Science in China Series A: Mathematics
Jul, 2009, Vol. 52, No. 7, 1446–1458
www.scichina.com math.scichina.com
www.springerlink.com

Semiparametric bounds of mean and variance for exotic options

LIU GuoQing[1†] & LI V. Wenbo[2]

[1] Department of Mathematics, Harbin Institute of Technology, Harbin 150006, China
[2] Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA
(email: liugq@hit.edu.cn, wli@math.udel.edu)

Abstract Finding semiparametric bounds for option prices is a widely studied pricing technique. We obtain closed-form semiparametric bounds of the mean and variance for the pay-off of two exotic (Collar and Gap) call options given mean and variance information on the underlying asset price. Mathematically, we extended domination technique by quadratic functions to bound mean and variances.

Keywords: semiparametric bounds, duality, moment problem, exotic options

MSC(2000): 60C05, 65K10, 49M29

1 Introduction

The problem of finding bounds for option prices under incomplete market conditions or an incomplete knowledge of the distribution of the price of the underlying assets has been studied extensively in the literature (see, e.g., Zuluaga, et al.[1], Vandenberghe, et al.[2], d'Aspremont and Ghaoui[3], Hobson, et al.[4], Laurence and Wang[5], Popescu[6], Bertsimas and Popescu[7, 8], Boyle and Lin[9], Grundy[10], Lo[11], Levy[12] and Ritchken[13], and the references therein). Here we study bounds on the expected pay-off and variance of two exotic call options, Collar option and Gap option, given only information on the moments of the underlying asset price at maturity. Following Zuluaga, et al.[1], which provide detailed justifications, these types of bounds are called semiparametric bounds. The interest in finding semiparametric bounds for option prices stems mainly from weaker assumptions about the distribution of the underlying asset price S.

The first, now classical, results in this area were derived by Scarf[14] in the context of an inventory control problem and used by Lo[11] on the pay-off function $\max(0, S-K)$ of a European call option when mean and second-order moment information about the asset price S at maturity is available. Namely, given $\mathbb{E}S = m_1$ and $\mathbb{E}S^2 = m_2$ of the stock price $S \geqslant 0$, the optimal upper bound on a European option with strike price $K > 0$ is

$$\mathbb{E}\max(0, S-K) \leqslant \begin{cases} 2^{-1}\left(m_1 - K + \sqrt{m_2 - 2m_1K + K^2}\right), & \text{if } K \geqslant m_2/2m_1, \\ m_1 - Km_1^2/m_2, & \text{if } K \leqslant m_2/2m_1. \end{cases} \tag{1.1}$$

Received February 15, 2009; accepted April 22, 2009
DOI: 10.1007/s11425-009-0105-8

† Corresponding author

This work was supported by National Science Foundation of the United States (Grant Nos. DMS-0720977 and DMS-0805929)

Citation: Liu G Q, Li W V. Semiparametric bounds of mean and variance for exotic options. Sci China Ser A, 2009, 52(7): 1446–1458, DOI: 10.1007/s11425-009-0105-8

Lo[11] used the bound in the following way: he assumed a certain model for the underlying risk-neutral asset distribution (e.g., lognormal or jump diffusion) and then obtained the risk-neutral moments from this model. The resulting option price bounds, and observed prices are then used to show how sensitive the option prices are to model misspecification.

Recently, Liu and Li[15] improved the bound in (1.1) under the additional condition that X is bounded by M or the condition that S has a unimodal distribution, or both. More importantly, they provided a sharper bound on the variance than the well-known estimate $\mathrm{Var}\max(0, S-K) \leqslant \mathrm{Var}(S) = m_2 - m_1^2$ when $S \in [0, M]$. Namely, given $\mathbb{E}S = m_1$ and $\mathbb{E}S^2 = m_2$ of the stock price $S \in [0, M]$,

$$\mathrm{Var}\max(0, S-K) \leqslant m_2 - m_1^2 - \frac{(m_2 - m_1 K)(2M - K)(K(M - m_1) - (Mm_1 - m_2))}{2M^2(M-K)}$$

for $K \in ((Mm_1 - m_2)/(M - m_1), m_2/m_1)$. It should be emphasized that the above result for variance is much harder to obtain from the point of view of probability theory. Additional study and applications for various options should be carried out. In general, semiparametric bounds can be used to set bounds for option prices under the risk-neutral measure pricing theory, and to examine the relationship between option prices and the true, as opposed to risk-neutral, distribution of the underlying asset, see, e.g., De la Pena, et al.[16] and Boyle and Lin[9].

In this paper, we do not consider any specific models for S or practical applications but only theoretical interests from probabilistic point of view. The goal is to provide sharper bounds on the mean and variance of the payoff function associated with a given call option. We first consider Collar option with shifted pay-off function

$$h_c(S) = \min(\max(0, S - K_1), K_2 - K_1) = (S - K_1)^+ - (S - K_2)^+ \tag{1.2}$$

with constants $K_2 > K_1 > 0$. The optimal upper and lower bounds for mean are given in Propositions 2.1 and 2.2. The variance estimates are given in Theorem 2.1. The precise statements are given in Section 2 together with their proofs. The techniques are based on domination by quadratic functions. The main difficulty in the proof of Theorem 2.1 is the construction of a majorizing quadratic function in two variables. We also consider similar estimates for the Gap option with pay-off function

$$h_g(S) = S1_{(S \geqslant K)}, \tag{1.3}$$

where $K > 0$ is a fixed constant. Precise statements are given in Section 3 together with their proofs.

The overall goal of this paper is to derive closed-form semiparametric bounds for mean and variance on two pay-off functions motivated by works on exotic call options. Our results are of interests to various applied areas of probability, statistics, economics, and operations research, since closed-form expressions are of both practical and theoretical significance, as they allow for easy computation of the bounds, and the performance of sensitivity analysis and optimization over the parameters involved in the problem.

2 Mean and variance bounds on collar option

We first consider the Collar option with shifted pay-off function given in (1.2). All possible range of parameters for K_1 and K_2 are considered for various bounds of means and precise distributions that attain these bounds are given.

Proposition 2.1. *Given any random variable $S \in [0, M]$ with $\mathbb{E}S = m_1$ and $\mathbb{E}S^2 = m_2$ and $0 \leqslant K_1 < K_2 < M$ fixed. Then the following holds.*

(i) *If $0 < K_2 \leqslant (Mm_1 - m_2)/(M - m_1)$, then trivially*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant K_2 - K_1$$

and equality holds if S takes only two values K_2 and $(m_2 - m_1K_2)/(m_1 - K_2)$, with $\mathbb{P}(S = K_2) = (m_2 - m_1^2)/(m_2 - m_1^2 + (m_1 - K_2)^2)$.

(ii) *If $(Mm_1 - m_2)/(M - m_1) < K_2 \leqslant m_2/m_1$, then*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant (K_2 - K_1)(Mm_1 + K_2m_1 - m_2)/MK_2$$

and the equality holds if S takes only three values, 0, K_2 and M, with $\mathbb{P}(S = 0) = ((M - m_1)K_2 - (Mm_1 - m_2))/MK_2$, $\mathbb{P}(S = K_2) = (Mm_1 - m_2)/(M - K_2)K_2$, and $\mathbb{P}(S = 0) + \mathbb{P}(S = K_2) = 1 - (m_2 - K_2m_1)/M(M - K_2) < 1$.

(iii) *If $m_2/m_1 \leqslant K_2, 0 \leqslant K_1 \leqslant m_2/2m_1$, then*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant m_1 - K_1m_1^2/m_2$$

and equality holds if S takes only two values 0 and m_2/m_1, with $\mathbb{P}(S = m_2/m_1) = m_1^2/m_2$.

(iv) *If $m_2/m_1 \leqslant K_2, m_2/2m_1 \leqslant K_1 \leqslant (K_2^2 - m_2)/2(K_2 - m_1)$, then*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant (m_1 - K_1 + L)/2,$$

where $L = (K_1^2 - 2m_1K_1 + m_2)^{1/2}$. The equality holds if S takes only two values, $K_1 - L$ and $K_1 + L$, with $\mathbb{P}(S = K_1 - L) = (m_2 - m_1^2)/2(L^2 - K_1L + m_1L)$.

(v) *If $m_2/m_1 \leqslant K_2, (K_2^2 - m_2)/2(K_2 - m_1) \leqslant K_1 < K_2$, then*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant (K_2 - K_1)(m_2 - m_1^2)/(K_2^2 - 2K_2m_1 + m_2)$$

and the equality holds if S takes only two values, $(K_2m_1 - m_2)/(K_2 - m_1)$ and K_2, with $\mathbb{P}(S = K_2) = (m_2 - m_1^2)/(K_2^2 - 2K_2m_1 + m_2)$.

Note that the trivial bound in (i) always holds but it is worse than bounds given in all other cases and this is the key point of this proposition.

Proof. We follow the general principle of dominating the function $\min(\max(0, s - K_1), K_2 - K_1)$ by quadratic function $Q(s) = \alpha + \beta s + \gamma s^2$, where α, β and γ are constants, such that $\min(\max(0, s - K_1), K_2 - K_1) \leqslant Q(s)$. Then

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant \min_Q \mathbb{E}Q(S) = \min_{\alpha,\beta,\gamma}(\alpha + \beta m_1 + \gamma m_2).$$

Denote

$$\psi(s) = \min(\max(0, s - K_1), K_2 - K_1).$$

In the case (i), it is trivial. Notice that since $0 < K_2 \leqslant (Mm_1 - m_2)/(M - m_1) < m_1$, then the value of optimal discrete distribution S satisfies $(m_2 - m_1K_2)/(m_1 - K_2) \in (K_2, M)$. Thus the optimal discrete distribution is well defined.

In the case (ii), we take

$$Q_2(s) = (K_2 - K_1)s(M + K_2 - s)/MK_2.$$

To check $\psi(s) \leqslant Q_2(s)$, divide the range $0 \leqslant s \leqslant M$ into three parts: (1) If $0 \leqslant s \leqslant K_1$, then $Q_2(s) - \psi(s) = Q_2(s) \geqslant 0$. (2) If $K_1 \leqslant s \leqslant K_2$, then

$$Q_2(s) - \psi(s) = (-(K_2 - K_1)s^2 + (K_2^2 - MK_1 - K_1K_2)s + MK_1K_2)/MK_2,$$

and thus

$$Q_2(K_1) - \psi(K_1) = (K_2 - K_1)K_1(M + K_2 - K_1)/MK_2 > 0$$

and $Q_2(K_2) - \psi(K_2) = 0$. Observing that $Q_2(s) - \psi(s)$ is concave, thus $Q_2(s) - \psi(s) \geqslant \min(Q_2(K_1) - \psi(K_1), Q_2(K_2) - \psi(K_2)) \geqslant 0$. (3) If $K_2 \leqslant s \leqslant M$, then $Q_2(s) - \psi(s) = (K_2 - K_1)(M - s)(s - K_2)/MK_2 \geqslant 0$. Therefore, $Q_2(s) - \psi(s) \geqslant 0$ in the whole range $0 \leqslant s \leqslant M$. Hence

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant \mathbb{E}Q_2(S) = (K_2 - K_1)(Mm_1 + K_2m_1 - m_2)/MK_2.$$

In the case (iii), let

$$Q_3(s) = K_1(m_1s - m_2)^2/m_2^2 + s - K_1.$$

To check $\psi(s) \leqslant Q_3(s)$, divide the range $0 \leqslant s \leqslant M$ into three parts: (1) If $0 \leqslant s \leqslant K_1$, then $Q_3(s) - \psi(s) = m_1^2K_1s^2/m_2^2 + (m_2 - 2m_1K_1)s/m_2 \geqslant 0$. (2) If $K_1 \leqslant s \leqslant K_2$, then $Q_3(s) - \psi(s) = K_1(m_1s - m_2)^2/m_2^2 \geqslant 0$. (3) If $K_2 \leqslant s \leqslant M$, then $Q_3(s) - \psi(s) = K_1(m_1s - m_2)^2/m_2^2 + s - K_2 \geqslant 0$. Hence

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant \mathbb{E}Q_3(S) = m_1 - K_1m_1^2/m_2.$$

In the case (iv), put $Q_4(s) = (s - K_1 + L)^2/4L$. To check $\psi(s) \leqslant Q_4(s)$, note that (1) If $0 \leqslant s \leqslant K_1$, then $Q_4(s) - \psi(s) = Q_4(s) \geqslant 0$. (2) If $K_1 \leqslant s \leqslant K_2$, then $Q_4(s) - \psi(s) = (s - K_1 - L)^2/4L > 0$. (3) If $K_2 \leqslant s \leqslant M$, then

$$Q_4(s) - \psi(s) = (s - K_1 - L)^2/4L + s - K_2 > 0.$$

Hence

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant \mathbb{E}Q_4(S) = (m_1 - K_1 + L)/2.$$

In the range $m_2/m_1 \leqslant K_2, (K_2^2 - m_2)/2(K_2 - m_1) \leqslant K_1 < K_2$, one can observe $0 < K_1 - L < m_1 < K_1 + L < K_2$. Since $(L + m_1 - K_1)(L - (m_1 - K_1)) = m_2 - m_1^2$, then $L(1 - \mathbb{P}(S = K_1 - L)) = (m_1 - K_1 + L)/2$ and thus $\mathbb{P}(S = K_1 - L) < 1$. So the optimal discrete distribution is well defined.

In the case (v), take

$$Q_5(s) = (K_2 - K_1)(K_2m_1 - m_2 - (K_2 - m_1)s)^2/(K_2^2 - 2K_2m_1 + m_2)^2.$$

We can check $\psi(s) \leqslant Q_5(s)$ as follows: (1) If $0 \leqslant s \leqslant K_1$, then $Q_5(s) - \psi(s) = Q_5(s) \geqslant 0$. (2) If $K_1 \leqslant s \leqslant K_2$, then the derivative

$$Q_5'(s) - \psi'(s) = \frac{(K_2 - K_1)(K_2 - m_1)^2(2(K_2 - K_1)(s - t_0) - (K_2 - t_0)^2)}{(K_2^2 - 2K_2m_1 + m_2)^2},$$

where $t_0 = (K_2m_1 - m_2)/(K_2 - m_1)$. One can observe $Q_5'(s) - \psi'(s)$ is increasing and then $Q_5'(s) - \psi'(s) \leqslant Q_5'(K_2) - \psi'(K_2) \leqslant 0$ by using the range of $m_2/m_1 \leqslant K_2, (K_2^2 - m_2)/2(K_2 - m_1) \leqslant K_1 < K_2$, then $Q_5(s) - \psi(s)$ is decreasing and $Q_5(s) - \psi(s) \geqslant Q_5(K_2) - \psi(K_2) = 0$. (3) if $K_2 \leqslant s \leqslant M$, then

$$Q_5(s) - \psi(s) = \frac{(K_2 - K_1)(K_2 - m_1)(s - K_2)((K_2 - m_1)(s + K_2 - 2m_1) + 2(m_2 - m_1^2))}{(K_2^2 - 2K_2m_1 + m_2)^2} > 0.$$

Hence

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \leqslant \mathbb{E}Q_5(S) = (K_2 - K_1)(m_2 - m_1^2)/(K_2^2 - 2K_2m_1 + m_2).$$

This finishes the proof.

Next we provide lower bounds which are of interests from probabilistic point of view.

Proposition 2.2. *Given any random variable $S \in [0, M]$ with $\mathbb{E}S = m_1$ and $\mathbb{E}S^2 = m_2$ and $0 \leqslant K_1 < K_2 < M$ fixed. Then the following holds.*

(i) *If $0 \leqslant K_1 < (Mm_1 - m_2)/(M - m_1), K_1 < K_2 < (m_2 - K_1^2)/2(m_1 - K_1)$, then*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant (K_2 - K_1)(1 - (m_2 - m_1^2)/(K_1^2 - 2K_1m_1 + m_2))$$

and the equality holds if S takes only two values, $(m_2 - K_1m_1)/(m_1 - K_1)$ and K_1, with $\mathbb{P}(S = K_1) = (m_2 - m_1^2)/(K_1^2 - 2K_1m_1 + m_2)$.

(ii) *If $0 \leqslant K_1 < (Mm_1 - m_2)/(M - m_1), (m_2 - K_1^2)/2(m_1 - K_1) \leqslant K_2 \leqslant (M^2 - m_2)/2(M - m_1)$, then*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant K_2 - K_1 - (K_2 - m_1 + T)/2,$$

where $T = (K_2^2 - 2m_1K_2 + m_2)^{1/2}$. The equality holds if S takes only two values, $K_2 - T$ and $K_2 + T$, with $\mathbb{P}(S = K_2 - T) = (m_2 - m_1^2)/2(T^2 - K_2T + m_1T)$.

(iii) *If $0 \leqslant K_1 < (Mm_1 - m_2)/(M - m_1), (M^2 - m_2)/2(M - m_1) \leqslant K_2 \leqslant M$, then*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant m_1 - K_1 - \frac{(M - K_2)(m_2 - m_1^2)}{M^2 - 2Mm_1 + m_2}$$

and the equality holds if S takes only two values, $(Mm_1 - m_2)/(M - m_1)$ and M, with $\mathbb{P}(S = M) = (m_2 - m_1^2)/(M^2 - 2Mm_1 + m_2)$.

(iv) *If $(Mm_1 - m_2)/(M - m_1) \leqslant K_1 \leqslant m_2/m_1$, then*

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant (K_2 - K_1)(m_2 - K_1m_1)/M(M - K_1)$$

and the equality holds if S takes only three values, 0, K_1 and M, with $\mathbb{P}(S = M) = (m_2 - K_1m_1)/M(M - K_1)$, $\mathbb{P}(S = K_1) = (Mm_1 - m_2)/(M - K_1)K_1$ and $\mathbb{P}(S = M) + \mathbb{P}(S = K_1) = (Mm_1 - m_2 + m_1K_1)/MK_1 < 1$.

(v) *If $K_1 \geqslant m_2/m_1$, then trivially $\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant 0$, and equality holds if S takes only two values 0 and m_2/m_1, with $\mathbb{P}(S = m_2/m_1) = m_1^2/m_2$.*

Proof. We follow the same idea as that in Proposition 2.1, that is, dominating the function $\min(\max(0, s - K_1), K_2 - K_1)$ by quadratic function $Q(s) = \alpha + \beta s + \gamma s^2$, where α, β and γ are constants, such that $\min(\max(0, s - K_1), K_2 - K_1) \geqslant Q(s)$. Then

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant \max_Q \mathbb{E}Q(S) = \max_{\alpha,\beta,\gamma}(\alpha + \beta m_1 + \gamma m_2).$$

Denote

$$\psi(s) = \min(\max(0, s - K_1), K_2 - K_1).$$

In the case (i), we take

$$Q_1(s) = K_2 - K_1 - (K_2 - K_1)(K_1 m_1 - m_2 - (K_1 - m_1)s)^2/(K_1^2 - 2K_1 m_1 + m_2)^2.$$

We can then check $\psi(s) \geqslant Q_1(s)$ as follows: (1) In the region $0 \leqslant s \leqslant K_1$, since $\psi(s) - Q_1(s)$ is monotone decreasing, then $\psi(s) - Q_1(s) \geqslant \psi(K_1) - Q_1(K_1) = 0$. (2) If $K_1 \leqslant s \leqslant K_2$, then $\psi(s) - Q_1(s)$ is a convex quadratic function and its symmetric axis is smaller than K_1 (since $K_2 < (m_2 - K_1^2)/2(m_1 - K_1)$). These together with $\psi(K_1) - Q_1(K_1) = 0$ gives $\psi(s) \geqslant Q_1(s)$. (3) If $K_2 \leqslant s \leqslant M$, then obviously $\psi(s) - Q_1(s) = K_2 - K_1 - Q_1(s) \geqslant 0$. Thus

$$\begin{aligned}\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) &\geqslant \mathbb{E}Q_1(S) \\ &= (K_2 - K_1)(1 - (m_2 - m_1^2)/(K_1^2 - 2K_1 m_1 + m_2)).\end{aligned}$$

In the case (ii), let

$$Q_2(s) = K_2 - K_1 - (s - K_2 - T)^2/4T,$$

where $T = (K_2^2 - 2m_1 K_2 + m_2)^{1/2}$. To check $\psi(s) \geqslant Q_2(s)$, observe that $Q_2(K_2 - T) = K_2 - T - K_1$, the derivative $Q_2'(K_2 - T) = 1$, $Q_2(K_2 + T) = K_2 - K_1$, the derivative $Q_2'(K_2 + T) = 0$ and by the range of K_2 in (ii), one can see $K_2 - T > K_1$, $K_2 + T < M$, hence $Q_2(s)$ is tangent with $\psi(s)$ at two points $K_2 - T$, $K_2 + T$. And then the concave function $Q_2(s) \leqslant \psi(s)$. Thus

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant \mathbb{E}Q_2(S) = K_2 - K_1 - (K_2 - m_1 + T)/2.$$

Notice that $T\mathbb{P}(S = K_2 - T) = (K_2 - m_1 + T)/2 < T$, thus $\mathbb{P}(S = K_2 - T) < 1$.

In the case (iii), take

$$Q_3(s) = s - K_1 - (M - K_2)(s - d)^2/(M - d)^2,$$

where $d = (Mm_1 - m_2)/(M - m_1)$. Then we can check $\psi(s) \geqslant Q_3(s)$ as follows: (1) In the region $0 \leqslant s \leqslant K_1$, since $\psi(s) - Q_3(s)$ is monotone decreasing, then $\psi(s) - Q_3(s) \geqslant \psi(K_1) - Q_3(K_1) \geqslant 0$. (2) If $K_1 \leqslant s \leqslant K_2$, then

$$\psi(s) - Q_3(s) = (M - K_2)(s - d)^2/(M - d)^2 \geqslant 0.$$

(3) If $K_2 \leqslant s \leqslant M$, then $\psi(s) - Q_3(s)$ is a convex quadratic function and its symmetric axis is larger than M, this together with $\psi(M) - Q_3(M) = 0$ and $\psi(K_2) - Q_3(K_2) > 0$ gives $\psi(s) - Q_3(s) \geqslant 0$. Thus,

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant \mathbb{E}Q_3(S) = m_1 - K_1 - \frac{(M - K_2)(m_2 - m_1^2)}{M^2 - 2Mm_1 + m_2}.$$

In the case (iv), we put

$$Q_4(s) = (K_2 - K_1)(s^2 - K_1 s)/M(M - K_1).$$

To check $\psi(s) \geqslant Q_4(s)$, note that (1) in the region $0 \leqslant s \leqslant K_1$, obviously $\psi(s) = 0 \geqslant Q_4(s)$. (2) If $K_1 \leqslant s \leqslant K_2$, then

$$\psi(s) - Q_4(s) = (s - K_1)(1 - s(K_2 - K_1)/M(M - K_1)) \geqslant 0.$$

(3) If $K_2 \leqslant s \leqslant M$,

$$\psi(s) - Q_4(s) = (K_2 - K_1)(M - s)(M + s - K_1)/M(M - K_1) \geqslant 0.$$

Thus,

$$\mathbb{E}\min(\max(0, S - K_1), K_2 - K_1) \geqslant \mathbb{E}Q_4(S) = (K_2 - K_1)(m_2 - K_1 m_1)/M(M - K_1).$$

This finishes the proof.

Next we present one of the main results of this paper which improves the simple bounds

$$\operatorname{Var}\min(\max(0, S - K_1), K_2 - K_1) \leqslant \operatorname{Var}\max(0, S - K_1) \leqslant \operatorname{Var}(S) = m_2 - m_1^2$$

on the variance for certain range of parameters. In particular, it is very good when $K_2 > K_1$ are close to each other. It should be pointed out that the only known results of this type, as far as we know, was given in [15] for European option. The method used here is similar but requires special constructions which are hard to find.

Theorem 2.1. *Given any random variable $S \in [0, M]$ with $\mathbb{E}S = m_1$ and $\mathbb{E}S^2 = m_2$ and $0 \leqslant K_1 < K_2 < M$ fixed. Then*

$$\begin{aligned}&\operatorname{Var}\min(\max(0, S - K_1), K_2 - K_1)\\ &\quad\leqslant (K_2 - K_1)^2\left(\frac{(2Mm_1 - m_1^2)}{2M^2} + \frac{(Mm_1 - m_2)(2M^2 - Mm_1 - m_2)}{2M^3 K_2} + \frac{(MK_1 m_1^2 - m_2^2)}{2M^3(M - K_1)}\right).\end{aligned}$$

Proof. After representing the variance with an independent copy, the key idea is finding good functions $Q(s, t)$ such that

$$\begin{aligned}&(\min(\max(0,s - K_1), K_2 - K_1) - \min(\max(0, t - K_1), K_2 - K_1))^2\\ &\qquad \leqslant Q(s, t) = a_1 s + a_2 t + a_3 s^2 + a_4 t^2 + a_5 st + a_6 s^2 t^2 + a_7,\end{aligned}$$

where a_i $(i = 1, \dots, 7)$ are constants. Then

$$\begin{aligned}&\operatorname{Var}\min(\max(0, S - K_1), K_2 - K_1)\\ &\quad= \frac{1}{2}\mathbb{E}(\min(\max(0, S - K_1), K_2 - K_1) - \min(\max(0, T - K_1), K_2 - K_1))^2\\ &\quad\leqslant \min_Q \frac{1}{2}\mathbb{E}Q(S, T),\end{aligned}$$

where S and T are i.i.d. random variables.

In our setting, we construct a special quadratic function

$$Q_0(s, t) = (K_2 - K_1)^2\left\{\left(\frac{(M - K_1 - K_2)t^2}{M^3 K_2(M - K_1)} - \frac{1}{MK_2}\right)s^2\right.$$

$$+\left(\frac{(K_1K_2-(M+K_2)(M-K_1))t}{M^2K_2(M-K_1)}+\frac{(M+K_2)}{MK_2}\right)s$$
$$+\frac{(M+K_2)t}{MK_2}-\frac{t^2}{MK_2}\bigg\}.$$

Denote

$$\psi(s,t)=(\min(\max(0,s-K_1),K_2-K_1)-\min(\max(0,t-K_1),K_2-K_1))^2$$

and

$$G_t(s)=Q_0(s,t)-\psi(s,t).$$

To check $G_t(s)\geqslant 0$, note that as $0\leqslant t\leqslant M$, the coefficient of s^2 in $Q_0(s,t)$ is negative, so $Q_0(s,t)$ is concave as a function of s for fixed t, we divide the range $0\leqslant t\leqslant M$ into three parts:

Case (i): Fixed $0\leqslant t\leqslant K_1$, then

$$\psi(s,t)=\begin{cases}0, & \text{if } 0\leqslant s\leqslant K_1,\\ (s-K_1)^2, & \text{if } K_1\leqslant s\leqslant K_2,\\ (K_2-K_1)^2, & \text{if } K_2\leqslant s\leqslant M.\end{cases}$$

Note that for fixed t, $\psi(s,t)$ is convex and increasing as a function of $s\in[K_1,K_2]$ and is constant for $s\in[0,K_1]$ and $s\in[K_2,M]$. On the other hand, $Q_0(s,t)$ is concave as a function of s for any fixed t. Thus to show $Q_0(s,t)\geqslant\psi(s,t)$, or $G_t(s)=Q_0(s,t)-\psi(s,t)\geqslant 0$, we only need to check it at three locations, namely $s=0,K_2,M$. It is easy to check

$$\begin{aligned}&G_t(0)=Q_0(0,t)-0\geqslant 0,\\ &G_t(M)=Q_0(M,t)-\psi(M,t)=(K_2-K_1)^2t(K_1-t)/M(M-K_1)\geqslant 0,\\ &G_t(K_2)=(K_2-K_1)^2t\left(\frac{(M^2-K_2^2)(M-t)}{M^3K_2}+\frac{K_2(MK_1-K_2t)}{M^3(M-K_1)}\right)\geqslant 0.\end{aligned}$$

And thus we see

$$Q_0(0,t)\geqslant\psi(0,t),\quad Q_0(K_2,t)\geqslant\psi(K_2,t),\quad Q_0(M,t)\geqslant\psi(M,t),$$

which finishes the case (i).

Case (ii). For fixed $K_1\leqslant t\leqslant K_2$, then

$$\psi(s,t)=\begin{cases}(t-K_1)^2, & \text{if } 0\leqslant s\leqslant K_1,\\ (s-t)^2, & \text{if } K_1\leqslant s\leqslant K_2,\\ (K_2-t)^2, & \text{if } K_2\leqslant s\leqslant M.\end{cases}$$

Note that for fixed t, $\psi(s,t)$ is convex as a function of $s\in[K_1,K_2]$ and is constant for $s\in[0,K_1]$ and $s\in[K_2,M]$. On the other hand, $Q_0(s,t)$ is concave as a function of s for any fixed t. Thus to show $Q_0(s,t)\geqslant\psi(s,t)$, or $G_t(s)=Q_0(s,t)-\psi(s,t)\geqslant 0$, we only need to check it at four locations, namely $s=0,K_1,K_2,M$. First note that

$$G_t(0)=-\left(\frac{(K_2-K_1)^2}{MK_2}+1\right)t^2+\left(2K_1+\frac{(K_2-K_1)^2(M+K_2)}{MK_2}\right)t-K_1^2,$$

which implies $G_t(0) \geqslant \min(G_{K_1}(0), G_{K_2}(0)) \geqslant 0$ for the concave function $G_t(0)$. Similarly, it is easy to see

$$\begin{aligned}G_t(M) &= -\bigg(\frac{(K_2-K_1)^2}{M(M-K_1)}+1\bigg)t^2 + \bigg(2K_2 + \frac{(K_2-K_1)^2K_1}{M(M-K_1)}\bigg)t + K_1^2 - 2K_1K_2\\ &\geqslant \min(G_{K_1}(M), G_{K_2}(M)) \geqslant 0.\end{aligned}$$

Next we have

$$\begin{aligned}G_t(K_1) = (K_2-K_1)^2\bigg\{&-\frac{(M-K_1)^2(M+K_1)+K_1^2K_2}{M^3K_2(M-K_1)}t^2\\ &+\frac{(M+K_2)(M-K_1)^2+K_1^2K_2}{M^2K_2(M-K_1)}t - \frac{(M-K_1)(K_2-K_1)}{MK_2}\bigg\}\\ &+(K_2-K_1)^2-(K_1-t)^2\end{aligned}$$

and thus the concavity implies

$$G_t(K_1) \geqslant \min(G_{K_1}(K_1), G_{K_2}(K_1)).$$

By simple computations,

$$G_{K_1}(K_1) = \frac{(K_2-K_1)^2K_1(K_2(K_1^2+2M^2-MK_1)+(M-K_1)^2(2M+K_1))}{M^3K_2} \geqslant 0$$

and

$$G_{K_2}(K_1) = \frac{(K_2-K_1)^2((M-K_1)^2K_1(M^2-K_2^2)+K_1^2K_2^2(M-K_2))}{M^3(M-K_1)K_2} \geqslant 0.$$

Finally, we have

$$\begin{aligned}G_t(K_2) &= (K_2-K_1)^2t\bigg(\frac{(M^2-K_2^2)(M-t)}{M^3K_2}+\frac{K_2(MK_1-K_2t)}{M^3(M-K_1)}\bigg)\\ &\quad+(K_2-K_1)^2-(K_2-t)^2,\end{aligned}$$

which implies by concavity $G_t(K_2) \geqslant \min(G_{K_1}(K_2), G_{K_2}(K_2))$. Once again, simple computations show that

$$\begin{aligned}G_{K_1}(K_2) &= \frac{(K_2-K_1)^2(M-K_2)K_1((M-K_1)^2(M+K_2)+K_1K_2^2)}{M^3(M-K_1)K_2} \geqslant 0,\\ G_{K_2}(K_2) &= \frac{(K_2-K_1)^2(M^2-K_2^2)((M-K_1)^2+M(M-K_2)+K_1K_2+K_2^2-K_1^2)}{M^3(M-K_1)} \geqslant 0,\end{aligned}$$

which implies $G_t(K_2) \geqslant 0$. Putting above together, one can obtain

$$Q_0(0,t) \geqslant \psi(0,t), \quad Q_0(K_1,t) \geqslant \psi(K_1,t), \quad Q_0(K_2,t) \geqslant \psi(K_2,t), \quad Q_0(M,t) \geqslant \psi(M,t),$$

which finishes the case (ii).

Case (iii). For fixed $K_2 \leqslant t \leqslant M$, one can obtain

$$\psi(s,t) = \begin{cases}(K_2-K_1)^2, & \text{if } 0 \leqslant s \leqslant K_1,\\ (K_2-s)^2, & \text{if } K_1 \leqslant s \leqslant K_2,\\ 0, & \text{if } K_2 \leqslant s \leqslant M.\end{cases}$$

Note that for fixed t, $\psi(s,t)$ is convex and decreasing as a function of $s \in [K_1, K_2]$ and is constant for $s \in [0, K_1]$ and $s \in [K_2, M]$. On the other hand, $Q_0(s,t)$ is concave as a function of s for any fixed t. Thus to show $Q_0(s,t) \geqslant \psi(s,t)$, or $G_t(s) = Q_0(s,t) - \psi(s,t) \geqslant 0$, we only need to check it at three locations, namely $s = 0, K_1, M$.

It is easy to check $G_t(0) = (K_2 - K_1)^2(M-t)(t-K_2)/MK_2 \geqslant 0$. Next we find

$$G_t(K_1) = (K_2 - K_1)^2\bigg\{ -\frac{(M-K_1)^2(M+K_1) + K_1^2K_2}{M^3K_2(M-K_1)}t^2 + \frac{(M+K_2)(M-K_1)^2 + K_1^2K_2}{M^2K_2(M-K_1)}t - \frac{(M-K_1)(K_2-K_1)}{MK_2}\bigg\},$$

which is a concave function and thus implies

$$G_t(K_1) \geqslant \min(G_{K_2}(K_1), G_M(K_1)) = 0.$$

Also,

$$G_t(M) = (K_2 - K_1)^2(M-t)(M+t-K_1)/M(M-K_1) \geqslant 0.$$

Hence,

$$Q_0(0,t) \geqslant \psi(0,t), \quad Q_0(K_1,t) \geqslant \psi(K_1,t), \quad Q_0(M,t) \geqslant \psi(M,t),$$

which gives $Q_0(s,t) \geqslant \psi(s,t)$. Putting three cases together, we obtain

$$\begin{aligned}
&\operatorname{Var}\min(\max(0, S-K_1), K_2-K_1) \\
&\quad= \frac{1}{2}\mathbb{E}(\min(\max(0,S-K_1),K_2-K_1) - \min(\max(0,T-K_1),K_2-K_1))^2 \\
&\quad= \frac{1}{2}\mathbb{E}\psi(S,T) \leqslant \frac{1}{2}\mathbb{E}Q_0(S,T) \\
&\quad= (K_2-K_1)^2\bigg(\frac{(2Mm_1 - m_1^2)}{2M^2} + \frac{(Mm_1 - m_2)(2M^2 - Mm_1 - m_2)}{2M^3K_2} + \frac{(MK_1m_1^2 - m_2^2)}{2M^3(M-K_1)}\bigg).
\end{aligned}$$

Note that for any fixed K_1, as K_2 approaches to K_1, the upper bound approaches to 0, smaller than trivial bound $m_2 - m_1^2$. We complete this proof.

3 Mean and variance bounds on gap option

In this section we consider the gap option with pay-off function given in (1.3). All possible range of parameters for $K > 0$ are considered for various bounds of means, and precise distributions that attain these bounds are given.

Proposition 3.1. *Given any random variable $S \in [0, M]$ with $\mathbb{E}S = m_1$ and $\mathbb{E}S^2 = m_2$ and $K \in [0, M]$ fixed. Then the following holds.*

(i) *If $K \leqslant m_2/m_1$, then trivially $\mathbb{E}S1_{(S\geqslant K)} \leqslant m_1$, and equality holds if S takes only two values 0 and m_2/m_1, with probabilities $\mathbb{P}(S = 0) = 1 - m_1^2/m_2$. Here $1_{(S\geqslant K)}$ is an indicator function which takes 1 for $(S \geqslant K)$ and 0 otherwise.*

(ii) *If $K \geqslant m_2/m_1$, then*

$$\mathbb{E}S1_{(S\geqslant K)} \leqslant K(m_2 - m_1^2)/(m_2 - 2m_1K + K^2)$$

and the equality holds if S takes only two values K and $(m_1K - m_2)/(K - m_1)$, with probabilities $\mathbb{P}(S = K) = (m_2 - m_1^2)/(m_2 - 2m_1K + K^2)$, $\mathbb{P}(S = (m_1K - m_2)/(K - m_1)) = (K - m_1)^2/(m_2 - 2m_1K + K^2)$.

Proof. For the case (ii), we follow the general principle of dominating the function $s1_{(s\geqslant K)}$ by quadratic function. Consider quadratic functions $Q(s)=\alpha+\beta s+\gamma s^2$, where α,β and γ are constants, such that $s1_{(s\geqslant K)}\leqslant Q(s)$. Then

$$\mathbb{E}S1_{(S\geqslant K)}\leqslant \min_{Q}\mathbb{E}Q(S)=\min_{\alpha,\beta,\gamma}(\alpha+\beta m_1+\gamma m_2).$$

In our setting, we take

$$Q(s)=Q_0(s)=K((K-m_1)s-(m_1K-m_2))^2/(m_2-2m_1K+K^2)^2.$$

To check $s1_{(s\geqslant K)}\leqslant Q_0(s)$, let

$$G(s)=Q_0(s)-s1_{(s\geqslant K)},$$

then we only need to check $G(s)\geqslant 0$. (1) If $s<K$, then obviously $G(s)=Q_0(s)-0\geqslant 0$. (2) If $s\geqslant K$, then

$$G(s)=Q_0(s)-s=\frac{K(s-K)^2(K-m_1)^2}{(m_2-2m_1K+K^2)^2}+\frac{(K^2-m_2)}{(m_2-2m_1K+K^2)}.$$

Since $K\geqslant m_2/m_1$, then $K^2\geqslant m_2$ which gives $G(s)\geqslant 0$. Hence

$$\mathbb{E}S1_{(S\geqslant K)}\leqslant \mathbb{E}Q_0(S)=K(m_2-m_1^2)/(m_2-2m_1K+K^2).$$

Notice that $K\geqslant m_2/m_1>m_1>(Mm_1-m_2)/(M-m_1)$, then the value of optimal discrete distribution S satisfies $0<(m_1K-m_2)/(K-m_1)<K$. And as $K\geqslant m_2/m_1$, the bound is smaller than trivial bound m_1. We complete the proof.

Next we provide lower bounds which are of interests from probabilistic point of view.

Proposition 3.2. *Given any random variable $S\in[0,M]$ with $\mathbb{E}S=m_1$ and $\mathbb{E}S^2=m_2$ and $K\in[0,M]$ fixed. Then*

(i) *If $0\leqslant K\leqslant (Mm_1-m_2)/(M-m_1)$, then*

$$\mathbb{E}S1_{(S>K)}\geqslant m_1-\frac{K(m_2-m_1^2)}{m_2-2Km_1+K^2}$$

and equality holds if S takes only two values K, and $(m_2-m_1K)/(m_1-K)$, with probability $\mathbb{P}(S=K)=(m_2-m_1^2)/(m_2-2m_1K+K^2)$.

(ii) *If $(Mm_1-m_2)/(M-m_1)\leqslant K\leqslant m_2/m_1$, then*

$$\mathbb{E}S1_{(S>K)}\geqslant (m_2-Km_1)/(M-K)$$

and the equality holds if S takes only three values, 0, K and M, with $\mathbb{P}(S=K)=(Mm_1-m_2)/K(M-K)$, $\mathbb{P}(S=M)=(m_2-m_1K)/M(M-K)$, and $\mathbb{P}(S=K)+\mathbb{P}(S=M)=(Mm_1-m_2+m_1K)/MK<1$.

(iii) *If $m_2/m_1\leqslant K\leqslant M$, then trivially $\mathbb{E}S1_{(S>K)}\geqslant 0$, and equality holds if S takes only two values 0 and m_2/m_1, with probability $\mathbb{P}(S=0)=1-m_1^2/m_2$.*

Proof. We use the same idea as in the proof of Proposition 3.1. In the case (i), we take

$$Q_1(s)=\frac{(s-K)(b^2-sK)}{(b-K)^2},$$

where $b = (m_2 - m_1K)/(m_1 - K)$. Note that $K < b < M$. To check $Q_1(s) \leqslant s1_{(s>K)}$, observe that if $0 \leqslant s \leqslant K$, then $Q_1(s) - s1_{(s>K)} = Q_1(s) \leqslant 0$; if $K \leqslant s \leqslant M$, then $Q_1(s) - s1_{(s>K)} = -K(s-b)^2/(b-K)^2 \leqslant 0$. Hence $Q_1(s) \leqslant s1_{(s>K)}$ holds and we obtain

$$\mathbb{E}S1_{(S>K)} \geqslant \mathbb{E}Q_1(S) = m_1 - \frac{K(m_2 - m_1^2)}{m_2 - 2Km_1 + K^2}.$$

In the case (ii), let

$$Q_2(s) = (s^2 - Ks)/(M - K).$$

To check $Q_2(s) \leqslant s1_{(s>K)}$, observe that if $0 \leqslant s \leqslant K$, then $Q_2(s) - s1_{(s>K)} = Q_2(s) \leqslant 0$; if $K \leqslant s \leqslant M$, then $Q_2(s) - s1_{(s>K)} = -s(M-s)/(M-K) \leqslant 0$. Hence $Q_2(s) \leqslant s1_{(s>K)}$ holds and we obtain

$$\mathbb{E}S1_{(S>K)} \geqslant \mathbb{E}Q_2(S) = (m_2 - Km_1)/(M - K).$$

This finishes the proof.

Next we present our improvement on the variance for certain range of parameters. This case is much simpler than the case in Theorem 2.1. Note first that we have the trivial bounds

$$\mathrm{Var}S1_{(S\geqslant K)} = \mathbb{E}S^2 1_{(S\geqslant K)} - (\mathbb{E}S1_{(S\geqslant K)})^2 \leqslant m_2 - K^2,$$

which is sharp for $S = K$. However, the information on the mean $\mathbb{E}S = m_1$ was lost. The estimates below can be very good for small value $K > 0$. It is easy to check that it is worse than the bound $m_2 - K^2$ for $K \geqslant m_1$.

Theorem 3.1. *Given any random variable $S \in [0, M]$ with $\mathbb{E}S = m_1$ and $\mathbb{E}S^2 = m_2$ and $K \in [0, M]$ fixed. Then*

$$\mathrm{Var}S1_{(S\geqslant K)} \leqslant m_2 - \max\left(K^2, \frac{(2M-K)m_2^2}{2M^2(M-K)} - \frac{Km_1^2}{2(M-K)}\right)$$

where $1_{(S\geqslant K)}$ is an indicator function which takes 1 for $(S \geqslant K)$ and 0 otherwise.

Proof. Let S and T are i.i.d. random variables, then

$$\mathrm{Var}S1_{(S\geqslant K)} = \frac{1}{2}\mathbb{E}(S1_{(S\geqslant K)} - T1_{(T\geqslant K)})^2.$$

Thus, if we find a good function $Q(s,t) = a_1s + a_2t + a_3s^2 + a_4t^2 + a_5st + a_6s^2t^2 + a_7$, such that $(s1_{(s\geqslant K)} - t1_{(t\geqslant K)})^2 \leqslant Q(s,t)$, then $\mathrm{Var}S1_{(S\geqslant K)} \leqslant \frac{1}{2}\mathbb{E}Q(S,T)$. In our setting, we take

$$Q(s,t) = s^2 + t^2 + \frac{Kst}{M-K} - \frac{(2M-K)s^2t^2}{M^2(M-K)}.$$

Denoting $\psi(s,t) = (s1_{(s\geqslant K)} - t1_{(t\geqslant K)})^2$, to check

$$\psi(s,t) = (s1_{(s\geqslant K)} - t1_{(t\geqslant K)})^2 \leqslant Q(s,t),$$

we divide the region $0 \leqslant s, t \leqslant M$ into four parts:

(1) If $0 \leqslant s, t < K$, then

$$Q(s,t) - \psi(s,t) = s^2\left(1 - \frac{t^2}{M^2}\right) + t^2 + \frac{st(MK - st)}{M(M-K)} \geqslant 0.$$

(2) If $0 \leqslant t < K, K \leqslant s \leqslant M$, then we have

$$Q(s,t) - \psi(s,t) = t^2\left(1 - \frac{s^2}{M^2}\right) + \frac{st(MK - st)}{M(M-K)} \geqslant 0.$$

(3) If $0 \leqslant s < K, K \leqslant t \leqslant M$, then we obtain

$$Q(s,t) - \psi(s,t) = s^2\left(1 - \frac{t^2}{M^2}\right) + \frac{st(MK - st)}{M(M-K)} \geqslant 0.$$

(4) If $K \leqslant s, t \leqslant M$, then

$$Q(s,t) - \psi(s,t) = \frac{(2M-K)st(M^2 - st)}{M^2(M-K)} \geqslant 0.$$

Hence, $\psi(s,t) = (s1_{(s \geqslant K)} - t1_{(t \geqslant K)})^2 \leqslant Q(s,t)$. Therefore

$$\begin{aligned} \mathrm{Var} S1_{(S \geqslant K)} &= \frac{1}{2}\mathbb{E}(S1_{(S \geqslant K)} - T1_{(T \geqslant K)})^2 \\ &\leqslant \frac{1}{2}\mathbb{E}Q(S,T) = m_2 - \left(\frac{(2M-K)m_2^2}{2M^2(M-K)} - \frac{Km_1^2}{2(M-K)}\right). \end{aligned}$$

Note that for small value $K > 0$, the far right is smaller than trivial bound $m_2 - K^2$. This finishes the proof.

References

1 Zuluaga L F, Pena J, Du D. Third-order extensions of Lo's semiparametric bound for European call options. *European J Oper Res*, **198**: 557–570 (2008)

2 Vandenberghe L, Boyd S, Comanor K. Generalized Chebyshev bounds via semidefinite programming. *SIAM Review*, **49**(1): 52–64 (2007)

3 d´Aspremont, Ghaoui L El. Static arbitrage bounds on basket option prices. *Math Program*, **106**(3): 467–489 (2006)

4 Hobson D, Laurence P M, Wang T H. Static-arbitrage upper bounds for the prices of basket options. *Quant Finance*, **5**(4): 329–342 (2005)

5 Laurence P M, Wang T H. Sharp upper and lower bounds for basket options. *Appl Math Finance*, **12**(3): 253–282 (2005)

6 Popescu I. A semidefinite programming approach to optimal moment bounds for convex classes of distributions. *Math Oper Res*, **30**: 1–24 (2005)

7 Bertsimas D, Popescu I. On the relation between option and stock prices: An optimization approach. *Operations Research*, **50**(2): 358–374 (2002)

8 Bertsimas D, Popescu I. Optimal inequalities in probability theory: a convex optimization approach. *SIAM J Optim*, **15**(3): 780–804 (2005)

9 Boyle P P, Lin X S. Bounds on contingent claims based on several assets. *J Financ Econ*, **46**(3): 383–400 (1997)

10 Grundy B. Option prices and the underlying assets return distribution. *J Finance*, **46**(3): 1045–1069 (1991)

11 Lo A. Semiparametric upper bounds for option prices and expected payoffs. *J Financ Econ*, **19**: 373–387 (1987)

12 Levy H. Upper and lower bounds of put and call option value: Stochastic dominance approach. *J Finance*, **40**(4): 1197–1217 (1985)

13 Ritchken P H. On option pricing bounds. *J Finance*, **40**(4): 1219–1233 (1985)

14 Scarf H. A min-max solution of an inventory problem. In: Arrow K J, Karlin S, Scarf H. eds. In Studies in the mathematical theory of inventory and production, Stanford, CA: Stanford University Press, 1958, 201–209

15 Liu G, Li W. Moment bounds for truncated random variables. *Statist Probab Lett*, to appear (2009)

16 De la Pena V H, Ibragimov R, Jordan S J. Option bounds. *J Appl Probab*, **41**A: 145–156 (2004)