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Moment bounds for truncated random variables

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ABSTRACT

Given any random variable $X \in [0, M]$ with $\mathbb{E}X = m_1$ and $\mathbb{E}X^2 = m_2$ fixed, various bounds are derived on the mean and variance of the truncated random variable $\max(0, X - K)$ with $K > 0$ given. The results are motivated by questions associated with European call options. The techniques are based on domination by quadratic functions and change of measures in the unimodal distribution case.

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1. Introduction

Bounds involving moments of random variables arise naturally in many areas of probability, statistics, economics, and operations research. There is also a long history of studying them, that goes back to the work of Chebyshev; see Shohat and Tamarkin (1943) for early history and developments. Various explicit and simpler bounds have been developed over the years by using the duality theory, in particular for the unimodal distribution; see Karlin and Studden (1966) and Dharmadhikari and Joag-Dev (1988). Recently, (Bertsimas and Popescu, 2005; Vandenberghe et al., 2007) presented nonlinear convex optimization, and in particular semidefinite programming approaches to generalized Chebyshev inequalities and related moment problems.

In this paper we are interested in the truncated random variable $\max(0, X - K)$ motivated by works on European call options, where X is the stock price and K is a fixed strike price. For example, given the mean $\mathbb{E}X = m_1$ and $\mathbb{E}X^2 = m_2$ of the stock price $X \geq 0$, the optimal upper bound on an option with strike K is given by:

$$\mathbb{E}\,\max(0, X - K) \leq \begin{cases} 2^{-1}\left(m_1 - K + \sqrt{m_2 - 2m_1K + K^2}\right) & \text{if } K \geq m_2/2m_1, \\ m_1 - Km_1^2/m_2 & \text{if } K \leq m_2/2m_1. \end{cases} \tag{1.1}$$

This bound is due to Scarf (1958) in the context of an inventory control problem. Lo (1987) observed the direct application of Scarf's result to option pricing. Various extensions based on optimization techniques can be found in Bertsimas and Popescu (2002) and Popescu (2005).

The goal of this paper is twofold. First, we improve the bound in (1.1) under the additional condition that X is bounded by M or the condition that X has a unimodal distribution, or both. These are given in Proposition 2 and Theorem 1. Second, we provide a sharper bound on variance than the well-known estimate $\mathrm{Var}\max(0, X - K) \leq \mathrm{Var}\,(X) = m_2 - m_1^2$ when $X \in [0, M]$. This is given in Theorem 2. The techniques are based on domination by quadratic functions and change of measures in the unimodal distribution case. The main difficulty in the proof of Theorem 2 is the construction of a majorizing quadratic function in two variables.

The rest of this paper is organized as follows. As a warmup, we present some explicit bounds in Section 2. The statements of our main results, Theorems 1 and 2, are given in Section 3 together with their proofs.

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2. Preliminaries

Before examining new inequalities in the next section, we begin with the following more or less known inequalities, which allow us to define terminology and illustrate our approach.

Proposition 1. *Given a non-constant random variable $X \in [0, M]$ with $\mathbb{E}X = m_1$ and $\mathbb{E}X^2 = m_2$ fixed.*

(i) *If $0 \le t \le (Mm_1 - m_2)/(M - m_1)$, then*

$$\mathbb{P}(X \le t) \le \frac{m_2 - m_1^2}{m_2 - 2m_1 t + t^2}$$

and the equality holds if X takes only two values, t and $(m_2 - m_1 t)/(m_1 - t)$, with $\mathbb{P}(X = t) = (m_2 - m_1^2)/(m_2 - 2m_1 t + t^2)$.

(ii) *If $(Mm_1 - m_2)/(M - m_1) \le t \le m_2/m_1$, then*

$$\mathbb{P}(X \le t) \le 1 - \frac{m_2 - m_1 t}{M(M - t)}$$

and the equality holds if X takes only three values, 0, t and M, with $\mathbb{P}(X = 0) = ((M - m_1)t - (Mm_1 - m_2))/Mt$, $\mathbb{P}(X = t) = (Mm_1 - m_2)/(M - t)t$.

(iii) *If $m_2/m_1 \le t \le M$, then $\mathbb{P}(X \le t) = 1$ for two point distribution at 0 and m_2/m_1, with $\mathbb{P}(X = m_2/m_1) = m_1^2/m_2$.*

Proof. Since part (iii) is trivial, we only show part (i) and (ii). There are at least two ways to prove part (i). One is based on the general principle given below in the proof of part (ii), so we omit it here. The other is based on the so-called shift Chebyshev inequality, a somewhat special technique. Namely, for any $\lambda > t$, we have by the Chebyshev inequality

$$\mathbb{P}(X \le t) = \mathbb{P}(\lambda - X \ge \lambda - t) \le \frac{\mathbb{E}(\lambda - X)^2}{(\lambda - t)^2} = \frac{\lambda^2 - 2\lambda m_1 + m_2}{(\lambda - t)^2}.$$

Hence

$$\mathbb{P}(X \le t) \le \inf_{\lambda > t} \frac{\lambda^2 - 2\lambda m_1 + m_2}{(\lambda - t)^2} = \frac{m_2 - m_1^2}{m_2 - 2m_1 t + t^2},$$

where the infimum is achieved at

$$\lambda = \lambda_0 = \frac{m_2 - m_1 t}{m_1 - t} > t \quad \text{for } t < \frac{Mm_1 - m_2}{M - m_1}$$

by simple calculus.

To prove (ii), we follow a general principle of dominating indicator function by the "best" quadratic function. To be more precise, consider quadratic functions $Q(x) = \alpha + \beta x + \gamma x^2$ such that indicator function $1(x \le t) \le Q(x)$. Then

$$\mathbb{P}(X \le t) = \mathbb{E}\,1_{(X \le t)} \le \min_{Q} \mathbb{E}\,Q(X) = \min_{\alpha,\beta,\gamma} (\alpha + \beta m_1 + \gamma m_2).$$

To find the minimum, one can consider two cases, $\gamma \ge 0$ and $\gamma < 0$. Simple calculation and comparison imply that the best Q is given by

$$Q(x) = Q_0(x) = 1 - \frac{x^2 - tx}{M(M - t)}.$$

Hence (ii) follows from $\mathbb{P}(X \le t) \le \mathbb{E}\,Q_0(X)$ and we finish the proof of Proposition 1. □

The sharp upper and lower bounds for $\mathbb{E}\max(0, X - K)$ will be given in the following two propositions.

Proposition 2. *Given a non-constant random variable $X \in [0, M]$ with $\mathbb{E}X = m_1$, $\mathbb{E}X^2 = m_2$ and a fixed constant $K \ge 0$.*

(i) *If $0 \le K \le m_2/2m_1$, then*

$$\mathbb{E}\max(0, X - K) \le m_1 - Km_1^2/m_2 \tag{2.2}$$

and the equality holds if X takes only two values, 0 and m_2/m_1, with $\mathbb{P}(X = 0) = 1 - m_1^2/m_2$.

(ii) *If $m_2/2m_1 \le K < (M^2 - m_2)/2(M - m_1)$, then*

$$\mathbb{E}\max(0, X - K) \le \frac{1}{2}(m_1 - K + L) \tag{2.3}$$

where $L = (K^2 - 2m_1 K + m_2)^{1/2}$. The equality holds if X takes only two values, $K - L$ and $K + L$, with $\mathbb{P}(X = K - L) = (m_2 - m_1^2)/2(L^2 - KL + m_1 L)$.

(iii) *If* $(M^2 - m_2)/2(M - m_1) \le K \le M$, *then*

$$\mathbb{E}\max(0, X - K) \le \frac{(M-K)(m_2 - m_1^2)}{M^2 - 2Mm_1 + m_2} \tag{2.4}$$

and the equality holds if X *takes only two values,* $(Mm_1 - m_2)/(M - m_1)$ *and* M*, with* $\mathbb{P}(X = M) = (m_2 - m_1^2)/(M^2 - 2Mm_1 + m_2)$.

Proof. We first prove part (iii). The basic idea is finding quadratic functions $Q(x) = \alpha + \beta x + \gamma x^2$ such that $\max(0, x - K) \le Q(x)$. Then

$$\mathbb{E}\max(0, X - K) \le \min_{Q} \mathbb{E}\, Q(X) = \min_{\alpha,\beta,\gamma} (\alpha + \beta m_1 + \gamma m_2).$$

By simple calculations, the minimum is achieved at the quadratic function

$$Q_3(x) = \frac{(M-K)(Mm_1 - m_2 - (M-m_1)x)^2}{(M^2 - 2Mm_1 + m_2)^2}.$$

Hence (2.4) follows from $\mathbb{E}\max(0, X - K) \le \mathbb{E}\, Q_3(X)$. Similarly, we can check easily that $\min_Q \mathbb{E}\, Q(X)$ is achieved at the quadratic function

$$Q_1(x) = \left(1 - \frac{2Km_1}{m_2}\right)x + \frac{Km_1^2}{m_2^2}x^2$$

in the case of (i) and at the quadratic function

$$Q_2(x) = (x - K + L)^2/4L$$

in the case of (ii). This finishes the proof of Proposition 2. □

Proposition 3. *Given a non-constant random variable* $X \in [0, M]$ *with* $\mathbb{E}X = m_1$, $\mathbb{E}X^2 = m_2$ *and a fixed constant* $K \ge 0$.

(i) *If* $0 \le K \le (Mm_1 - m_2)/(M - m_1)$, *then trivially* $\mathbb{E}\max(0, X - K) \ge m_1 - K$ *and the equality holds if* X *takes only two values,* $(Mm_1 - m_2)/(M - m_1)$ *and* M*, with* $\mathbb{P}(X = M) = (m_2 - m_1^2)/(M^2 - 2Mm_1 + m_2)$.

(ii) *If* $(Mm_1 - m_2)/(M - m_1) \le K < m_2/m_1$, *then*

$$\mathbb{E}\max(0, X - K) \ge (m_2 - m_1 K)/M$$

and the equality holds if X *takes only three values,* 0, K *and* M*, with* $\mathbb{P}(X = K) = (Mm_1 - m_2)/K(M - K)$, *and* $\mathbb{P}(X = M) = (m_2 - m_1 K)/M(M - K)$.

(iii) *If* $m_2/m_1 \le K \le M$, *then trivially* $\mathbb{E}\max(0, X - K) \ge 0$ *and the equality holds if* X *takes only two values,* m_2/m_1 *and* 0, *with* $\mathbb{P}(X = m_2/m_1) = m_1^2/m_2$.

Proof. We only need to show part (ii). The idea is similar to the proof of Proposition 2. One can easily check that in part (ii)

$$\max(0, x - K) \ge x(x - K)/M$$

and hence, the result follows. □

Note that the lower bound in part (ii), $(m_2 - m_1 K)/M$ improves the trivial estimates $\max(0, m_1 - K)$ in part (i) and (iii).

3. Main results and proofs

The main results of this paper are the following estimates on the mean (under unimodal distribution) and the variance of European call option $\max(0, X - K)$ given the first and second moments of X supported on $[0, M]$.

Let us first recall the definition of a unimodal distribution in order to study its moment bounds. A distribution function $F(x)$ is called a unimodal distribution with mode m if there exists a smallest m such that $F(x)$ is convex on $(-\infty, m)$ and concave on $(m, +\infty)$. Note that if $F(x)$ has a density function $\varphi(x)$, then $\varphi(x)$ is nondecreasing for $x < m$ and nonincreasing for $x > m$.

Theorem 1. *Given a non-constant random variable* $X \in [0, M]$ *with* $\mathbb{E}X = m_1$, $\mathbb{E}X^2 = m_2$ *and a fixed constant* $K \ge 0$. *Assume* X *is unimodal distributed with mode* m.

(i) *If* $0 \le K \le m$, *then*

$$\mathbb{E}\max(0, X - K) \le \frac{M(m-K)^2 + (2m_1 - m)(Mm - K^2)}{2Mm}.$$

(ii) *If* $m < K \le 2m_1 - m$, *then*

$$\mathbb{E}\max(0, X - K) \le \frac{K^2 - 2(2m_1 - m)K + 3m_2 - 2m_1 m}{2(K - m)}.$$

(iii) *If* $\max(m, 2m_1 - m) < K \le M$, *then*

$$\mathbb{E}\max(0, X-K) \le \frac{3m_2 - 2m_1 m - (2m_1 - m)^2}{2(K-m)}.$$

Proof. The basic idea, as had been used in Karlin and Studden (1966), is considering a new "modified" probability measure defined by

$$\mathrm{d}H(x) = (m-x)d\left(\frac{\mathrm{d}F(x)}{\mathrm{d}x}\right).$$

Using integration by parts for Stieltjes integral, it follows that

$$\int_0^M \mathrm{d}H(x) = \int_0^M \mathrm{d}F(x) = 1,$$
$$\int_0^M x\mathrm{d}H(x) = \int_0^M (2x-m)\mathrm{d}F(x) = 2m_1 - m,$$
$$\int_0^M x^2\mathrm{d}H(x) = \int_0^M (3x^2 - 2mx)\mathrm{d}F(x) = 3m_2 - 2m_1 m.$$

Next, we rewrite

$$\mathbb{E}\max(0, X-K) = \int_K^M (x-K)\mathrm{d}F(x) = \int_0^M \psi(x)\mathrm{d}H(x),$$

where in the case $0 \le K \le m$,

$$\psi(x) = \begin{cases} (m-K)^{2/2}(m-x) & \text{if } 0 \le x \le K, \\ (m+x-2K)/2 & \text{if } K \le x \le M; \end{cases} \tag{3.5}$$

and in the case $m < K \le M$,

$$\psi(x) = \begin{cases} 0 & \text{if } 0 \le x < K, \\ (x-K)^{2/2}(x-m) & \text{if } K \le x \le M. \end{cases} \tag{3.6}$$

The above can be easily checked by the integration by parts. For example, in the case of $0 \le K \le m$,

$$\begin{aligned}\int_0^M \psi(x)\mathrm{d}H(x) &= \int_0^K \frac{(m-K)^2}{2}\mathrm{d}\left(\frac{\mathrm{d}F(x)}{\mathrm{d}x}\right) + \int_K^M \frac{(m+x-2K)(m-x)}{2}\mathrm{d}\left(\frac{\mathrm{d}F(x)}{\mathrm{d}x}\right) \\ &= \int_K^M (x-K)\mathrm{d}F(x) = \mathbb{E}\max(0, X-K).\end{aligned}$$

Next one needs to find a quadratic function $Q(x) = \alpha + \beta x + \gamma x^2$ such that $\psi(x) \le Q(x)$. Then

$$\begin{aligned}\mathbb{E}\max(0, X-K) &= \int_0^M \psi(x)\mathrm{d}H(x) \\ &\le \min_Q \int_0^M Q(x)\mathrm{d}H(x) \\ &= \min_{\alpha,\beta,\gamma}(\alpha + \beta(2m_1 - m) + \gamma(3m_2 - 2m_1 m)).\end{aligned}$$

In the case (i), $0 \le K \le m$, we take a special quadratic (linear here) function

$$Q_1(x) = (M(m-K)^2 + (Mm - K^2)x)/2Mm \ge \psi(x)$$

where $\psi(x)$ is defined in (3.5). To see the domination, we observe that $\psi(0) = Q_1(0)$, $\psi(K) \le Q_1(K)$, $\psi(M) = Q_1(M)$, and $\psi(x)$ is increasing and convex in $0 \le x \le K$. Thus

$$\begin{aligned}\mathbb{E}\max(0, X-K) &\le \int_0^M Q_1(x)\mathrm{d}H(x) \\ &= (M(m-K)^2 + (2m_1 - m)(Mm - K^2))/2Mm.\end{aligned}$$

which finishes the proof of (i).

In the case $m < K \le M$, we take a special family of quadratic functions

$$Q(x, s) = (x-s)^2/2(K-m)$$

for $s \le K$. Then one can easily check $\psi(x) \le Q(x,s)$, where $\psi(x)$ is defined in (3.6). Hence

$$\begin{aligned}\mathbb{E}\max(0, X-K) &\le \min_{s\le K}\int_0^M Q(x,s)\mathrm{d}H(x)\\ &= \min_{s\le K}\frac{s^2-2(2m_1-m)s+3m_2-2m_1m}{2(K-m)}.\end{aligned}$$

Noting that the symmetric axis of quadratic function $s^2-2(2m_1-m)s+3m_2-2m_1m$ is $2m_1-m$, and $s\le K$, hence in the case (ii), we take $s=K$; in the case (iii), we take $s=2m_1-m$. This ends the proof of Theorem 1. □

Next we investigate the variance estimate for $\max(0, X-K)$ given $\mathbb{E}X = m_1$ and $\mathbb{E}X^2 = m_2$. It is well known that $\operatorname{Var}\max(0, X-K) \le \operatorname{Var}X = m_2 - m_1^2$. The following theorem provides a refinement for K in the given range.

Theorem 2. *Given a non-constant random variable $X \in [0, M]$ with $\mathbb{E}X = m_1$, $\mathbb{E}X^2 = m_2$ and a fixed constant $K \in ((Mm_1-m_2)/(M-m_1), m_2/m_1)$, then*

$$\operatorname{Var}\max(0, X-K) \le m_2 - m_1^2 - \frac{(m_2-m_1K)(2M-K)(K(M-m_1)-(Mm_1-m_2))}{2M^2(M-K)}.$$

Proof. After representing the variance with an independent copy, the key idea is finding good quadratic functions $Q(x,y)$ such that

$$(\max(x,K)-\max(y,K))^2 \le Q(x,y) = \alpha(y)+\beta(y)x+\gamma(y)x^2, \tag{3.7}$$

where $\alpha(y)$, $\beta(y)$ and $\gamma(y)$ are also quadratic functions. Then

$$\begin{aligned}\operatorname{Var}\max(0, X-K) &= \frac{1}{2}\mathbb{E}\,(\max(X,K)-\max(Y,K))^2\\ &\le \min_Q \frac{1}{2}\mathbb{E}\,Q(X,Y),\end{aligned}$$

where X and Y are i.i.d. random variables. In our setting, we construct a special quadratic function

$$Q_0(x,y) = (x-y)^2 - \frac{(x-K)(x-M)y(y-K)(2M-K)}{M^2(M-K)}. \tag{3.8}$$

Note this is the hardest part of this proof. To check Q_0 in (3.8) satisfying (3.7), we divide the region $0 \le x, y \le M$ into four parts. In the regions $0 \le x, y \le K$ and $K \le x, y \le M$, (3.7) clearly holds.

Let

$$\psi(x,y) = (\max(x,K)-\max(y,K))^2 \quad \text{and} \quad G(x,y) = Q_0(x,y)-\psi(x,y).$$

In the region $K \le x \le M$, $0 \le y \le K$, we fix y and rewrite $G(x,y)$ as a quadratic function of x,

$$G(x,y) = (r(y)-1)x^2 - ((r(y)-1)(K+M)+2(y-K))x + (r(y)-1)KM + y^2 - K^2$$

where $r(y)-1 = y(K-y)(2M-K)/M^2(M-K) \ge 0$. To check (3.7), we observe by simple calculation that the symmetric axis of $G(x,y)$ in x is smaller than K, and hence

$$G(x,y) \ge G(K,y) = (K-y)^2 \ge 0.$$

In the region $0 \le x \le K$, $K \le y \le M$, we fix x and rewrite $G(x,y) = Q_0(x,y) - \psi(x,y)$ as a quadratic function of y,

$$G(x,y) = (-h(x))y^2 - (-h(x)K + 2x - 2K)y + x^2 - K^2, \tag{3.9}$$

where $h(x) = (M-x)(K-x)(2M-K)/M^2(M-K) \ge 0$. To check (3.7), we observe that $G(x,y)$ is concave in y as a quadratic function of y. Hence for $x \in [0,K]$ fixed and $K \le y \le M$,

$$\begin{aligned}G(x,y) &\ge \min(G(x,K), G(x,M))\\ &= \min((x-K)^2, (K-x)x(M-K)/M) \ge 0.\end{aligned}$$

Putting things together, we obtain

$$\begin{aligned}\operatorname{Var}\max(0, X-K) &\le \frac{1}{2}\mathbb{E}\,Q_0(X,Y)\\ &= \frac{1}{2}\mathbb{E}\,(X-Y)^2 - \frac{1}{2}\mathbb{E}\,\frac{(X-K)(X-M)Y(Y-K)(2M-K)}{M^2(M-K)}\\ &= m_2 - m_1^2 - \frac{(m_2-m_1K)(2M-K)(K(M-m_1)-(Mm_1-m_2))}{2M^2(M-K)}\end{aligned}$$

which finishes the proof. Note that for K in the range $((Mm_1 - m_2)/(M - m_1), m_2/m_1)$, the last line is smaller than the standard bound $m_2 - m_1^2$. $\square$

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