^{\zeta'} \big(x_{\pi'(2k)} - x_{\pi'(2k-1)}\big) \\
\times \prod_{k=1}^{n} u^{\zeta'} \big(x_{l_{2\sigma(k)-1}} - x_{l_{2\sigma(k-1)-1}}\big) \, dx_j.
\end{aligned} \tag{4.32}$$

Using the Cauchy–Schwarz inequality as in (4.23), we see that

$$|E_{2,h;A,A'}| \le C h^{3n + |A^c| + |A'^c|}. \tag{4.33}$$

It now follows from (4.30) that

$$E_{2,h} = O(h^{3n+1}). \tag{4.34}$$

We now consider

$$\overline{B}_h(\pi, \pi', e, e) := \int \prod_{i=1}^{n} \big(1_{\{|x_{l_{2i}} - x_{l_{2i-1}}| \le h\}}\big) \widetilde{\mathcal{T}}_h(x; \pi, \pi', e, e) \prod_{j=1}^{2n} dx_j. \tag{4.35}$$

Using (4.18) and (4.24) we see that

$$\overline{B}_h(\pi,\pi',e,e)=\int \widetilde{\mathcal{T}}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n}dx_j+O(h^{3n+1}). \tag{4.36}$$

Using (4.29) we see that

$$\begin{aligned}\int \widetilde{\mathcal{T}}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n}dx_j\\ &=\int\prod_{i=1}^{n}\Delta^h\Delta^{-h}u^{\zeta}\big(x_{l_{2i}}-x_{l_{2i-1}}\big)\\ &\quad\times\prod_{j=1}^{n}u^{\zeta}\big(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}}\big)\prod_{i=1}^{n}\Delta^h\Delta^{-h}u^{\zeta'}(x_{l_{2i}}-x_{l_{2i-1}})\\ &\quad\times\prod_{j=1}^{n}u^{\zeta'}\big(x_{l_{2\sigma'(j)-1}}-x_{l_{2\sigma'(j-1)-1}}\big)\prod_{i=1}^{2n}dx_i.\end{aligned}\tag{4.37}$$

We make the change of variables $x_{l_{2i}}\to x_{l_{2i}}+x_{l_{2i-1}}$, $i=1,\ldots,n$ and write this as

$$\begin{aligned}&\int\prod_{i=1}^{n}\Delta^h\Delta^{-h}u^{\zeta}(x_{l_{2i}})\prod_{j=1}^{n}u^{\zeta}\big(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}}\big)\\ &\qquad\times\prod_{i=1}^{n}\Delta^h\Delta^{-h}u^{\zeta'}(x_{l_{2i}})\prod_{j=1}^{n}u^{\zeta'}\big(x_{l_{2\sigma'(j)-1}}-x_{l_{2\sigma'(j-1)-1}}\big)\prod_{i=1}^{2n}dx_i.\end{aligned}$$

We now rearrange the integrals with respect to $x_{l_2},x_{l_4},\ldots,x_{l_{2n}}$ and get

$$\begin{aligned}\int \widetilde{\mathcal{T}}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n}dx_j\\ &=\left(\int(\Delta^h\Delta^{-h}u^{\zeta}(x))(\Delta^h\Delta^{-h}u^{\zeta'}(x))\,dx\right)^n\\ &\quad\times\int\prod_{j=1}^{n}u^{\zeta}\big(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}}\big)u^{\zeta'}\big(x_{l_{2\sigma'(j)-1}}-x_{l_{2\sigma'(j-1)-1}}\big)\\ &\quad\times\prod_{i=1}^{n}dx_{l_{2i-1}}.\end{aligned}\tag{4.38}$$

Writing $y_{\sigma(j)} = x_{l_{2\sigma(j)-1}}$, $y_{\sigma'(j)} = x_{l_{2\sigma'(j)-1}}$ and using (3.8), we can write this as

$$\begin{aligned}
&\int \widetilde{\mathcal{T}}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n} dx_j \\
&\quad = \left(\frac{8h^3(1+O(h))}{3}\right)^n \\
&\qquad \times \int \prod_{j=1}^{n} u^{\zeta}\bigl(y_{\sigma(j)} - y_{\sigma(j-1)}\bigr) u^{\zeta'}\bigl(y_{\sigma'(j)} - y_{\sigma'(j-1)}\bigr) \prod_{i=1}^{n} dy_i .
\end{aligned} \tag{4.39}$$

Considering (4.28), (4.34) and (4.39) we see that

$$\begin{aligned}
&\int \mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n} dx_j \\
&\quad = \left(\frac{8h^3}{3}\right)^n \int \prod_{j=1}^{n} u^{\zeta}\bigl(y_{\sigma(j)} - y_{\sigma(j-1)}\bigr) u^{\zeta'}\bigl(y_{\sigma'(j)} - y_{\sigma'(j-1)}\bigr) \\
&\qquad \times \prod_{i=1}^{n} dy_i + O(h^{3n+1}).
\end{aligned} \tag{4.40}$$

In the first paragraph of this subsection we explain what we mean by $(\pi,\pi') \sim \mathcal{P}$, for a pairing $\mathcal{P} = \{(l_{2i-1}, l_{2i}), 1 \le i \le n\}$ of the integers $[1, 2n]$ and permutations π, π' of $[1, 2n]$ that are compatible with $\mathcal{P}$. Obviously, there are many such pairs. There are 2^{2n} ways we can interchange the two elements of each pair $\pi(2j-1), \pi(2j)$, and $\pi'(2j-1), \pi'(2j)$ without changing (4.40). Furthermore, by permuting the pairs $\{\pi(2j-1), \pi(2j)\}$ we give rise to all possible permutations σ of $[1, n]$, and similarly for π'. Consequently,

$$\begin{aligned}
&\sum_{(\pi,\pi')\sim\mathcal{P}} \int \mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n} dx_j \\
&\quad = \left(\frac{32h^3}{3}\right)^n \sum_{\sigma,\sigma'} \int \prod_{j=1}^{n} u^{\zeta}\bigl(y_{\sigma(j)} - y_{\sigma(j-1)}\bigr) u^{\zeta'}\bigl(y_{\sigma'(j)} - y_{\sigma'(j-1)}\bigr) \prod_{i=1}^{n} dy_i \\
&\qquad + O(h^{3n+1}) \\
&\quad = \left(\frac{32h^3}{3}\right)^n E\left\{\left(\int L^x_{\lambda_\zeta} \widetilde{L}^x_{\tilde{\lambda}_{\zeta'}}\, dx\right)^n\right\} + O(h^{3n+1}).
\end{aligned} \tag{4.41}$$

Here the sum in the second line runs over all permutations σ, σ' of $\{1, \dots, n\}$ and $\sigma(0) = \sigma'(0) = 0$. The final line of (4.41) follows from the Kac moment formula (4.3).

Since there are $(2n)!/(2^n n!)$ pairings of the $2n$ elements $\{1,\dots,2n\}$ we obtain (4.13).

In the next two subsections we obtain (4.14).

4.2. *$a=a'=e$ without compatible permutations.* Consider the multigraph $G_{\pi,\pi'}$ whose vertices consist of $\{1,\dots,2n\}$ and assign an edge between the vertices $\pi(2j-1)$ and $\pi(2j)$ for each $j=1,\dots,n$ and similarly between $\pi'(2j-1)$ and $\pi'(2j)$ for each $j=1,\dots,n$. Each vertex is connected to two edges, and it is possible to have two edges between any two vertices i,j. Note that the connected components C_j, $j=1,\dots,k$ of $G_{\pi,\pi'}$ consist of cycles. (For example, in Section 4.1, all the cycles are of order two.)

Let $C_j=\{j_1,\dots,j_{l(j)}\}$ be written in cyclic order where $l(j)=|C_j|$. Clearly $\sum_{j=1}^k l(j)=2n$. We show that when all the cycles are not of order two, as they are in the case of compatible permutations considered in Section 4.1, then

$$\left|\int \mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n} dx_j\right| \le Ch^{3n+1}. \tag{4.42}$$

Since we only need an upper bound, we take absolute values in the integrand and get

$$\begin{aligned} &\left|\int \mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n} dx_j\right| \\ &\quad\le \int \prod_{j=1}^{k}\bigl(w(x_{j_2}-x_{j_1})\cdots w\bigl(x_{j_{l(j)}}-x_{j_{l(j)-1}}\bigr)w\bigl(x_{j_1}-x_{j_{l(j)}}\bigr)\bigr) \qquad (4.43)\\ &\qquad\times \prod_{j=1}^{n} u\bigl(x_{\pi(2j-1)}-x_{\pi(2j-2)}\bigr)u\bigl(x_{\pi'(2j-1)}-x_{\pi'(2j-2)}\bigr)\prod_{j=1}^{2n} dx_j, \end{aligned}$$

where we use the notation $u(x)$ to denote either $u^{\zeta}(x)$ or $u^{\zeta'}(x)$, and $w(x)$ to denote either $w^{\zeta}(x)$ or $w^{\zeta'}(x)$. [$w^{\zeta}(x)$ is defined in (4.20).] Note that we group the functions w according to the cycles.

For each $j=1,\dots,k$ we set $y_{j_i}=x_{j_i}-x_{j_{i-1}}$, $i=2,\dots,l(j)$, and note that $\sum_{i=2}^{l(j)} y_{j_i}=-(x_{j_1}-x_{j_{l(j)}})$. It is easy to see that the $2n-k$ variables $\{y_{j_i}\mid j=1,\dots,k; i=2,\dots,l(j)\}$ are linearly independent. We then choose additional k variables $z_l; l=1,\dots,k$ from amongst the variables $\{x_{\pi(2j-1)}-x_{\pi(2j-2)}, x_{\pi'(2j-1)}-x_{\pi'(2j-2)}; 1\le j\le n\}$ so that $\{y_{j_i}\mid j=1,\dots,k; i=2,\dots,l(j)\}\cup\{z_l\mid l=1,\dots,k\}$ are linearly independent and generate $\{x_1,\dots,x_{2n}\}$. We make this change of variables and use the fact that $u(x)$ is bounded and integrable, followed by (3.6) and (3.10), to see that

$$
\begin{aligned}
&\left|\int \mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n} dx_j\right| \\
&\quad\le C\prod_{j=1}^{k}\left(\int w(y_{j_2})\cdots w(y_{j_{l(j)}})w\left(\sum_{i=2}^{l(j)} y_{j_i}\right)\prod_{i=2}^{l(j)} dy_{j_i}\right) \\
&\quad\le C\prod_{j=1}^{k}\sup_x |w(x)|\left(\int w(y)\,dy\right)^{l(j)-1} \\
&\quad\le C\prod_{j=1}^{k} h^{1+2(l(j)-1)} = C\prod_{j=1}^{k} h^{2l(j)-1}.
\end{aligned}
\tag{4.44}
$$

(Note that the only dependence on ζ and ζ' is in the constant C.)

Since $\sum_{j=1}^{k} l(j) = 2n$, we see from (4.44) that

$$
\left|\int \mathcal{T}_h(x;\pi,\pi',e,e)\prod_{j=1}^{2n} dx_j\right| \le Ch^{4n-k} = Ch^{3n}h^{n-k}. \tag{4.45}
$$

It is easily seen that for noncompatible permutations we have $k < n$, which proves (4.42).

4.3. *When $a = a' = e$ does not hold.* We now consider all partitions π and π' when $a = a' = e$ does not hold. Consider the basic formula (4.9). Since we only need an upper bound, we take absolute values in the integrand as in (4.43). Since $a = a' = e$ does not hold there are terms in which only one Δ^h is applied to a u^ζ or $u^{\zeta'}$.

We use the notation u and w defined right after (4.43). If there are $k < 2n$ factors of the type w, then there are $2(2n-k)$ factors of the type $\Delta^{\pm h}u$. We use (3.5) to pull out a factor of

$$
h^{2(2n-k)} \tag{4.46}
$$

from the basic formula (4.9), and are left with an integral like the one on the right-hand side of (4.43) except that there are k factors of the form w which may be linked in chains as well as in cycles, and there are $4n-k$ factors of type u. We denote this integral by J_h.

As in (4.43), we arrange the w factors into cycles and chains. We then change variables and integrate the w factors. As in (4.44) a cycle of length l gives a contribution that is bounded by $Ch^{1+2(l-1)} = Ch^{2l-1}$. In addition, by (3.10), chain of length l' gives a contribution that is bounded by $Ch^{2l'}$.

If there are j cycles of lengths $l(i), i = 1, \dots, j$ and j' chains of lengths $l'(i), i = 1, \dots, j'$ we have

$$\sum_{i=1}^{j} l(i) + \sum_{i=1}^{j'} l'(i) = k. \tag{4.47}$$

Therefore,

$$J_h \le C h^{(2\sum_{i=1}^{j} l(i))-j} h^{2\sum_{i=1}^{j'} l'(i)} \le C h^{2k-j}. \tag{4.48}$$

Together with (4.46) this shows that

$$\left| \int \mathcal{T}_h(x; \pi, \pi', a, a') \prod_{j=1}^{2n} dx_j \right| \le C h^{4n-j} = C h^{3n} h^{n-j}. \tag{4.49}$$

As in (4.45) we see that

$$\left| \int \mathcal{T}_h(x; \pi, \pi', a, a') \prod_{j=1}^{2n} dx_j \right| \le C h^{3n+1}. \tag{4.50}$$

We have established (4.13) when m is even. We now show that we get the same estimates when u^{ζ} and $u^{\zeta'}$ are replaced by $u^{\zeta,\sharp}$ and $u^{\zeta',\sharp}$ [see (4.8) and (4.9)].

The key observation that explains this is that in applying the product formula (4.5), the only terms of the form $u^{\zeta}(x - y)$ that may have x replaced by $x + h$ are those to which Δ_x^h is not applied. Similarly y may be replaced by $y + h$ only if Δ_y^h is not applied to a term of the form $u^{\zeta}(x - y)$. Consequently, in evaluating (4.10) with $\mathcal{T}_h$ replaced by $\mathcal{T}_h'$ we still have $\Delta^h \Delta^{-h} u^{\zeta,\sharp} = \Delta^h \Delta^{-h} u^{\zeta}$ and similarly for $\Delta^h \Delta^{-h} u^{\zeta',\sharp}$.

It is easy to see that the presence of the terms in $u^{\zeta,\sharp}$ or $\Delta^{\pm h} u^{\zeta,\sharp}$, or in $u^{\zeta',\sharp}$ or $\Delta^{\pm h} u^{\zeta',\sharp}$ have no effect on the integrals that are $O(h^{3n+1/2})$ as $h \to 0$. [That is, the terms that are equal to 0 in (4.10).] This is because in evaluating these expressions we either integrate over all of R^1 or else use bounds that hold on all of R^1.

They do have an effect on the terms for which the limit in (4.10) are not zero. For example, instead of the right-hand side of (4.40), we now have

$$\begin{aligned} &\left(\frac{8h^3}{3}\right)^n \int \prod_{j=1}^{n} u^{\zeta,\sharp}(y_{\sigma(j)} - y_{\sigma(j-1)}) u^{\zeta',\sharp}(y_{\sigma'(j)} - y_{\sigma'(j-1)}) \\ &\qquad \times \prod_{i=1}^{n} dy_i + O(h^{3n+1}). \end{aligned} \tag{4.51}$$

Suppose that $u^{\zeta,\sharp}(y_{\sigma(i)} - y_{\sigma(i-1)}) = u^{\zeta}(y_{\sigma(i)} - y_{\sigma(i-1)} \pm h)$. We write this term as

$$u^{\zeta,\sharp}(y_{\sigma(i)} - y_{\sigma(i-1)}) = u^{\zeta}(y_{\sigma(i)} - y_{\sigma(i-1)}) + \Delta^{\pm h} u^{\zeta}(y_{\sigma(i)} - y_{\sigma(i-1)}) \tag{4.52}$$

and similarly for $u^{\zeta',\sharp}$. Substituting these expressions into (4.51) and using (3.5), it is easy to see that (4.51) is asymptotically equivalent to the right-hand side of (4.13) when m is even. (The error term may be different.) Thus we see that replacing u^{ζ} and $u^{\zeta'}$ by $u^{\zeta,\sharp}$ and $u^{\zeta',\sharp}$ does not change (4.10) when m is even.

It is rather obvious that the limit in (4.10) is zero when m is odd because in this case we can not construct a graph with all cycles of order 2. The extension of this limit when u^{ζ} and $u^{\zeta'}$ are replaced by $u^{\zeta,\sharp}$ and $u^{\zeta',\sharp}$ follows as above.

5. Proof of Lemma 2.3. For $h=0$ it suffices to show that

$$G_0(s,t) := E\left\{\left(\int L_s^x \widetilde{L}_t^x\,dx\right)^n\right\}$$

is a nonnegative, polynomially-bounded, continuous, increasing function of (s,t). The fact that $G_0(s,t)$ is a nonnegative, increasing function of (s,t) follows immediately from the fact that the local times L_s^x and $\widetilde{L}_t^x$ have these properties.

To prove continuity we note that for all $|r|,|r'|\le r_0$, $\int L_{s+r}^x \widetilde{L}_{t+r'}^x\,dx \le \int L_{s+r_0}^x \widetilde{L}_{t+r_0}^x\,dx$. Therefore continuity follows from the Dominated Convergence theorem and the continuity of local times once we show that for all s,t,

$$\int L_s^x \widetilde{L}_t^x\,dx \tag{5.1}$$

has all moments. It follows from the Cauchy–Schwarz inequality, the scaling relationship (1.9) and (2.2), that

$$\begin{aligned}
E\left\{\left(\int L_s^x \widetilde{L}_t^x\,dx\right)^n\right\} &\le E\left\{\left(\int (L_s^x)^2\,dx\right)^{n/2}\right\}E\left\{\left(\int (\widetilde{L}_t^x)^2\,dx\right)^{n/2}\right\}\\
&\le (st)^{3n/4}E\left\{\left(\int (L_1^x)^2\,dx\right)^{n/2}\right\}E\left\{\left(\int (\widetilde{L}_1^x)^2\,dx\right)^{n/2}\right\}\\
&\le C(st)^{3n/4}.
\end{aligned}\tag{5.2}$$

In addition to showing that (5.1) has all moments, this also shows that $G_0(s,t)$ is a polynomially bounded function of (s,t).

We now consider $F_h(s,t)$ for $h>0$. It suffices to show that

$$G_h(s,t) := E\left(\left(\int (L_s^{x+h}-L_s^x)(\widetilde{L}_t^{x+h}-\widetilde{L}_t^x)\,dx\right)^m\right) \tag{5.3}$$

is a nonnegative, polynomially-bounded, continuous, increasing function of (s,t).

Let W_t denote Brownian motion and let $f\in\mathcal{S}(R^1)$ be a positive symmetric function supported on $[-1,1]$ with $\int f(x)\,dx=1$. Set $f_\varepsilon(x)=f(x/\varepsilon)/\varepsilon$

and

$$L^x_{s,\varepsilon} = \int_0^s f_\varepsilon(W_r - x)\,dr. \tag{5.4}$$

It follows from [9], Lemma 2.4.1, that

$$\begin{aligned} &E\Biggl(\prod_{j=1}^m (L_s^{x_j+h} - L_s^{x_j})(\widetilde{L}_t^{x_j+h} - \widetilde{L}_t^{x_j})\Biggr) \\ &\quad = \lim_{\varepsilon\to 0} E\Biggl(\prod_{j=1}^m (L_{s,\varepsilon}^{x_j+h} - L_{s,\varepsilon}^{x_j})(\widetilde{L}_{t,\varepsilon}^{x_j+h} - \widetilde{L}_{t,\varepsilon}^{xj})\Biggr) \\ &\quad = \lim_{\varepsilon\to 0} E\Biggl(\prod_{j=1}^m (L_{s,\varepsilon}^{x_j+h} - L_{s,\varepsilon}^{x_j})\Biggr) E\Biggl(\prod_{j=1}^m (\widetilde{L}_{t,\varepsilon}^{x_j+h} - \widetilde{L}_{t,\varepsilon}^{x_j})\Biggr). \end{aligned} \tag{5.5}$$

Using the Fourier transform,

$$L^{x+h}_{s,\varepsilon} - L^x_{s,\varepsilon} = \int e^{-ipx}(e^{-iph} - 1)\widehat{f}(\varepsilon p)\int_0^s e^{ipW_r}\,dr\,dp, \tag{5.6}$$

we have

$$\begin{aligned} E\Biggl(\prod_{j=1}^m (L_{s,\varepsilon}^{x_j+h} - L_{s,\varepsilon}^{x_j})\Biggr) &= \int_{R^m} \prod_{j=1}^m e^{-ip_jx_j}(e^{-ip_jh} - 1)\widehat{f}(\varepsilon p_j) \\ &\qquad \times \int_{[0,s]^m} E\Biggl(\prod_{j=1}^m e^{ip_jW_{r_j}}\Biggr)\prod_{j=1}^m dr_j\,dp_j. \end{aligned} \tag{5.7}$$

Note that

$$\begin{aligned} &\int_{[0,s]^m} E\Biggl(\prod_{j=1}^m e^{ip_jW_{r_j}}\Biggr)\prod_{j=1}^m dr_j \\ &\quad = \sum_\pi \int_{\{0\le r_1\le\cdots\le r_m\le s\}} E\Biggl(\prod_{j=1}^m e^{ip_{\pi(j)}W_{r_j}}\Biggr)\prod_{j=1}^m dr_j \\ &\quad = \sum_\pi \int_{\{0\le r_1\le\cdots\le r_m\le s\}} E\Biggl(\prod_{j=1}^m e^{i(\sum_{k=j}^m p_{\pi(k)})(W_{r_j} - W_{r_{j-1}})}\Biggr)\prod_{j=1}^m dr_j \\ &\quad = \sum_\pi \int_{\{0\le r_1\le\cdots\le r_m\le s\}} \prod_{j=1}^m e^{-(\sum_{k=j}^m p_{\pi(k)})^2(r_j - r_{j-1})}\prod_{j=1}^m dr_j. \end{aligned} \tag{5.8}$$

Since this is bounded and integrable in $p_1,\dots,p_m$ and $\widehat{f}(\varepsilon p) \le C$, we can take the limit as ε goes to zero in (5.7) and hence in (5.5), to see that

$$
\begin{aligned}
&E\left(\prod_{j=1}^{m}(L_s^{x_j+h}-L_s^{x_j})(\widetilde{L}_t^{x_j+h}-\widetilde{L}_t^{x_j})\right)\\
&\quad=\int_{R^{2m}}\prod_{j=1}^{m}e^{-i(p_j+p_j')x_j}(e^{-ip_jh}-1)(e^{-ip_j'h}-1)\\
&\qquad\times\int_{[0,s]^m}E\left(\prod_{j=1}^{m}e^{ip_jW_{r_j}}\right)\\
&\qquad\qquad\times\int_{[0,t]^m}E\left(\prod_{j=1}^{m}e^{ip_j'W_{r_j'}}\right)\prod_{j=1}^{m}dr_j'\,dr_j\,dp_j\,dp_j'.
\end{aligned}\tag{5.9}
$$

It now follows from Parseval's theorem that

$$
\begin{aligned}
G_h(s,t)&=\int E\left(\prod_{j=1}^{m}(L_s^{x_j+h}-L_s^{x_j})(\widetilde{L}_t^{x_j+h}-\widetilde{L}_t^{x_j})\right)\prod_{j=1}^{m}dx_j\\
&=\frac{1}{(2\pi)^m}\int_{R^m}\prod_{j=1}^{m}|e^{-ip_jh}-1|^2\\
&\qquad\times\int_{[0,s]^m}E\left(\prod_{j=1}^{m}e^{ip_jW_{r_j}}\right)\\
&\qquad\qquad\times\int_{[0,t]^m}E\left(\prod_{j=1}^{m}e^{ip_jW_{r_j'}}\right)\prod_{j=1}^{m}dr_j'\,dr_j\,dp_j.
\end{aligned}\tag{5.10}
$$

The fact that $G_h(s,t)$ is a nonnegative, increasing function of (s,t) follows from this and (5.8). The fact that $G_h(s,t)$ is a polynomially-bounded, continuous function of (s,t), follows, as in the proof for $G_0(s,t)$, if we note that by translation invariance $\int(L_s^{x+h})^2\,dx=\int(L_s^x)^2\,dx$ so that, as in (5.2),

$$
\begin{aligned}
E\left\{\left(\int L_s^{x+h}\widetilde{L}_t^x\,dx\right)^n\right\}&\le E\left\{\left(\int(L_s^{x+h})^2\,dx\right)^{n/2}\right\}E\left\{\left(\int(\widetilde{L}_t^x)^2\,dx\right)^{n/2}\right\}\\
&\le E\left\{\left(\int(L_s^{x})^2\,dx\right)^{n/2}\right\}E\left\{\left(\int(\widetilde{L}_t^x)^2\,dx\right)^{n/2}\right\}\\
&\le C(st)^{3n/4}.
\end{aligned}\tag{5.11}
$$

6. Proof of Theorem 1.1. The proof of Theorem 1.1 follows from the next lemma exactly as in the proof of Theorem 1.2 on page 400.

LEMMA 6.1. *For each integer $m \ge 0$ and $t \in R_+$,*

$$\lim_{h\to 0} E\left(\left(\frac{\int (L_t^{x+h} - L_t^x)^2\,dx - 4h}{h^{3/2}}\right)^m\right) = \begin{cases} \dfrac{(2n)!}{2^n n!}\left(\dfrac{64}{3}\right)^n E\left\{\left(\displaystyle\int (L_t^x)^2\,dx\right)^n\right\}, & \text{if } m = 2n, \\ 0, & \text{otherwise.}\end{cases} \tag{6.1}$$

We use the next lemma in the proof of Lemma 6.1. It is proved in Section 7.

LEMMA 6.2. *Let λ_ζ be an exponential random variable with mean $1/\zeta$. For each integer $m \ge 0$,*

$$\lim_{h\to 0} E\left(\left(\frac{\int (L_{\lambda_\zeta}^{x+h} - L_{\lambda_\zeta}^x)^2\,dx - 4h\lambda_\zeta}{h^{3/2}}\right)^m\right) = \begin{cases} \dfrac{(2n)!}{2^n n!}\left(\dfrac{64}{3}\right)^n E\left\{\left(\displaystyle\int (L_{\lambda_\zeta}^x)^2\,dx\right)^n\right\}, & \text{if } m = 2n, \\ 0, & \text{otherwise.}\end{cases} \tag{6.2}$$

PROOF OF LEMMA 6.1. We write (6.2) as

$$\begin{aligned} &\lim_{h\to 0} \int_0^\infty e^{-\zeta s} E\left(\left(\frac{\int (L_s^{x+h} - L_s^x)^2\,dx - 4hs}{h^{3/2}}\right)^m\right) ds \\ &\qquad = \int_0^\infty e^{-\zeta s} E\left\{\eta^m \left(\frac{64}{3}\int (L_s^x)^2\,dx\right)^{m/2}\right\} ds. \end{aligned} \tag{6.3}$$

Let

$$\widehat{F}_{m,h}(s) := E\left(\left(\frac{\int (L_s^{x+h} - L_s^x)^2\,dx - 4hs}{h^{3/2}}\right)^m\right), \qquad h > 0, \tag{6.4}$$

and

$$\widehat{F}_{m,0}(s) := E\left\{\eta^m \left(\frac{64}{3}\int (L_s^x)^2\,dx\right)^{m/2}\right\}.$$

Then (6.3) can be written as

$$\lim_{h\to 0} \int_0^\infty e^{-\zeta s} \widehat{F}_{m,h}(s)\,ds = \int_0^\infty e^{-\zeta s} \widehat{F}_{m,0}(s)\,ds. \tag{6.5}$$

We consider first the case when m is even and write $m = 2n$. In this case $\widehat{F}_{2n,h}(s) \ge 0$ and the extended continuity theorem [5], Chapter XIII, Section 1, Theorem 2a, applied to (6.5) implies that

$$\lim_{h\to 0} \int_0^t \widehat{F}_{2n,h}(s)\,ds = \int_0^t \widehat{F}_{2n,0}(s)\,ds \tag{6.6}$$

for all t. In particular,

$$\lim_{h\to 0}\int_t^{t+\delta}\widehat{F}_{2n,h}(s)\,ds=\int_t^{t+\delta}\widehat{F}_{2n,0}(s)\,ds. \tag{6.7}$$

It follows from the proof of Lemma 5 that $\widehat{F}_{2n,0}(t)$ is continuous in t. Consequently,

$$\lim_{\delta\to 0}\lim_{h\to 0}\frac{1}{\delta}\int_t^{t+\delta}\widehat{F}_{2n,h}(s)\,ds=\widehat{F}_{2n,0}(t). \tag{6.8}$$

When $t=0$ we get

$$\lim_{\delta\to 0^+}\lim_{h\to 0^+}\frac{1}{\delta}\int_0^{\delta}\widehat{F}_{2n,h}(s)\,ds=0. \tag{6.9}$$

To obtain (6.1) when m is even we must show that

$$\lim_{h\to 0}\widehat{F}_{2n,h}(t)=\widehat{F}_{2n,0}(t). \tag{6.10}$$

This follows from (6.8) once we show that

$$\lim_{\delta\to 0}\lim_{h\to 0}\frac{1}{\delta}\int_t^{t+\delta}\widehat{F}_{2n,h}(s)\,ds=\lim_{h\to 0}\widehat{F}_{2n,h}(t). \tag{6.11}$$

We proceed to obtain (6.11).

For $s\ge t$, we write

$$\begin{aligned}
&\int (L_s^{x+h}-L_s^x)^2\,dx-4hs\\
&\qquad=\biggl\{\int (L_t^{x+h}-L_t^x)^2\,dx-4ht\biggr\}\\
&\qquad\quad+\biggl\{\int [(L_s^{x+h}-L_s^x)-(L_t^{x+h}-L_t^x)]^2\,dx-4h(s-t)\biggr\}\\
&\qquad\quad+2\biggl\{\int (L_t^{x+h}-L_t^x)[(L_s^{x+h}-L_s^x)-(L_t^{x+h}-L_t^x)]\,dx\biggr\}.
\end{aligned} \tag{6.12}$$

We use the triangle inequality with respect to the norm $\|\cdot\|_{2n}$ to see that

$$\begin{aligned}
&\widehat{F}^{1/(2n)}_{2n,h}(s)\\
&\quad\le \widehat{F}^{1/(2n)}_{2n,h}(t)+\biggl\{E\biggl[\frac{1}{h^{3/2}}\biggl\{\int [(L_s^{x+h}-L_s^x)-(L_t^{x+h}-L_t^x)]^2\,dx\\
&\qquad\qquad -4h(s-t)\biggr\}\biggr]^{2n}\biggr\}^{1/(2n)}\\
&\qquad+2\biggl\{E\biggl[\frac{1}{h^{3/2}}\biggl\{\int (L_t^{x+h}-L_t^x)[(L_s^{x+h}-L_s^x)\\
&\qquad\qquad -(L_t^{x+h}-L_t^x)]\,dx\biggr\}\biggr]^{2n}\biggr\}^{1/(2n)}.
\end{aligned} \tag{6.13}$$

Note that

$$\int [(L_s^{x+h} - L_s^x) - (L_t^{x+h} - L_t^x)]^2 \, dx \stackrel{\mathcal{L}}{=} \int (L_{s-t}^{x+h} - L_{s-t}^x)^2 \, dx \tag{6.14}$$

and

$$\begin{aligned} &\int [L_t^{x+h} - L_t^x][(L_s^{x+h} - L_s^x) - (L_t^{x+h} - L_t^x)] \, dx \\ &\qquad \stackrel{\mathcal{L}}{=} \int [L_t^{x+h} - L_t^x][\widetilde{L}_{s-t}^{x+h} - \widetilde{L}_{s-t}^x] \, dx. \end{aligned} \tag{6.15}$$

Hence we can write (6.13) as

$$\begin{aligned} \widehat{F}_{2n,h}^{1/(2n)}(s) &\le \widehat{F}_{2n,h}^{1/(2n)}(t) + \widehat{F}_{2n,h}^{1/(2n)}(s-t) \\ &\qquad + 2\left\{ E\left[\frac{1}{h^{3/2}} \int [L_t^{x+h} - L_t^x][\widetilde{L}_{s-t}^{x+h} - \widetilde{L}_{s-t}^x] \, dx \right]^{2n} \right\}^{1/(2n)}. \end{aligned} \tag{6.16}$$

We now use the triangle inequality with respect to the norm in $L^{2n}([t, t + \delta], \delta^{-1} \, ds)$ to see that

$$\begin{aligned} &\left\{ \frac{1}{\delta} \int_t^{t+\delta} \widehat{F}_{2n,h}(s) \, ds \right\}^{1/(2n)} \\ &\quad \le \widehat{F}_{2n,h}^{1/(2n)}(t) + \left\{ \frac{1}{\delta} \int_t^{t+\delta} \widehat{F}_{2n,h}(s-t) \, ds \right\}^{1/(2n)} \\ &\qquad + 2\left\{ \frac{1}{\delta} \int_t^{t+\delta} E\left[\frac{1}{h^{3/2}} \int [L_t^{x+h} - L_t^x][\widetilde{L}_{s-t}^{x+h} - \widetilde{L}_{s-t}^x] \, dx \right]^{2n} ds \right\}^{1/(2n)}. \end{aligned} \tag{6.17}$$

A similar argument starting with (6.12) shows that

$$\begin{aligned} &\left\{ \frac{1}{\delta} \int_t^{t+\delta} \widehat{F}_{2n,h}(s) \, ds \right\}^{1/(2n)} \\ &\quad \ge \widehat{F}_{2n,h}^{1/(2n)}(t) - \left\{ \frac{1}{\delta} \int_t^{t+\delta} \widehat{F}_{2n,h}(s-t) \, ds \right\}^{1/(2n)} \\ &\qquad - 2\left\{ \frac{1}{\delta} \int_t^{t+\delta} E\left[\frac{1}{h^{3/2}} \int [L_t^{x+h} - L_t^x][\widetilde{L}_{s-t}^{x+h} - \widetilde{L}_{s-t}^x] \, dx \right]^{2n} ds \right\}^{1/(2n)}. \end{aligned} \tag{6.18}$$

Since

$$\frac{1}{\delta} \int_t^{t+\delta} \widehat{F}_{2n,h}(s-t) \, ds = \frac{1}{\delta} \int_0^{\delta} \widehat{F}_{2n,h}(s) \, ds, \tag{6.19}$$

we see from (6.9) that to prove (6.11), it only remains to show that

$$\lim_{\delta \to 0^+} \limsup_{h \to 0} \frac{1}{\delta} \int_t^{t+\delta} E\left[\frac{1}{h^{3/2}} \int [L_t^{x+h} - L_t^x][\widetilde{L}_{s-t}^{x+h} - \widetilde{L}_{s-t}^x] \, dx \right]^{2n} ds = 0. \tag{6.20}$$

By the monotonicity property of $F_h(s,t;m)$ given in Lemma 2.3,

$$
\begin{aligned}
&\frac{1}{\delta}\int_t^{t+\delta} E\biggl[\frac{1}{h^{3/2}}\int [L_t^{x+h}-L_t^x][\widetilde{L}_{s-t}^{x+h}-\widetilde{L}_{s-t}^x]\,dx\biggr]^{2n}\,ds \\
&\qquad \le E\biggl[\frac{1}{h^{3/2}}\int_{-\infty}^{\infty}[L_t^{x+h}-L_t^x][\widetilde{L}_\delta^{x+h}-\widetilde{L}_\delta^x]\,dx\biggr]^{2n}.
\end{aligned}
\tag{6.21}
$$

Thus (6.20) follows from the fact that

$$
\lim_{\delta\to 0^+}\limsup_{h\to 0^+} E\biggl[\frac{1}{h^{3/2}}\int (L_t^{x+h}-L_t^x)(\widetilde{L}_\delta^{x+h}-\widetilde{L}_\delta^x)\,dx\biggr]^{2n}=0 \tag{6.22}
$$

which, itself, is a simple consequence of (2.12) and Lemma 2.3. Thus we obtain (6.10) and hence (6.1) when m is even.

In order to obtain (6.10) when m is odd we first show that

$$
\sup_{h>0}\widehat{F}_{2n,h}(t)\le Ct^{3n/2}. \tag{6.23}
$$

To see this we observe that by first changing variables and then using the scaling relationship (1.9) with $h=\sqrt{t}$, we have

$$
\begin{aligned}
\int (L_t^{x+h}-L_t^x)^2\,dx &= \sqrt{t}\int \bigl(L_t^{\sqrt{t}(x+ht^{-1/2})}-L_t^{\sqrt{t}x}\bigr)^2\,dx \\
&= t^{3/2}\int (L_1^{x+ht^{-1/2}}-L_1^x)^2\,dx.
\end{aligned}
\tag{6.24}
$$

Therefore,

$$
\begin{aligned}
\frac{\int (L_t^{x+h}-L_t^x)^2\,dx-4ht}{h^{3/2}} &\stackrel{\mathcal{L}}{=} \frac{t^{3/2}(\int (L_1^{x+ht^{-1/2}}-L_1^x)^2\,dx-4ht^{-1/2})}{h^{3/2}} \\
&= t^{3/4}\frac{(\int (L_1^{x+ht^{-1/2}}-L_1^x)^2\,dx-4ht^{-1/2})}{(ht^{-1/2})^{3/2}}
\end{aligned}
\tag{6.25}
$$

so that for any integer m,

$$
\widehat{F}_{m,h}(t)=t^{3m/4}\widehat{F}_{m,ht^{-1/2}}(1). \tag{6.26}
$$

Therefore to prove (6.23) we need only show that

$$
\sup_t\sup_{h>0}\widehat{F}_{2n,ht^{-1/2}}(1)\le C. \tag{6.27}
$$

It follows from (6.10) that for some $\delta>0$,

$$
\sup_{\{t,h|ht^{-1/2}\le\delta\}}\widehat{F}_{2n,ht^{-1/2}}(1)\le C. \tag{6.28}
$$

On the other hand, for $ht^{-1/2} \geq \delta$,

$$
\begin{aligned}
&\left|\frac{(\int (L_1^{x+ht^{-1/2}} - L_1^x)^2\,dx - 4ht^{-1/2})}{(ht^{-1/2})^{3/2}}\right| \\
&\quad\leq \delta^{-3/2}\int (L_1^{x+ht^{-1/2}} - L_1^x)^2\,dx + 4\delta^{-1/2} \qquad (6.29)\\
&\quad\leq 4\delta^{-3/2}\int (L_1^x)^2\,dx + 4\delta^{-1/2} < \infty
\end{aligned}
$$

since $\int (L_1^x)^2\,dx$ has finite moments [see (2.2)]. Using (6.28) and (6.29) we get (6.27) and hence (6.23). It then follows from the Cauchy–Schwarz inequality that

$$\sup_{h>0} |\widehat{F}_{m,h}(t)| \leq Ct^{3m/4} \tag{6.30}$$

for all integers m.

We next show that for any integer m, the family of functions $\{\widehat{F}_{m,h}(t); h\}$ is equicontinuous in t, that is, for each t and $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$\sup_{\{s\,|\,|s-t|\leq\delta\}} \sup_{h>0} |\widehat{F}_{m,h}(t) - \widehat{F}_{m,h}(s)| \leq \varepsilon. \tag{6.31}$$

Let

$$\Phi_h(t) := \frac{\int (L_t^{x+h} - L_t^x)^2\,dx - 4ht}{h^{3/2}}. \tag{6.32}$$

Applying the identity $A^m - B^m = \sum_{j=0}^{m-1} A^j(A-B)B^{m-j-1}$ with $A = \Phi_h(t)$, $B = \Phi_h(s)$ gives

$$\widehat{F}_{m,h}(t) - \widehat{F}_{m,h}(s) = \sum_{j=0}^{m-1} \Phi_h(t)^j\big(\Phi_h(t) - \Phi_h(s)\big)\Phi_h(s)^{m-j-1}. \tag{6.33}$$

Consequently by using the Cauchy–Schwarz inequality twice and (6.30), we see that

$$
\begin{aligned}
&\sup_{\{s\,|\,|s-t|\leq\delta\}} \sup_{h>0} |\widehat{F}_{m,h}(t) - \widehat{F}_{m,h}(s)| \\
&\quad\leq C_{t,m} \sup_{\{s\,|\,|s-t|\leq\delta\}} \sup_{h>0} \|\Phi_h(t) - \Phi_h(s)\|_2. \qquad (6.34)
\end{aligned}
$$

Using (6.12)–(6.15), we see that to obtain (6.31) it suffices to show that for some $\delta > 0$,

$$\sup_{\{s\,|\,s\leq\delta\}} \sup_{h>0} \widehat{F}_{2,h}(s) \leq \varepsilon, \tag{6.35}$$

and for any $T < \infty$,

$$\sup_{\{t\leq T\}} \sup_{\{s\leq\delta\}} \sup_{h>0} E\left[\frac{1}{h^{3/2}}\int (L_t^{x+h} - L_t^x)(\widetilde{L}_s^{x+h} - \widetilde{L}_s^x)\,dx\right]^2 \leq \varepsilon. \tag{6.36}$$

By (6.23),

$$\sup_{h>0} F_{2,h}(s) \le C s^{3/2} \tag{6.37}$$

which immediately gives (6.35). Furthermore, applying the Cauchy–Schwarz inequality in (5.10) and using (5.8) to see that

$$\int_{[0,t]^m} E\left(\prod_{j=1}^m e^{ip_j W_{r_j}}\right) \prod_{j=1}^m dr_j \tag{6.38}$$

is positive and increasing in t, we see that for all $t \le T$,

$$\begin{aligned} & E\left[\frac{1}{h^{3/2}} \int (L_t^{x+h} - L_t^x)(\widetilde{L}_s^{x+h} - \widetilde{L}_s^x)\,dx\right]^2 \\ &\quad \le \left(E\left[\frac{1}{h^{3/2}} \int (L_t^{x+h} - L_t^x)(\widetilde{L}_t^{x+h} - \widetilde{L}_t^x)\,dx\right]^2\right)^{1/2} \\ &\qquad \times \left(E\left[\frac{1}{h^{3/2}} \int (L_s^{x+h} - L_s^x)(\widetilde{L}_s^{x+h} - \widetilde{L}_s^x)\,dx\right]^2\right)^{1/2} \\ &\quad \le \left(E\left[\frac{1}{h^{3/2}} \int (L_T^{x+h} - L_T^x)(\widetilde{L}_T^{x+h} - \widetilde{L}_T^x)\,dx\right]^2\right)^{1/2} \\ &\qquad \times \left(E\left[\frac{1}{h^{3/2}} \int (L_s^{x+h} - L_s^x)(\widetilde{L}_s^{x+h} - \widetilde{L}_s^x)\,dx\right]^2\right)^{1/2}. \end{aligned} \tag{6.39}$$

Using the scaling relationship, as in (6.25), we see that

$$\begin{aligned} & E\left[\frac{1}{h^{3/2}} \int (L_s^{x+h} - L_s^x)(\widetilde{L}_s^{x+h} - \widetilde{L}_s^x)\,dx\right]^2 \\ &\quad = s^{3/4} E\left[\frac{1}{(hs^{-1/2})^{3/2}} \int (L_1^{x+hs^{-1/2}} - L_1^x)(\widetilde{L}_1^{x+hs^{-1/2}} - \widetilde{L}_1^x)\,dx\right]^2. \end{aligned} \tag{6.40}$$

Following the proof of (6.27) we see that the expectation is bounded in s and h. Therefore, by taking δ sufficiently small we get (6.36). This establishes (6.31).

We now obtain (6.1) when m is odd. By equicontinuity, for any sequence $h_n \to 0$, we can find a subsequence $h_{n_j} \to 0$, such that

$$\lim_{j\to\infty} \widehat{F}_{m,h_{n_j}}(t) \tag{6.41}$$

converges to a continuous function which we denote by $\overline{F}_m(t)$. It remains to show that

$$\overline{F}_m(t) \equiv 0. \tag{6.42}$$

Let

$$G_{m,h}(t) := e^{-t} \widehat{F}_{m,h}(t) \quad \text{and} \quad \overline{G}_m(t) := e^{-t} \overline{F}_m(t). \tag{6.43}$$

By (6.30),

$$\sup_{h>0}\sup_t |G_{m,h}(t)| \le C \quad \text{and} \quad \lim_{t\to\infty}\sup_{h>0} G_{m,h}(t) = 0. \tag{6.44}$$

It then follows from (6.5) and the Dominated Convergence Theorem that for all $\zeta > 0$,

$$\int_0^\infty e^{-\zeta s}\overline{G}_m(s)\,ds = 0. \tag{6.45}$$

We obtain (6.42) by showing that $\overline{G}_m(s) \equiv 0$.

It follows from (6.44) that $\overline{G}_m(t)$ is a continuous, bounded function on R_+ that goes to zero as $t \to \infty$. By the Stone–Weierstrass theorem (see [7], Lemma 5.4), for any $\varepsilon > 0$, we can find a finite linear combination of the form $\sum_{i=1}^n c_i e^{-\zeta_i s}$ such that

$$\sup_t \left|\overline{G}_m(t) - \sum_{i=1}^n c_i e^{-\zeta_i t}\right| \le \varepsilon. \tag{6.46}$$

Therefore, by (6.45),

$$\begin{aligned}
\int_0^\infty e^{-s}\overline{G}_m^2(s)\,ds &= \int_0^\infty e^{-s}\left(\sum_{i=1}^n c_i e^{-\zeta_i s}\right)\overline{G}_m(s)\,ds \\
&\quad + \int_0^\infty e^{-s}\left(\overline{G}_m(s) - \sum_{i=1}^n c_i e^{-\zeta_i s}\right)\overline{G}_m(s)\,ds \\
&= \int_0^\infty e^{-s}\left(\overline{G}_m(s) - \sum_{i=1}^n c_i e^{-\zeta_i s}\right)\overline{G}_m(s)\,ds \\
&\le 2\varepsilon\left(\int_0^\infty e^{-s}\overline{G}_m^2(s)\,ds\right)^{1/2}
\end{aligned} \tag{6.47}$$

by the Cauchy–Schwarz inequality and (6.46). Thus $\int_0^\infty e^{-s}\overline{G}_m^2(s)\,ds = 0$ which implies that $\overline{G}_m(s) \equiv 0$. □

7. Proof of Lemma 6.2. Our goal is to obtain the asymptotic behavior of the mth moment of

$$\frac{\int (L^{x+h}_{\lambda_\zeta} - L^x_{\lambda_\zeta})^2\,dx - 4h\lambda_\zeta}{h^{3/2}} \tag{7.1}$$

as $h \to 0$. In the numerator we have the term $4h\lambda_\zeta$. Note that by Lemma 8.1, this is necessary in order that the expected value of the numerator goes to 0. Since we have $h^{3/2}$ in the denominator in (7.1), and $O(h/h^{3/2}) = O(h^{-1/2})$, we must show that in the expansion of the expectation of the mth moment of (7.1), the terms that would cause it to blow up are canceled. We do this in the first part of this proof.

Note that

$$\int L^x_{\lambda_\zeta}\,dx=\lambda_\zeta. \tag{7.2}$$

Using this and (3.4), we write the left-hand side of (6.2) as

$$\lim_{h\to 0} E\biggl(\biggl(\frac{\int (L^{x+h}_{\lambda_\zeta}-L^x_{\lambda_\zeta})^2\,dx-2\Delta^h\Delta^{-h}u^\zeta(0)\int L^x_{\lambda_\zeta}\,dx}{h^{3/2}}\biggr)^m\biggr). \tag{7.3}$$

For any integer m we have

$$\begin{aligned}
&E\biggl(\biggl(\int (L^{x+h}_{\lambda_\zeta}-L^x_{\lambda_\zeta})^2\,dx-2\Delta^h\Delta^{-h}u^\zeta(0)\int L^x_{\lambda_\zeta}\,dx\biggr)^m\biggr)\\
&\quad=E\biggl(\prod_{i=1}^m\biggl(\int (L^{x_i+h}_{\lambda_\zeta}-L^{x_i}_{\lambda_\zeta})^2\,dx_i-2\Delta^h\Delta^{-h}u^\zeta(0)\int L^{x_i}_{\lambda_\zeta}\,dx_i\biggr)\biggr)\\
&\quad=\sum_{A\subseteq\{1,\dots,m\}}(-1)^{|A^c|}E\biggl(\biggl(\prod_{i\in A}\int (L^{x_i+h}_{\lambda_\zeta}-L^{x_i}_{\lambda_\zeta})^2\,dx_i\biggr)\\
&\qquad\qquad\times\biggl(\prod_{i\in A^c}2\Delta^h\Delta^{-h}u^\zeta(0)\int L^{x_i}_{\lambda_\zeta}\,dx_i\biggr)\biggr).
\end{aligned} \tag{7.4}$$

We now show that there are many cancelations in the final equation in (7.4) that eliminate the problematical terms we discussed in the beginning of this proof, and also significantly simplifies it.

Consider a generic term in the final equation in (7.4) without the integrals. To clarify what is going on, we calculate

$$E\biggl(\prod_{i\in A}(L^{x_i+h}_{\lambda_\zeta}-L^{x_i}_{\lambda_\zeta})(L^{y_i+h}_{\lambda_\zeta}-L^{y_i}_{\lambda_\zeta})\prod_{i\in A^c}2\Delta^h\Delta^{-h}u^\zeta(0)L^{x_i}_{\lambda_\zeta}\biggr), \tag{7.5}$$

keeping in mind that $y_i=x_i$ for all $1\le i\le m$. Using the Kac moment formula (3.2), we have

$$\begin{aligned}
&E\biggl(\prod_{i\in A}(L^{x_i+h}_{\lambda_\zeta}-L^{x_i}_{\lambda_\zeta})(L^{y_i+h}_{\lambda_\zeta}-L^{y_i}_{\lambda_\zeta})\prod_{i\in A^c}2\Delta^h\Delta^{-h}u^\zeta(0)L^{x_i}_{\lambda_\zeta}\biggr)\\
&\quad=\biggl(\prod_{i\in A}\Delta^h_{x_i}\Delta^h_{y_i}\biggr)E\biggl(\prod_{i\in A}L^{x_i}_{\lambda_\zeta}L^{y_i}_{\lambda_\zeta}\prod_{i\in A^c}2\Delta^h\Delta^{-h}u^\zeta(0)L^{x_i}_{\lambda_\zeta}\biggr)\\
&\quad=(2\Delta^h\Delta^{-h}u^\zeta(0))^{|A^c|}\biggl(\prod_{i\in A}\Delta^h_{x_i}\Delta^h_{y_i}\biggr)\sum_{\sigma\in\mathcal{B}_A}\prod_{j=1}^{m+|A|}u^\zeta\bigl(\sigma(j)-\sigma(j-1)\bigr),
\end{aligned} \tag{7.6}$$

where the sum runs over $\mathcal{B}_A$, the set of all bijections,

$$\sigma:[1,\dots,m+|A|]\mapsto\{x_i,y_i,i\in A\}\cup\{x_i,i\in A^c\}. \tag{7.7}$$

As we did in the beginning of Section 4 we use the product rule,

$$\Delta_x^h\{f(x)g(x)\} = \{\Delta_x^h f(x)\}g(x+h) + f(x)\{\Delta_x^h g(x)\}, \tag{7.8}$$

to expand the last line of (7.6) into a sum of many terms, over all $\sigma \in \mathcal{B}_A$, and all ways to allocate each difference operator, $\Delta_{x_i}^h$ and $\Delta_{y_j}^h$, $i, j \in A$, to the terms $u^\zeta(\sigma(j) - \sigma(j-1))$ in which $\sigma(j-1)$ and/or $\sigma(j)$ are contained in A. After setting all $y_i = x_i$ we can then write (7.6) as

$$\begin{aligned}(2\Delta^h\Delta^{-h}u^\zeta(0))^{|A^c|} \sum_{\sigma\in\mathcal{B}_A, a} \prod_{j=1}^{m+|A|} &(\Delta_{\sigma(j)}^h)^{a_1(j)}(\Delta_{\sigma(j-1)}^h)^{a_2(j)} \\ &\times u^{\zeta,\sharp}(\sigma(j)-\sigma(j-1))\big|_{y_i=x_i,\forall i},\end{aligned} \tag{7.9}$$

where the sum runs over $\sigma \in \mathcal{B}_A$ and all $a = (a_1, a_2) : [1, \dots, m + |A|] \mapsto \{0, 1\} \times \{0, 1\}$ with the restriction that for each $i \in A$ there is exactly one factor of the form $\Delta_{x_i}^h$ and one factor of the form $\Delta_{y_i}^h$, and there are no such factors for $i \in A^c$. [Here we define $(\Delta_{x_i}^h)^0 = 1$ and $(\Delta_0^h) = 1$.] In this formula, $u^{\zeta,\sharp}(x)$ can take any of the values $u^\zeta(x)$, $u^\zeta(x+h)$ or $u^\zeta(x-h)$. [This is because we use (7.8) to pass from the last line of (7.6) to (7.9). We consider all three possibilities in the subsequent proofs.] It is important to recognize that in (7.9) each of the difference operators is applied to only one of the terms $u^{\zeta,\sharp}(\cdot)$.

We get the simplification of the final equation in (7.4) because many terms in the expansion of (7.6) for different sets A and $\sigma \in \mathcal{B}_A$ are the same, and when they are added, as they are in the final equation in (7.4), they cancel. We now make this precise.

Fix $A \subseteq \{1, \dots, m\}$ and consider a particular bijection $\sigma \in \mathcal{B}_A$. Consider (7.9) for this A and σ. For $i \in A$ we say that x_i is a bound variable, if x_i and y_i are adjacent, that is, if either $(x_i, y_i) = (\sigma(j-1), \sigma(j))$ or $(y_i, x_i) = (\sigma(j-1), \sigma(j))$ for some j. Furthermore, for a given $\sigma \in \mathcal{B}_A$ that contains bound variables, and a given a, we say that a bound variable x_i is a singular variable if both $\Delta_{x_i}^h$ and $\Delta_{y_i}^h$ are applied to the factor $u^\zeta(x_i - y_i)$.

Note that by (7.8) an h is not added to x in any $u^\zeta(\cdot)$ to which Δ_x^h is applied. Consequently,

$$\Delta_{x_i}^h\Delta_{y_i}^h u^{\zeta,\sharp}(x_i - y_i)|_{y_i=x_i} = \Delta^h\Delta^{-h}u^\zeta(0). \tag{7.10}$$

Continuing, we emphasize that the property that x_i is a bound variable depends only on σ. The property that x_i is a a singular variable depends on the pair σ, a. Let

$$S(\sigma, a) = \{i | x_i \text{ is a singular variable for } \sigma, a\}. \tag{7.11}$$

Consider a term in (7.9), with $S(\sigma, a) = J \subseteq A$. Then for each $i \in J$ we have a unique $k_i \in [1, m + |A|]$ such that $\{\sigma(k_i - 1), \sigma(k_i)\} = \{x_i, y_i\}$. Let $K = \{k_i, i \in$

$J\}$. Using (7.10), we see that the contribution of σ, a in the second line in (7.9) is

$$V(\sigma,a) := \prod_{j=1}^{m+|A|} \big(\Delta^h_{\sigma(j)}\big)^{a_1(j)} \big(\Delta^h_{\sigma(j-1)}\big)^{a_2(j)} u^{\zeta,\sharp}\big(\sigma(j)-\sigma(j-1)\big)\big|_{y_i=x_i,\forall i}$$

$$= (\Delta^h\Delta^{-h}u^\zeta(0))^{|J|} \prod_{j=1, j\notin K}^{m+|A|} \big(\Delta^h_{\sigma(j)}\big)^{a_1(j)} \big(\Delta^h_{\sigma(j-1)}\big)^{a_2(j)} \times u^{\zeta,\sharp}\big(\sigma(j)-\sigma(j-1)\big)\big|_{y_i=x_i,\forall i}. \tag{7.12}$$

Let $\mathcal{I}(\sigma)$ denote the set of all $\sigma' \in \mathcal{B}_A$ which can be obtained from σ by interchanging $\sigma(k_i-1)$ and $\sigma(k_i)$ for some set of the elements $i \in J$. Clearly $V(\sigma',a) = V(\sigma,a)$ for all $\sigma' \in \mathcal{I}(\sigma)$. Since $|\mathcal{I}(\sigma)| = 2^{|J|}$ we see that the contribution in the second line in (7.9) obtained by summing over all $\sigma' \in \mathcal{I}(\sigma)$ is

$$V(\mathcal{I}(\sigma),a) = (2\Delta^h\Delta^{-h}u^\zeta(0))^{|J|} \prod_{j=1, j\notin K}^{m+|A|} \big(\Delta^h_{\sigma(j)}\big)^{a_1(j)} \big(\Delta^h_{\sigma(j-1)}\big)^{a_2(j)} \times u^{\zeta,\sharp}\big(\sigma(j)-\sigma(j-1)\big)\big|_{y_i=x_i,\forall i}. \tag{7.13}$$

In what follows, given $\sigma \in \mathcal{B}_A$, we write it as a vector $(\sigma(1),\dots,\sigma(m+|A|)) \in R^{m+|A|}$. For any $J \subseteq A$ we define $\sigma_{A-J} \in \mathcal{B}_{A-J}$, by deleting the components y_i, $i \in J$ from $(\sigma(1),\dots,\sigma(m+|A|))$. We only use this latter notation when J is contained in the set of singular variables of some σ, a.

As an example of the relationship between σ and $\mathcal{I}(\sigma)$, let $m=3$, $A=\{1,2,3\}$, $\sigma=(x_1,x_2,y_2,y_3,x_3,y_1)$ and $J=\{2,3\}$. Then $\mathcal{I}(\sigma)$ consists of the four bijections

$$\begin{aligned} \sigma = \sigma_1 &= (x_1,x_2,y_2,y_3,x_3,y_1),\\ \sigma_2 &= (x_1,y_2,x_2,y_3,x_3,y_1),\\ \sigma_3 &= (x_1,x_2,y_2,x_3,y_3,y_1),\\ \sigma_4 &= (x_1,y_2,x_2,x_3,y_3,y_1). \end{aligned} \tag{7.14}$$

Also, in the notation just defined, $\sigma_{\{1,3\}} = (x_1,x_2,y_3,x_3,y_1)$, $\sigma_{\{1,2\}} = (x_1,x_2,y_2,x_3,y_1)$ and $\sigma_{\{1\}} = (x_1,x_2,x_3,y_1)$.

In the notation just defined, we write (7.13) as

$$\begin{aligned} V(\mathcal{I}(\sigma),a) &= (2\Delta^h\Delta^{-h}u^\zeta(0))^{|J|}\\ &\quad\times \prod_{j=1}^{m+|A-J|} \big(\Delta^h_{\sigma_{A-J}(j)}\big)^{a'_1(j)} \big(\Delta^h_{\sigma_{A-J}(j-1)}\big)^{a'_2(j)}\\ &\quad\times u^{\zeta,\sharp}\big(\sigma_{A-J}(j)-\sigma_{A-J}(j-1)\big)\big|_{y_i=x_i,\forall i}, \end{aligned} \tag{7.15}$$

where a' is obtained from $a = \{(a_1(j), a_2(j))\}_{j=1}^{m+|A|}$ by deleting from a the pairs $(a_1(j), a_2(j))$ for $j \in K$, and renumbering the remaining terms in increasing order.

Note that in applying the product formula for difference operators (7.8) we can choose which function plays the role of f, and which the role of g. When x_i is a bound variable, that is both x_i, y_i appear in the same $u^\zeta(\cdot)$, and we apply (7.8) to expand $\Delta^h_{x_i}$, we take g to be $u^\zeta(y_i - x_i)$. That is, we take

$$\begin{aligned}
&\Delta^h_{x_i} u^\zeta(x_i - a) u^\zeta(x_i - y_i) \\
&\qquad = \Delta^h_{x_1} u^\zeta(x_i - a) u^\zeta(x_i + h - y_i) + u^\zeta(x_i - a)\Delta^h_{x_1} u^\zeta(x_i - y_i),
\end{aligned} \tag{7.16}$$

and similarly when we apply (7.8) to expand $\Delta^h_{y_i}$. Thus if x_i is a singular variable and we apply $\Delta^h_{x_i}\Delta^h_{y_i}$ by the above rule, and then set $y_i = x_i$, the term that contains $\Delta^h \Delta^{-h} u^\zeta(0)$ is

$$u^\zeta(x_i - a)\Delta^h \Delta^{-h} u^\zeta(0) u^\zeta(b - x_i). \tag{7.17}$$

Note that there are no $\pm h$ terms added to the y_i or x_i. Because of this we see that

$$\begin{aligned}
&\sum_{\{a \mid S(\sigma,a) = J \subseteq A\}} V(\mathcal{I}(\sigma), a) \\
&\qquad = (2\Delta^h \Delta^{-h} u^\zeta(0))^{|J|} \\
&\qquad\quad \times \left\{ \left(\prod_{i \in A-J}{}' \Delta^h_{x_i}\Delta^h_{y_i} \right) u^\zeta(\sigma_{A-J}(j) - \sigma_{A-J}(j-1)) \right\}\Bigg|_{y_i = x_i, \forall i},
\end{aligned} \tag{7.18}$$

where the notation $\prod'$ indicates that when we use (7.8) to expand the second line of (7.18), we do not apply both $\Delta^h_{x_i}\Delta^h_{y_i}$ to the same factor $u^\zeta(\cdot)$. This is because all the singular variables have been removed from the $S(\sigma, a)$. The significance of this representation is that it does not contain any ambiguous terms $u^{\zeta,\sharp}(\cdot)$.

For $J \subseteq A$, let $\psi \in \mathcal{B}_{A-J}$. We write ψ as a vector in $R^{m+|A-J|}$ whose components consist of a permutation of the $m + |A - J|$ elements $\{x_i, y_i, i \in A - J\} \cup \{x_i, i \in (A - J)^c\}$. Let σ be obtained from this vector by inserting a component y_i, following x_i, for each $i \in J$. Considering the way σ_{A-J} was obtained from σ [see the paragraph following the one containing (7.14)], it clear that for this σ, we have $\sigma_{A-J} = \psi$. It then follows from this and (7.18) that we can rewrite (7.9) as

$$\begin{aligned}
&\sum_{J \subseteq A} (2\Delta^h \Delta^{-h} u^\zeta(0))^{|A^c|} (2\Delta^h \Delta^{-h} u^\zeta(0))^{|J|} \\
&\qquad \times \left\{ \left(\prod_{i \in A-J}{}' \Delta^h_{x_i}\Delta^h_{y_i} \right) \sum_{\sigma \in \mathcal{B}_{A-J}} \prod_{j=1}^{m+|A-J|} u^\zeta(\sigma(j) - \sigma(j-1)) \right\}\Bigg|_{y_i = x_i, \forall i} \\
&\quad = \sum_{J \subseteq A} (2\Delta^h \Delta^{-h} u^\zeta(0))^{|(A-J)^c|}
\end{aligned} \tag{7.19}$$

$$\times \left\{ \left(\prod_{i \in A-J}' \Delta_{x_i}^h \Delta_{y_i}^h \right) \right.$$

$$\left. \times \sum_{\sigma \in \mathcal{B}_{A-J}} \prod_{j=1}^{m+|A-J|} u^{\zeta}\big(\sigma(j) - \sigma(j-1)\big) \right\} \Big|_{y_i = x_i, \forall i}.$$

Hence by (7.4)–(7.9), for any integer m, we have

$$\begin{aligned} &E\left(\left(\int (L_{\lambda_\zeta}^{x+h} - L_{\lambda_\zeta}^{x})^2 \, dx - 2\Delta^h \Delta^{-h} u^{\zeta}(0) \int L_{\lambda_\zeta}^{x} \, dx \right)^m \right) \\ &\quad = \sum_{A \subseteq \{1,\dots,m\}} (-1)^{|A^c|} \sum_{J \subseteq A} \int \phi(A-J) \, dx, \end{aligned} \tag{7.20}$$

where the set function ϕ is defined by

$$\begin{aligned} \phi(D) &:= (2\Delta^h \Delta^{-h} u^{\zeta}(0))^{|D^c|} \\ &\quad \times \left\{ \left(\prod_{i \in D}' \Delta_{x_i}^h \Delta_{y_i}^h \right) \sum_{\sigma \in \mathcal{B}_D} \prod_{j=1}^{m+|D|} u^{\zeta}\big(\sigma(j) - \sigma(j-1)\big) \right\} \Big|_{y_i = x_i, \forall i}. \end{aligned} \tag{7.21}$$

It follows from Principle of Inclusion–Exclusion ([13], page 66, formula (8)) that

$$\sum_{A \subseteq \{1,\dots,m\}} (-1)^{|A^c|} \sum_{J \subseteq A} \phi(A-J) = \phi(\{1,\dots,m\}). \tag{7.22}$$

Referring to (7.20)–(7.22) we see that to estimate (7.4) we need only consider $A = \{1,\dots,m\}$ and those cases in which each of the $2m$ difference operators Δ^h are assigned either to a unique factor $u^{\zeta}(\cdot)$, or if two difference operators are assigned to the same factor $u^{\zeta}(\cdot)$, it is not of the form $u^{\zeta}(0)$. Therefore, we see that

$$\begin{aligned} &E\left(\left(\int (L_{\lambda_\zeta}^{x+h} - L_{\lambda_\zeta}^{x})^2 \, dx - 2\Delta^h \Delta^{-h} u^{\zeta}(0) \int L_{\lambda_\zeta}^{x} \, dx \right)^m \right) \\ &\quad = 2^m \sum_{\pi \in \mathcal{D}, a} \int \mathcal{T}_h^{\sharp}(x; \pi, a) \, dx, \end{aligned} \tag{7.23}$$

where

$$\mathcal{T}_h^{\sharp}(x; \pi, a) = \prod_{j=1}^{2m} \big(\Delta_{x_{\pi(j)}}^h\big)^{a_1(j)} \big(\Delta_{x_{\pi(j-1)}}^h\big)^{a_2(j)} u^{\zeta,\sharp}\big(x_{\pi(j)} - x_{\pi(j-1)}\big), \tag{7.24}$$

and the sum runs over $\mathcal{D}$, the set of all maps $\pi : [1,\dots,2m] \mapsto [1,\dots,m]$ with $|\pi^{-1}(i)| = 2$ for each i, and all $a = (a_1, a_2) : [1,\dots,2m] \mapsto \{0,1\} \times \{0,1\}$ with the property that for each i there are exactly two factors of the form $\Delta_{x_i}^h$ in (7.24), and if $a(j) = (1,1)$ for any j, then $x_{\pi(j)} \neq x_{\pi(j-1)}$. The factor 2^m in (7.23) comes from the fact that $|\pi^{-1}(i)| = 2$ for each i.

It follows from (1.4), (7.3) and (7.23) that to obtain (6.2) it suffices to show that

$$\lim_{h\to 0} h^{-3/2} 2^m \sum_{\pi\in\mathcal{D},a} \int \mathcal{T}_h^{\sharp}(x;\pi,a)\,dx \tag{7.25}$$

is equal to the right-hand side of (6.2). To simplify the proof we first show this with $\mathcal{T}_h^{\sharp}(x;\pi,a)$ replaced by

$$\mathcal{T}_h(x;\pi,a) = \prod_{j=1}^{2m} \big(\Delta^h_{x_{\pi(j)}}\big)^{a_1(j)} \big(\Delta^h_{x_{\pi(j-1)}}\big)^{a_2(j)} u^{\zeta}\big(x_{\pi(j)} - x_{\pi(j-1)}\big). \tag{7.26}$$

At the conclusion of this proof we explain why we have the same limits when $\mathcal{T}_h(\cdot)$ is replaced by $\mathcal{T}_h^{\sharp}(\cdot)$.

From this point on, the proof is very similar to the proof of Lemma 2.2. Let $m = 2n$. Consider the multigraph G_π whose vertices consist of $\{1,\dots,2n\}$ and we assign an edge between the vertices $\pi(2j-1)$ and $\pi(2j)$ for each $j = 1,\dots,2n$. Each vertex is connected to two edges, and it is possible to have two edges between any two vertices i, j. Note that the connected components C_j, $j = 1,\dots,k$ of G_π consist of cycles.

7.1. *$a = e$ and all cycles are of order two.* When $a = e$ (defined just before Section 4.1), we have

$$\mathcal{T}_h(x;\pi,e) = \prod_{j=1}^{2n} u^{\zeta}\big(x_{\pi(2j-1)} - x_{\pi(2j-2)}\big)\Delta^h\Delta^{-h} u^{\zeta}\big(x_{\pi(2j)} - x_{\pi(2j-1)}\big). \tag{7.27}$$

Assume now that, in addition, all cycles are of order two.

Let $\mathcal{P} = \{(l_{2i-1}, l_{2i}), 1 \le i \le n\}$ be a pairing of the integers $[1, 2n]$. Let $\pi \in \mathcal{D}$ [defined just after (7.24)] be such that for each $1 \le j \le 2n$, $\{\pi(2j-1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\}$ for some, necessarily unique, $1 \le i \le n$. In this case we say that π is compatible with the pairing $\mathcal{P}$ and write this as $\pi \sim \mathcal{P}$. [Note that when we write $\{\pi(2j-1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\}$ we mean as two sets, so, according to what π is, we may have $\pi(2j-1) = l_{2i-1}$, $\pi(2j) = l_{2i}$ or $\pi(2j-1) = l_{2i}$, $\pi(2j) = l_{2i-1}$.] Whenever $\pi \in \mathcal{D}$ is such that G_π consists only of cycles of order two, $\pi \sim \mathcal{P}$, for some pairing $\mathcal{P}$ of the integers $[1, 2n]$. In this case we have

$$\mathcal{T}_h(x;\pi,e) = \prod_{i=1}^{n} \big(\Delta^h\Delta^{-h} u^{\zeta}(x_{l_{2i}} - x_{l_{2i-1}})\big)^2 \prod_{j=1}^{2n} u^{\zeta}\big(x_{\pi(2j-1)} - x_{\pi(2j-2)}\big). \tag{7.28}$$

Following the proof of Lemma 2.2 we first show that

$$\int \mathcal{T}_h(x;\pi,e) \prod_{j=1}^{2n} dx_j = \int \mathcal{T}_{1,h}(x;\pi,a) \prod_{j=1}^{2n} dx_j + O(h^{3n+1}), \tag{7.29}$$

where

$$
\begin{aligned}
\mathcal{T}_{1,h}(x;\pi,e) = \prod_{i=1}^{n} &\big(1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\le h\}}\big)\big(\Delta^h\Delta^{-h}u^\zeta(x_{l_{2i}}-x_{l_{2i-1}})\big)^2 \\
&\times \prod_{j=1}^{2n} u^\zeta\big(x_{\pi(2j-1)}-x_{\pi(2j-2)}\big).
\end{aligned}
\tag{7.30}
$$

To prove (7.29) we proceed as in (4.16)–(4.19), and see that it suffices to show that for $A\subseteq[1,\dots,n]$ and $|A^c|\ge 1$,

$$
\begin{aligned}
\int \prod_{i\in A} 1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\le h\}} &\prod_{i\in A^c} 1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\ge h\}}\big(\Delta^h\Delta^{-h}u^\zeta(x_{l_{2i}}-x_{l_{2i-1}})\big)^2 \\
&\times \prod_{j=1}^{2n} u^\zeta\big(x_{\pi(2j-1)}-x_{\pi(2j-2)}\big)\prod_{j=1}^{2n} dx_j = O(h^{3n+1}).
\end{aligned}
\tag{7.31}
$$

To show this we first choose j_k, $k=1,\dots,n$, so that

$$
\big\{x_{\pi(2j_k-1)}-x_{\pi(2j_k-2)}, k=1,\dots,n\big\}\cup\{x_{l_{2i}}-x_{l_{2i-1}}, i=1,\dots,n\}
\tag{7.32}
$$

spans R^{2n}. Let y_i, $i=1,\dots,2n$, denote the $2n$ variables in (7.32). We make the change of variables in (7.31) to $\{y_1,\dots,y_{2n}\}$. We then bound those terms in $u^\zeta(x_{\pi(2j-1)}-x_{\pi(2j-2)})$, $j=1,\dots,2n$, that do not map into $u^\zeta(y_i)$, for some $i=1,\dots,2n$ [see (3.1)]. We are then left with an easy integral and using (3.8), and (3.9) and the fact that $u^\zeta(\cdot)$ is integrable, we get (7.31).

Analogous to (4.25) and (4.26), we now study

$$
\int \mathcal{T}_{1,h}(x;\pi,e)\prod_{j=1}^{2n} dx_j.
\tag{7.33}
$$

Recall that for each $1\le j\le 2n$, $\{\pi(2j-1),\pi(2j)\}=\{l_{2i-1},l_{2i}\}$, for some $1\le i\le n$. We identify these relationships by setting $i=\sigma(j)$ when $\{\pi(2j-1),\pi(2j)\}=\{l_{2i-1},l_{2i}\}$. In the present situation, in which all cycles are of order two, we have $\sigma:[1,2n]\mapsto[1,n]$, with $|\sigma^{-1}(i)|=2$, for each $1\le i\le n$. We write

$$
\begin{aligned}
&\prod_{j=1}^{2n} u^\zeta\big(x_{\pi(2j-1)}-x_{\pi(2j-2)}\big) \\
&\quad= \prod_{j=1}^{2n}\big(u^\zeta\big(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}}\big)+\Delta^{h_j}u^\zeta\big(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}}\big)\big),
\end{aligned}
\tag{7.34}
$$

where $h_j=(x_{\pi(2j-1)}-x_{l_{2\sigma(j)-1}})+(x_{l_{2\sigma(j-1)-1}}-x_{\pi(2j-2)})$. Note that because of the presence of the term $\prod_{i=1}^n(1_{\{|x_{l_{2i}}-x_{l_{2i-1}}|\le h\}})$ in the integral in (7.33), we need only be concerned with values of $|h_j|\le 2h$, $1\le j\le 2n$.

Following (4.28)–(4.33), we see that

$$\begin{aligned}
&\int \mathcal{T}_{1,h}(x;\pi,e)\prod_{j=1}^{2n} dx_j \\
&\quad = \int \prod_{i=1}^{n}(\Delta^h\Delta^{-h}u^\zeta(x_{l_{2i}}-x_{l_{2i-1}}))^2 \prod_{j=1}^{2n} u^\zeta(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}})\prod_{j=1}^{2n} dx_j \qquad (7.35)\\
&\qquad + O(h^{3n+1}),
\end{aligned}$$

where $x_{-1}=0$.

We now estimate the integral in (7.35). Using translation invariance and then (3.8), we have

$$\begin{aligned}
&\int \prod_{i=1}^{n}(\Delta^h\Delta^{-h}u^\zeta(x_{l_{2i}}-x_{l_{2i-1}}))^2 \prod_{j=1}^{2n} u^\zeta(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}})\prod_{j=1}^{2n} dx_j \\
&\quad = \int \prod_{i=1}^{n}(\Delta^h\Delta^{-h}u^\zeta(x_{l_{2i}}))^2 \prod_{j=1}^{2n} u^\zeta(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}})\prod_{k=1}^{2n} dx_{l_k} \qquad (7.36)\\
&\quad = (8/3+O(h))^n h^{3n}\int \prod_{j=1}^{2n} u^\zeta(x_{l_{2\sigma(j)-1}}-x_{l_{2\sigma(j-1)-1}})\prod_{k=1}^{n} dx_{l_{2k-1}}.
\end{aligned}$$

We set $y_k=x_{l_{2k-1}}$ and write the last line of (7.36) as

$$(8/3)^n h^{3n}\int \prod_{j=1}^{2n} u^\zeta(y_{\sigma(j)}-y_{\sigma(j-1)})\prod_{k=1}^{n} dy_k + O(h^{3n+1}). \qquad (7.37)$$

It follows from (7.29) and (7.35)–(7.37) that

$$\begin{aligned}
&\int \mathcal{T}_{h}(x;\pi,e)\prod_{j=1}^{2n} dx_j \\
&\quad = (8/3)^n h^{3n}\int \prod_{j=1}^{2n} u^\zeta(y_{\sigma(j)}-y_{\sigma(j-1)})\prod_{k=1}^{n} dy_k + O(h^{3n+1}), \qquad (7.38)
\end{aligned}$$

where $y_0=0$.

Let $\mathcal{M}$ denote the set of maps σ from $[1,\dots,2n]$ to $[1,\dots,n]$ such that $|\sigma^{-1}(i)|=2$ for all i. For each pairing $\mathcal{P}$ of $[1,\dots,2n]$, any $\pi\in\mathcal{D}$ that is compatible with $\mathcal{P}$ (i.e. $\pi\sim\mathcal{P}$) gives rise to such a map $\sigma\in\mathcal{M}$. Furthermore, any of the 2^{2n} maps in $\mathcal{D}$ obtained from π by permuting the 2 elements in any of the $2n$ pairs $\{\pi(2j-1),\pi(2j)\}$, give rise to the same map σ. In addition, for any $\sigma'\in\mathcal{M}$, we can reorder the $2n$ pairs of π to obtain a new $\pi'\sim\mathcal{P}$ which gives rise to σ'. Thus we have shown that

$$
\begin{aligned}
&\sum_{\pi\sim\mathcal{P}}\int \mathcal{T}_h(x;\pi,e)\prod_{j=1}^{2n} dx_j \\
&\quad = \left(\frac{32}{3}h^3\right)^n \sum_{\sigma\in\mathcal{M}}\int \prod_{j=1}^{2n} u^\zeta\big(y_{\sigma(j)}-y_{\sigma(j-1)}\big)\prod_{k=1}^{n} dy_k + O(h^{3n+1}) \\
&\quad = \left(\frac{16}{3}h^3\right)^n E\left\{\left(\int (L^x_{\lambda_\zeta})^2\,dx\right)^n\right\} + O(h^{3n+1}),
\end{aligned}
\tag{7.39}
$$

where the last line follows from Kac's moment formula. The factor 2^{-n} that appears in the transition from the second to the third line in (7.39) is due to the fact that $|\sigma^{-1}(i)| = 2$ for each i [see (7.23)].

Let $\mathcal{G}_2$ denote the set of $\pi\in\mathcal{D}$ such that all cycles of the graph G_π have order two. Since every such π is compatible with some pairing $\mathcal{P}$, and there are $\frac{(2n)!}{2^n n!}$ such pairings, we see that

$$
\begin{aligned}
\sum_{\pi\in\mathcal{G}_2}\int \mathcal{T}_h(x;\pi,e)\prod_{j=1}^{2n} dx_j = {}&\frac{(2n)!}{2^n n!}\left(\frac{16}{3}h^3\right)^n E\left\{\left(\int (L^x_{\lambda_\zeta})^2\,dx\right)^n\right\} \\
&+ O(h^{3n+1}).
\end{aligned}
\tag{7.40}
$$

7.2. *$a = e$ and all cycles are not of order two and $a \neq e$.* We follow closely the argument in Section 4.2 to show that

$$
\sum_{\pi\notin\mathcal{G}_2}\left|\int \mathcal{T}(x;\pi,e)\prod_{j=1}^{2n} dx_j\right| = O(h^{3n+1}). \tag{7.41}
$$

Let the cycles $C_j = \{j_1,\dots,j_{l(j)}\}$ of G_π be written in cyclic order where $l(j) = |C_j|$. Note that $\sum_{j=1}^k l(j) = 2n$.

Since we only need an upper bound, we take absolute values in the integrand to see that

$$
\begin{aligned}
&\left|\int \mathcal{T}_h(x;\pi,e)\prod_{j=1}^{2n} dx_j\right| \\
&\quad \le \int \prod_{j=1}^{k}\big(w^\zeta(x_{j_2}-x_{j_1})\cdots w^\zeta(x_{j_{l(j)}}-x_{j_{l(j)-1}})w^\zeta(x_{j_1}-x_{j_{l(j)}})\big) \\
&\qquad\quad \times \prod_{j=1}^{2n} u^\zeta\big(x_{\pi(2j-1)}-x_{\pi(2j-2)}\big)\prod_{j=1}^{2n} dx_j,
\end{aligned}
\tag{7.42}
$$

where $w^\zeta(x)$ is defined in (4.20). Note that we group the functions w according to the cycles.

We now follow the paragraph containing (4.44) verbatim until the end of Section 4.2, except that we replace u and w by u^{ζ} and w^{ζ}, to get (7.41).

When $a \neq e$,

$$\sum_{\pi} \sum_{a \neq e} \left| \int \mathcal{T}_h(x; \pi, a) \prod_{j=1}^{2n} dx_j \right| = O(h^{3n+1}). \tag{7.43}$$

This follows easily by obvious modifications of the proof in Section 4.3, similar to the modifications of the proof in Section 4.2 that gives (7.41).

We now note that it follows from the arguments in the final three paragraphs of the proof of Lemma 2.2 on page 416, that for m even we obtain the same asymptotic behavior when we replace $\mathcal{T}_h(x; \pi, a)$ by $\mathcal{T}_h^{\sharp}(x; \pi, a)$, and also, that we get the right-hand side of (6.2) for odd moments.

Summing up, we have shown that the only nonzero limits in (7.25) come from (7.40) when m is even. Using this in (7.25), in which we multiply by 2^{2n}, we see that (7.25) is equal to the right-hand side of (6.2).

8. Expectation.

LEMMA 8.1. *For $h \geq 0$,*

$$E\left(\int (L_1^{x+h} - L_1^x)^2 \, dx\right) = 4h + O(h^2) \tag{8.1}$$

as $h \to 0$. Equivalently,

$$E\left(\int (L_t^{x+1} - L_t^x)^2 \, dx\right) = 4t + O(t^{1/2}) \tag{8.2}$$

as $t \to \infty$.

PROOF. By the Kac moment formula,

$$\begin{aligned} E&\left(\int (L_1^{x+h} - L_1^x)^2 \, dx\right) \\ &= 2 \int \int_{\{\sum_{i=1}^2 r_i \leq 1\}} \Delta^h p_{r_1}(x) \Delta^h p_{r_2}(0) \, dr_1 \, dr_2 \, dx \\ &\quad + 2 \int \int_{\{\sum_{i=1}^2 r_i \leq 1\}} p_{r_1}(x) \Delta^h \Delta^{-h} p_{r_2}(0) \, dr_1 \, dr_2 \, dx. \end{aligned} \tag{8.3}$$

When we integrate with respect to x, we get zero in the first integral and one in the second. Consequently,

$$\begin{aligned} E\left(\int (L_1^{x+h} - L_1^x)^2 \, dx\right) &= 2 \int_{\{\sum_{i=1}^2 r_i \leq 1\}} \Delta^h \Delta^{-h} p_{r_2}(0) \, dr_1 \, dr_2 \\ &= 4 \int_0^1 (1 - r)\big(p_r(0) - p_r(h)\big) \, dr. \end{aligned} \tag{8.4}$$

Since

$$\int_0^1 r \frac{1-e^{-h^2/2r}}{\sqrt{r}}\,dr \le \int_0^1 r \frac{h^2/2r}{\sqrt{r}}\,dr = O(h^2) \tag{8.5}$$

and

$$\int_1^\infty \frac{1-e^{-h^2/2r}}{\sqrt{r}}\,dr \le \int_1^\infty \frac{h^2/2r}{\sqrt{r}}\,dr = O(h^2), \tag{8.6}$$

we see that to prove (8.1) it suffices to show that

$$\int_0^\infty \bigl(p_r(0) - p_r(h)\bigr)\,dr = h + O(h^2). \tag{8.7}$$

This follows from (3.1) since

$$\int_0^\infty \bigl(p_r(0) - p_r(h)\bigr)\,dr = \lim_{\alpha\to 0}\int_0^\infty e^{-\alpha r}\bigl(p_r(0) - p_r(h)\bigr)\,dr. \tag{8.8}$$

Thus we get (8.1); (8.2) follows from the scaling property (1.9). □

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X. CHEN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TENNESSEE
KNOXVILLE, TENNESSEE 37996
USA
E-MAIL: xchen@math.utk.edu

W. V. LI
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE 19716
USA
E-MAIL: wli@math.udel.edu

M. B. MARCUS
DEPARTMENT OF MATHEMATICS
CITY COLLEGE
CITY UNIVERSITY OF NEW YORK
NEW YORK, NEW YORK 10031
USA
E-MAIL: mbmarcus@optonline.net

J. ROSEN
DEPARTMENT OF MATHEMATICS
COLLEGE OF STATEN ISLAND
CITY UNIVERSITY OF NEW YORK
STATEN ISLAND, NEW YORK 10314
USA
E-MAIL: jrosen30@optimum.net