Large deviations for local times and intersection local times of fractional Brownian motions and Riemann-Liouville processes

Xia Chen* Wenbo V. Li† Jan Rosiński‡ Qi-Man Shao§

September 28, 2009

Abstract

In this paper we prove exact forms of large deviations for local times and intersection local times of fractional Brownian motions and Riemann–Liouville processes. We also show that a fractional Brownian motion and the related Riemann–Liouville process behave like constant multiples of each other with regard to large deviations for their local and intersection local times. As a consequence of our large deviation estimates, we derive laws of iterated logarithm for the corresponding local times. The key points of our methods: (1) logarithmic *superadditivity* of a normalized sequence of moments of exponentially randomized local time of a fractional Brownian motion; (2) logarithmic *subadditivity* of a normalized sequence of moments of exponentially randomized intersection local time of Riemann–Liouville processes; (3) comparison of local and intersection local times based on embedding of a part of a fractional Brownian motion into the reproducing kernel Hilbert space of the Riemann–Liouville process.

Key-words: local time, intersection local time, large deviations, fractional Brownian motion, Riemann–Liouville process, law of iterated logarithm.

AMS subject classification (2010): 60G22, 60J55, 60F10, 60G15, 60G18.

*Research partially supported by NSF grant DMS-0704024.
†Research partially supported by NSF grant DMS–0805929.
‡Research partially supported by NSA grant MSPF-50G-049.
§Research partially supported by Hong Kong RGC CERG 602206 and 602608.

Contents

1 Introduction

Let $B^H(t)$, $t \geq 0$ be a standard d-dimensional fractional Brownian motion with index $H \in (0,1)$. That is, $B^H(t)$ is a zero-mean Gaussian process with stationary increments and covariance function

$$\mathbb{E}\left[B^H(t)B^H(s)^\top\right] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\} I_d,$$

where I_d is the identity matrix of size d. $B^H(t)$ is also a self-similar process with index H. The local time $L_t^x(B^H)$ of $B^H(t)$ at $x \in \mathbb{R}^d$ is defined heuristically as

$$L_t^x(B^H) = \int_0^t \delta_x\left(B^H(s)\right)\, ds, \quad t \geq 0.$$

It is know that $L_t^x(B^H)$ exists and is jointly continuous in (t,x) as long as $Hd < 1$. By the self-similarity of a fractional Brownian motion, $L_t^x(B^H) \stackrel{d}{=} t^{1-Hd} L_1^{x/t^H}(B^H)$. In particular,

$$L_t^0(B^H) \stackrel{d}{=} t^{1-Hd} L_1^0(B^H). \tag{1.1}$$

Our first goal is to investigate large deviations associated with tail probabilities of $L_t^0(B^H)$. By the scaling given above, we may consider only $t = 1$. In the classical case, when $H = 1/2$ and $d = 1$, it is well known, see the book of Revuz and Yor [38], p240, that $L_1^0(B^{1/2}) \overset{d}{=} |U|$ with $U \sim \mathcal{N}(0,1)$. Consequently,

$$\lim_{a\to\infty} a^{-2} \log \mathbb{P}\left\{L_1^0(B^{1/2}) \ge a\right\} = -\frac{1}{2}.$$

In Theorem 2.1 we prove that for a fractional Brownian motion a nontrivial limit

$$\lim_{a\to\infty} a^{-1/Hd} \log \mathbb{P}\{L_1^0(B^H) \ge a\}$$

exists and we give bounds for this limit.

Closely related to the fractional Brownian motion is the Riemann–Liouville process $W^H(t)$ with index $H > 0$ which is defined as a stochastic convolution

$$W^H(t) = \int_0^t (t-s)^{H-1/2}\, dB(s)\,, \quad t \ge 0, \tag{1.2}$$

where $B(t)$ is a d-dimensional standard Brownian motion. $\{W^H(t)\}_{t\ge 0}$ is a self-similar zero-mean Gaussian process with index H, as is $B^H(t)$, but $W^H(t)$ does not have stationary increments and there is no upper bound restriction on index $H > 0$. If $L_t^0(W^H)$ denotes the local time of $W^H(t)$ at 0, then by the self-similarity we also have

$$L_t^0(W^H) \overset{d}{=} t^{1-Hd} L_1^0(W^H)\,. \tag{1.3}$$

The relation between $W^H(t)$ and $B^H(t)$ becomes transparent when we write a moving average representation of $B^H(t)$, $t \in \mathbb{R}$, in the form

$$B^H(t) = c_H \int_{-\infty}^t \left[(t-s)^{H-1/2} - (-s)_+^{H-1/2}\right]\, dB(s), \tag{1.4}$$

where

$$c_H = \sqrt{2H}\, 2^H B\,(1-H, H+1/2)^{-1/2}\ , \tag{1.5}$$

Here $B(\cdot,\cdot)$ denotes the beta function, and $B(t)$, $t \in \mathbb{R}$ is a standard d-dimensional Brownian motion (see Lemma A1 for the analytic derivation of c_H.) Then we have a decomposition

$$c_H^{-1} B^H(t) = W^H(t) + Z^H(t), \tag{1.6}$$

where

$$Z^H(t) = \int_{-\infty}^0 \left[(t-s)^{H-1/2} - (-s)^{H-1/2}\right]\, dB(s) \tag{1.7}$$

is a process independent of $W^H(t)$.

This moving average representation for fractional Brownian motion was introduced in the pioneering work of Mandelbrot and Van Ness [34] and used extensively by many authors, sometimes with different normalizing constant c_H in (1.5) (e.g., Li and Linde [30] uses $\Gamma(H+1/2)^{-1}$ for c_H).

We will show that paths of $Z^H(t)$, away from $t=0$, can be matched with functions in the reproducing kernel Hilbert space of $W^H(t)$ (Proposition 3.5, Section 3.2). This and the independence of $Z^H(t)$ from $W^H(t)$ will allow us to show that large deviation constants of tail probabilities of $L_1^0(W^H)$ and of $L_1^0(c_H^{-1}B^H) = c_H^d L_1^0(B^H)$ are the same (Theorem 2.2). In this context we also want to mention Theorem 3.22 of Xiao, [44], who established bounds for tail probabilities of the local time L_1^0 of the general Gaussian processes in the form

$$-C_1 \leq \liminf_{a\to\infty} \frac{1}{\phi(a)} \log\{L_1^0 \geq a\} \leq \limsup_{a\to\infty} \frac{1}{\phi(a)} \log\{L_1^0 \geq a\} \leq -C_2$$

and raised a question on the existence of the limit (Question 3.25, [44]).

Next we will consider p independent copies $B_1^H(t), \dots, B_p^H(t)$ of a standard d-dimensional fractional Brownian motion $B^H(t)$. Throughout this paper

$$p^* := p/(p-1)$$

will stand for the conjugate to $p>1$. Our next and main goal is to investigate large deviations for intersection local time $\alpha^H(\cdot)$ of $B_1^H(t), \cdots, B_p^H(t)$, which is a random measure on $(\mathbb{R}^+)^p$ given heuristically by

$$\alpha^H(A) = \int_A \prod_{j=1}^{p-1} \delta_0\big(B_j^H(s_j) - B_{j+1}^H(s_{j+1})\big)\, ds_1 \cdots ds_p, \quad A \subset (\mathbb{R}^+)^p.$$

Quantities measuring the amount of self-intersection of a random walk, or of mutual intersection of several independent random walks, have been studied intensively for more than twenty years, see e.g. [15], [28], [27], [35], [21], [8], [9]. This research is motivated by the role these quantities play in quantum field theory, see e.g. [16], in our understanding of self-avoiding walks and polymer models, see e.g.[33], [23], or in the analysis of stochastic processes in random environments, see e.g. [22] [18], [2], [17]. In the latter models dependence between a moving particle and a random environment frequently comes from the particle's ability to revisit sites with an attractive (in some sense) environment. Consequently, measures of self-intersection quantify the degree of dependence between movement and environment. Typically, in high dimensions, this dependence gets weaker, as the movements become more transient and self-intersections less likely. Investigation of large deviations for intersection local times is closely related to asymptotics of the partition functions in above models.

There are two equivalent ways to construct $\alpha^H(A)$ rigorously. In the first way, $\alpha^H(A)$ is defined as the local time at zero of the multi-parameter process

$$X(t_1, \cdots, t_p) = \big(B_1^H(t_1) - B_2^H(t_2), \cdots, B_{p-1}^H(t_{p-1}) - B_p^H(t_p)\big) \quad (t_1, \cdots, t_p) \in (\mathbb{R}^+)^p \tag{1.8}$$

More precisely, consider the occupation measure

$$\mu_A(B) = \int_A \mathbf{1}_B\big(B_1^H(s_1) - B_2^H(s_2), \cdots, B_{p-1}^H(s_{p-1}) - B_p^H(s_p)\big)\, ds_1 \cdots ds_p, \quad B \subset \mathbb{R}^{d(p-1)}.$$

By Theorem 7.1, as $Hd < p^*$, there is a density function $\alpha^H(A, \cdot)$ of $\mu_A(\cdot)$ such that if $A = [0, t_1] \times \cdots \times [0, t_p]$, then $\alpha^H([0, t_1] \times \cdots \times [0, t_p], x)$ is jointly continuous in $(t_1, \cdots, t_p, x)$. We define $\alpha^H(A) := \alpha^H(A, 0)$.

For the second way of constructing $\alpha^H(A)$, write for any $\epsilon > 0$

$$\alpha_\epsilon^H(A) = \int_{\mathbb{R}^d} \int_A \prod_{j=1}^p p_\epsilon\big(B_j^H(s_j) - x\big)\, ds_1 \cdots ds_p\, dx, \tag{1.9}$$

where p_ϵ are probability densities approximating δ_0 as $\epsilon \to 0$. Notice that

$$\begin{aligned}\alpha_\epsilon^H(A) &= \int_A h_\epsilon\big(B_1^H(s_1) - B_2^H(s_2), \cdots, B_{p-1}^H(s_{p-1}) - B_p^H(s_p)\big)\, ds_1 \cdots ds_p \\ &= \int_{\mathbb{R}^{d(p-1)}} h_\epsilon(x) \alpha^H(A, x) dx\end{aligned}$$

where

$$h_\epsilon(x_1, \cdots, x_{p-1}) = \int_{\mathbb{R}^d} p_\epsilon(-x) \prod_{j=1}^{p-1} p_\epsilon\Big(\sum_{k=j}^{p-1} x_k - x\Big)$$

is an probability density on $\mathbb{R}^{d(p-1)}$ approaching $\delta_0(x_1, \cdots, x_{p-1})$ as $\epsilon \to 0^+$.

By the continuity of $\alpha^H(A, x)$, $\lim_{\epsilon \to 0^+} \alpha_\epsilon^H(A) = \alpha^H(A)$ almost surely. Applying Lemma 3.1 to the Gaussian field given in (1.8), the convergence is also in $\mathcal{L}^m$ for all positive m. This way of constructing $\alpha^H(A)$ justifies the symbolic notation

$$\alpha^H(A) = \int_{\mathbb{R}^d} \int_A \prod_{j=1}^p \delta_0\big(B_j^H(s_j) - x\big)\, ds_1 \cdots ds_p\, dx.$$

In the special case $p = 2$ and $Hd < 2$, Nualart and Ortiz-Latorre [36] proved that $\alpha_\epsilon^H\big([0, t_1] \times [0, t_2]\big)$ converges in $\mathcal{L}^2$ as $\epsilon \to 0^+$, with

$$p_\epsilon(x) = (2\epsilon\pi)^{-d/2} \exp\{-|x|^2/2\epsilon\}. \tag{1.10}$$

For the Riemann-Liouville process $W^H(t)$ an analogous construction of the intersection local time

$$\begin{aligned}\tilde{\alpha}^H(A) &= \int_A \prod_{j=1}^{p-1} \delta_0\big(W_j^H(s_j) - W_{j+1}^H(s_{j+1})\big)\, ds_1 \cdots ds_p \\ &= \int_{\mathbb{R}^d} \int_A \prod_{j=1}^{p} \delta_0\big(W_j^H(s_j) - x\big)\, ds_1 \cdots ds_p\, dx, \quad A \subset (\mathbb{R}^+)^p\end{aligned}$$

can be done under the same condition $Hd < p^*$.

By the self-similarity of $B^H(t)$ and $W^H(t)$, for any $t > 0$

$$\alpha^H\big([0,t]^p\big) \stackrel{d}{=} t^{p-Hd(p-1)} \alpha^H\big([0,1]^p\big) \tag{1.11}$$

and

$$\tilde{\alpha}^H\big([0,t]^p\big) \stackrel{d}{=} t^{p-Hd(p-1)} \tilde{\alpha}^H\big([0,1]^p\big). \tag{1.12}$$

Finally, we would like to discuss this research in a more general context of Markovian versus non-Markovian structures. Naturally, most of the existing results on large deviation for (intersection) local time have been obtained for Markov processes such as Brownian motions, Lévy stable processes, general Lévy processes, and random walks. The underlying Markovian structure has been essential for the methods in these studies; see Chen [9] for references and a systematical account of such works. Departures from Markovian models are often driven by the underlying physics to match the required level of dependence (memory) and smoothness/roughness of sample paths. Fractional Brownian motion and Riemann–Liouville processes are the most natural candidates as extensions of Brownian motion into the non-Markovian world. They offer the existence of the intersection local time for any number p of processes in any dimension d as long as H is sufficiently small. Therefore, they may help scientists to build more realistic and robust models while posing serious challenge to mathematicians due to the non-Markovian nature.

In this paper, we mainly use Gaussian techniques motivated from the study of continuity properties of local time, and more generally, from theory of Gaussian processes. It is also helpful to see connections between small ball probability estimates and tail behavior of the local time. Indeed, large value of the local time at zero means that the process stayed for a long time in a small neighborhood of zero. By this analogy, Propositions 3.1 and 3.2 can be motivated by the corresponding results for small balls (see comments preceding these propositions in Section 3.1).

2 Main results

Theorem 2.1 *Let $B^H(t)$ be a standard d-dimensional fractional Brownian motion with index H such that $Hd < 1$. Then the limit*

$$\lim_{a\to\infty} a^{-1/(Hd)} \log \mathbb{P}\{L_1^0(B^H) \geq a\} = -\theta(H,d) \tag{2.1}$$

exists and $\theta(H,d)$ satisfies the following bounds

$$\left(\pi c_H^2/H\right)^{1/(2H)} \theta_0(Hd) \leq \theta(H,d) \leq (2\pi)^{1/(2H)}\theta_0(Hd), \tag{2.2}$$

where c_H is given by (1.5) and

$$\theta_0(\kappa) = \kappa \left(\frac{(1-\kappa)^{1-\kappa}}{\Gamma(1-\kappa)}\right)^{1/\kappa}. \tag{2.3}$$

Notice that in the classical case of one-dimensional Brownian motion, (2.2) becomes the equality. The fact that the lower bound is less than or equal to the upper bound in (2.2) is equivalent to $c_H^2 \leq 2H$, which can also be seen directly. Indeed, from (3.16)

$$\frac{c_H^2}{2H} = \mathrm{Var}(B^H(1)|B^H(s), s \leq 0) \leq \mathrm{Var}(B^H(1)) = 1. \tag{2.4}$$

The equality only holds for a Brownian motion, i.e., $H = 1/2$.

Theorem 2.2 *Let $W^H(t)$ be a d-dimensional Riemann–Liouville process as in* (1.2) *such that $Hd < 1$. Then the limit*

$$\lim_{a\to\infty} a^{-1/(Hd)} \log \mathbb{P}\{L_1^0(W^H) \geq a\} = -\tilde{\theta}(H,d), \tag{2.5}$$

exists with

$$\tilde{\theta}(H,d) = (c_H)^{-1/H}\, \theta(H,d)\,, \tag{2.6}$$

where $\theta(H,d)$ is as in Theorem 2.1 and c_H is given by (1.5).

Theorem 2.3 *Let $\tilde{\alpha}^H(\cdot)$ be the intersection local time of p-independent d-dimensional Riemann–Liouville process $W_1^H(t), \cdots, W_p^H(t)$, where $Hd < p^*$. Then the limit*

$$\lim_{a\to\infty} a^{-p^*/(Hdp)} \log \mathbb{P}\big\{\tilde{\alpha}^H\big([0,1]^p\big) \geq a\big\} = -\widetilde{K}(H,d,p) \tag{2.7}$$

exists and $\widetilde{K}(H,d,p)$ satisfies the following bounds

$$\begin{aligned} p\frac{Hd}{p^*}\Big(1-\frac{Hd}{p^*}\Big)^{1-\frac{p^*}{Hd}} \Big(\frac{\pi}{H}\Big)^{\frac{1}{2H}} p^{-\frac{p^*}{2Hp}}\Gamma\Big(1-\frac{Hd}{p^*}\Big)^{-\frac{p^*}{Hd}} &\leq \widetilde{K}(H,d,p) \\ \leq p\frac{Hd}{p^*}\Big(1-\frac{Hd}{p^*}\Big)^{1-\frac{p^*}{Hd}} \left(\frac{2\pi}{c_H^2 p^*}\right)^{\frac{1}{2H}} \left(\int_0^\infty \big(1+t^{2H}\big)^{-d/2} e^{-t}dt\right)^{-\frac{p^*}{Hd}} & \end{aligned} \tag{2.8}$$

where c_H is given by (1.5).

There is a direct way to show that the lower bound is less than or equal to the upper bound in (2.8). Observe that by Hölder inequality,

$$1 + t^{2H} \geq p^{1/p}(p^*)^{1/p^*} t^{2H/p^*}$$

which leads to

$$\int_0^\infty \big(1+t^{2H}\big)^{-d/2} e^{-t} dt \leq p^{-d/(2p)}(p^*)^{-d/(2p^*)}\Gamma(1 - Hd/p^*).$$

After cancellation on both sides of (2.8), the problem is then reduced to examining the relation $c_H^2 \leq 2H$, which is given in (2.4).

Theorem 2.4 *Let $\alpha^H(\cdot)$ be the intersection local time of p-independent standard d-dimensional fractional Brownian motions $B_1^H(t), \cdots, B_p^H(t)$, where $Hd < p^*$. Then the limit*

$$\lim_{a\to\infty} a^{-p^*/(Hdp)} \log \mathbb{P}\big\{\alpha^H\big([0,1]^p\big) \geq a\big\} = -K(H,d,p) \tag{2.9}$$

exists with

$$K(H,d,p) = c_H^{1/H} \tilde{K}(H,d,p). \tag{2.10}$$

Our results seem to be closely related to the large deviations of the self-intersection local times heuristically written as

$$\beta^H\big([0,t]_<^p\big) = \int_{[0,t]_<^p} \prod_{j=1}^{p-1} \delta_0\big(B^H(s_j) - B^H(s_{j+1})\big) ds_1 \cdots ds_p$$

where

$$[0,t]_<^p = \big\{(s_1, \cdots, s_p) \in [0,t]^p;\ \ s_1 < \cdots < s_p\big\}.$$

In the case when $Hd < 1$, we can rewrite

$$\beta^H\big([0,t]_<^p\big) = \frac{1}{p!}\int_{\mathbb{R}^d} \big[L_t^x(B^H)\big]^p dx.$$

To see the connection between α^H and β^H, notice that by Holder inequality and arithmetic and geometric mean inequality,

$$\big(\alpha^H\big([0,1]^p\big)\big)^{1/p} = \bigg(\int_{\mathbb{R}^d} \prod_{j=1}^p L_1^x(B_j^H) dx\bigg)^{1/p} \leq \frac{1}{p}\sum_{j=1}^p \bigg(\int_{\mathbb{R}^d} \big[L_1^x(B_j^H)\big]^p dx\bigg)^{1/p}.$$

Thus, for any $\theta > 0$

$$\mathbb{E}\exp\left\{\theta a^{\frac{p^*-Hd}{Hdp}}\Big(\alpha^H\big([0,1]^p\big)\Big)^{1/p}\right\} \le \left[\mathbb{E}\exp\left\{\theta p^{-1} a^{\frac{p^*-Hd}{Hdp}}\bigg(\int_{\mathbb{R}^d}\big[L_1^x(B^H)\big]^p\,dx\bigg)^{1/p}\right\}\right]^p.$$

On the other hand, by Theorem 2.4 and Varadhan's integral lemma,

$$\begin{aligned}
&\lim_{a\to\infty} a^{-p^*/(Hdp)}\log\mathbb{E}\exp\left\{\theta p^{-1} a^{\frac{p^*-Hd}{Hdp}}\Big(\alpha^H\big([0,1]^p\big)\Big)^{1/p}\right\}\\
&=\sup_{\lambda>0}\Big\{\theta p^{-1}\lambda^{1/p} - K(H,d,p)\lambda^{p^*/Hdp}\Big\}\\
&=(Hd/(p^*K(H,d,p)))^{Hd/(p^*-Hd)}(1-Hd/p^*)(\theta/p)^{p^*/(p^*-Hd)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\liminf_{a\to\infty} a^{-p^*/(Hdp)}\log\mathbb{E}\exp\left\{\theta a^{\frac{p^*-Hd}{Hdp}}\bigg(\int_{\mathbb{R}^d}\big[L_1^x(B^H)\big]^p\,dx\bigg)^{1/p}\right\}\\
&\ge p^{-1}(Hd/(p^*K(H,d,p)))^{Hd/(p^*-Hd)}(1-Hd/p^*)(\theta/p)^{p^*/(p^*-Hd)}.
\end{aligned}\tag{2.11}$$

If this can be strengthened into equality with limits, then by Gärtner-Ellis theorem, for any $\lambda > 0$,

$$\begin{aligned}
&\lim_{a\to\infty} a^{-p^*/(Hdp)}\log\mathbb{P}\left\{\bigg(\int_{\mathbb{R}^d}\big[L_1^x(B^H)\big]^p\,dx\bigg)^{1/p}\ge \lambda a^{1/p}\right\}\\
&=-\sup_{\theta>0}\left\{\lambda\theta - p^{-1}(Hd/(p^*K(H,d,p)))^{\frac{Hd}{p^*-Hd}}(1-Hd/p^*)(\theta/p)^{\frac{p^*}{p^*-Hd}}\right\}\\
&=-p^{-1}K(H,d,p)\lambda^{p^*/(Hd)}
\end{aligned}$$

In particular,

$$\lim_{a\to\infty} a^{-p^*/(Hdp)}\log\mathbb{P}\left\{\int_{\mathbb{R}^d}\big[L_1^x(B^H)\big]^p\,dx\ge a\right\} = -p^{-1}K(H,d,p).\tag{2.12}$$

The conjecture (2.12) is partially supported by a recent result of Hu, Nualart and Song (Theorem 1, [25]) which states that when $Hd < 1$ and $p = 2$

$$\mathbb{E}\left\{\int_{\mathbb{R}^d}\big[L_1^x(B^H)\big]^2 dx\right\}^n \le C^n(n!)^{Hd}\qquad n=1,2,\cdots$$

for some $C > 0$. Indeed, a standard application of Chebyshev inequality and Stirling formula leads to the upper bound of the form

$$\limsup_{a\to\infty} a^{-1/(Hd)}\log\mathbb{P}\left\{\int_{\mathbb{R}^d}\big[L_1^x(B^H)\big]^2 dx\ge \lambda a\right\}\le -l,$$

where l is a positive constant. This rate of decay of tail probabilities is sharp by comparing it with (2.11) for $p = 2$.

In the case $Hd \geq 1$, $\beta^H\big([0,t]_<^p\big)$ can not be properly defined. On the other hand, this problem can be fixed in some cases by renormalization. For simplicity we consider the case $p = 2$. Hu and Nualart prove (Theorem 1, [24]) that for $1 \leq Hd < 3/2$, the renormalized self-intersection local time formally given as

$$\begin{aligned}\gamma^H\big([0,t]_<^2\big) = &\iint_{\{0\leq r<s\leq t\}} \delta_0\big(B^H(r) - B^H(s)\big)\,drds\\ &- \mathbb{E}\iint_{\{0\leq r<s\leq t\}} \delta_0\big(B^H(r) - B^H(s)\big)\,drds\end{aligned}$$

exists with the scaling property

$$\gamma^H\big([0,t]_<^2\big) \overset{d}{=} t^{2-Hd}\gamma^H\big([0,1]_<^2\big) \tag{2.13}$$

We also point that an earlier work by Rosen ([39]) in the special case $d = 2$.

Based on a similar but more heuristic reasoning, it seems plausible to expect that

$$\lim_{a\to\infty} a^{-1/(Hd)} \log \mathbb{P}\Big\{\gamma^H\big([0,1]_<^2\big) \geq a\Big\} = -2^{(Hd)^{-1}-1}K(H,d,2) \tag{2.14}$$

We refer the interested reader to Theorem 4, [25] for some exponential integrabilities established by Hu, Nualart and Song based on Clark-Ocone's formula.

We leave these problems to the future investigation.

Our large deviations estimates can be applied to obtain the law of the iterated logarithm.

Theorem 2.5 *When $Hd < 1$,*

$$\limsup_{t\to\infty} t^{-(1-Hd)}(\log\log t)^{-Hd} L_t^0(B^H) = \theta(H,d)^{-Hd} \quad a.s. \tag{2.15}$$

When $Hd < p^*$,

$$\limsup_{t\to\infty} t^{-p(1-Hd/p^*)}(\log\log t)^{-Hd(p-1)}\alpha^H\big([0,t]^p\big) = K(H,d,p)^{-Hd(p-1)} \quad a.s. \tag{2.16}$$

$$\limsup_{t\to\infty} t^{-p(1-Hd/p^*)}(\log\log t)^{-Hd(p-1)}\widetilde{\alpha}^H\big([0,t]^p\big) = \widetilde{K}(H,d,p)^{-Hd(p-1)} \quad a.s. \tag{2.17}$$

Theorem 2.5 will be proved in section 6. The proof of the lower bound appears to be highly non-trivial due to long-range dependency of the model. The approach relies on a quantified use of Cameron-Martin formula.

Since all main theorems stated in this section have been known in the classic case $H = 1/2$ (see, e.g., [8] and [11]), we assume $H \neq 1/2$ in the remaining of the paper.

3 Basic Tools

In this section we provide some basic results that will be used in our proofs. We state them separately for a convenient reference.

3.1 Comparison of local times

We will give general comparison results for local times for Gaussian processes. They are based on the standard Fourier analytic approach but go far beyond, motivated mainly by similar small deviation estimates. We start with an outline of the analytic method typically used in the study of local times for Gaussian processes, in particular on its the moments, see Berman [6] and Xiao [44].

For a fixed sample function and fixed time $t > 0$, the Fourier transform on space variable $x \in \mathbb{R}^d$ is the function of $\lambda \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} e^{i\lambda\cdot x} L(t,x)dx = \int_0^t e^{i\lambda\cdot X(s)}ds.$$

Thus the local time $L(t,x)$ can be expressed as the inverse Fourier transform:

$$L(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\lambda\cdot x}\int_0^t e^{i\lambda\cdot X(s)}\,dsd\lambda.$$

The m-th power of $L(t,x)$ is

$$L(t,x)^m = \frac{1}{(2\pi)^{md}}\int_{\mathbb{R}^{md}} e^{-ix\cdot\sum_{k=1}^m\lambda_k}\int_{[0,t]^m}\exp\big(i\sum_{k=1}^m\lambda_k\cdot X(s_k)\big)\,ds_1\cdots ds_m d\lambda_1\cdots d\lambda_m.$$

Take the expected value under the sign of integration: the second exponential in the above integral is replaced by the joint characteristic function of $X(s_1),\cdots,X(s_m)$. In the Gaussian case, we obtain

$$\begin{aligned}&\mathbb{E}L(t,x)^m\\ &= \frac{1}{(2\pi)^{md}}\int_{\mathbb{R}^{md}} e^{-ix\cdot\sum_{k=1}^m\lambda_k}\int_{[0,t]^m}\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{k=1}^m\lambda_k\cdot X(s_k)\big)\Big)\,ds_1\cdots ds_m d\lambda_1\cdots d\lambda_m.\end{aligned}$$

Interchanging integration and applying the characteristic function inversion formula, we can get more explicit (but somewhat less useful) expression in terms of integration associated with $\det(\mathbb{E}X(s_i)X(s_j))^{-1/2}$. Estimates of the moments of local time $L(t,x)$ thus depend on the rate of decrease to 0 of $\det(\mathbb{E}X(s_i)X(s_j))$ as $s_j - s_{j-1}\to 0$ for some j. Here in our approach, we have to make proper adjustment by approximating $L(t,x)$.

Consider now a random fields $X(\mathbf{t})$ taking values in $\mathbb{R}^d$, where $\mathbf{t} = (t_1, \ldots, t_p) \in (\mathbb{R}^+)^p$. For a fixed Borel set $A \subset (\mathbb{R}^+)^p$, recall that the local time formally given as

$$L_X(A, x) = \int_A \delta_x\big(X(\mathbf{s})\big)\, d\mathbf{s} \tag{3.1}$$

is defined as the density of the occupation measure

$$\mu_A(B) = \int_A 1_B\big(X(\mathbf{s})\big) d\mathbf{s} \qquad B \subset \mathbb{R}^d$$

if $\mu_A(\cdot)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Given a non-degenerate Gaussian probability density $h(x)$ on $\mathbb{R}^d$ and $\epsilon > 0$, the function $h_\epsilon(x) = \epsilon^{-d/2} h(\epsilon^{-1/2}x)$ is also a probability density. Define the smoothed local time

$$L_X(A, x, \epsilon) = \int_A h_\epsilon\big(X(\mathbf{s}) - x\big)\, d\mathbf{s}. \tag{3.2}$$

Our first proposition provides moment comparison (3.6) which can be viewed as analogy of Anderson's inequality in the small ball analog: For independent Gaussian vectors X, Y, X symmetric,

$$\mathbb{P}(\|X + Y\| \le \epsilon) \le \mathbb{P}(\|X\| \le \epsilon).$$

See Li and Shao [32] for various application of this useful inequality.

Proposition 3.1 *Let $A \subset (\mathbb{R}^+)^p$ be a fixed bounded Borel set. Let $X(\mathbf{t})$ ($\mathbf{t} = (t_1, \ldots, t_p) \in (\mathbb{R}^+)^p$) be a zero-mean $\mathbb{R}^d$-valued Gaussian random field with the local time $L_X(A, x)$ continuous in $x \in \mathbb{R}^d$. Assume that for every $m = 1, 2, \ldots$*

$$\int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \exp\left\{ -\frac{1}{2}\mathrm{Var}\left(\sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\right)\right\} < \infty\,. \tag{3.3}$$

Then $L(A, 0) \in \mathcal{L}^m$ (i.e., finite m-th moment), with

$$\mathbb{E}L_X(A, 0)^m = \frac{1}{(2\pi)^{md}} \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \tag{3.4}$$
$$\times \exp\left\{ -\frac{1}{2}\mathrm{Var}\left(\sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\right)\right\}$$

and

$$\lim_{\epsilon \to 0^+} \mathbb{E}\big|L_X(A, 0, \epsilon) - L_X(A, 0)\big|^m = 0. \tag{3.5}$$

If $Y(\mathbf{t})$ ($\mathbf{t} = (t_1, \ldots, t_p) \in (\mathbb{R}^+)^p$) is another zero-mean $\mathbb{R}^d$-valued Gaussian random field independent of $X(\mathbf{t})$ such that the local time $L_{X+Y}(A, x)$ of $X(\mathbf{t}) + Y(\mathbf{t})$ is continuous in x, then

$$\mathbb{E}\left[L_{X+Y}(A, 0)^m\right] \leq \mathbb{E}\left[L_X(A, 0)^m\right]. \tag{3.6}$$

Proof: By Fourier inversion, we have from (3.2)

$$L_X(A, 0, \epsilon) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\lambda \exp\Big\{ -\frac{\epsilon}{2}(\lambda \cdot \Gamma\lambda)\Big\} \int_A e^{-i\lambda \cdot X(\mathbf{s})} d\mathbf{s}$$

where Γ is the covariance matrix of Gaussian density $h(x)$. Using Fubini theorem,

$$\begin{aligned}
\mathbb{E}L_X(A, 0, \epsilon)^m &= \frac{1}{(2\pi)^{md}} \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \\
&\quad \times \exp\Big\{ -\frac{\epsilon}{2} \sum_{k=1}^m \lambda_k \cdot \Gamma\lambda_k \Big\} \mathbb{E}\exp\Big\{ -i \sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\Big\} \qquad (3.7) \\
&= \frac{1}{(2\pi)^{md}} \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \\
&\quad \times \exp\Big\{ -\frac{\epsilon}{2} \sum_{k=1}^m \lambda_k \cdot \Gamma\lambda_k \Big\} \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\Big)\Big\}.
\end{aligned}$$

By monotonic convergence theorem, the right hand side converges to the right hand side of (3.4) as $\epsilon \to 0^+$. In particular, the family

$$\mathbb{E}L_X(A, 0, \epsilon)^m \quad (\epsilon > 0)$$

is bounded for $m = 1, 2, \cdots$. Consequently, this family is uniformly integrable for $m = 1, 2, \cdots$. Therefore, (3.4) and (3.5) follow from the fact that $L_X(A, 0, \epsilon)$ converges to $L_X(A, 0)$, which is led by the continuity of $L_X(A, x)$.

Finally, (3.6) follows from the comparison

$$\begin{aligned}
&\int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda \cdot \big(X(\mathbf{s}_k) + Y(\mathbf{s}_k)\big)\Big)\Big\} \\
&\leq \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda \cdot X(\mathbf{s}_k)\Big)\Big\}.
\end{aligned}$$

□

In certain situations we can also reverse bound in (3.6) as a result of the Cameron-Martin Formula. In small ball setting, this is motivated by the Chen-Li's inequality [10] which can be used to estimate small ball probabilities under any norm via a relatively easier L_2-norm estimate. See also the survey of Li and Shao [32]. Let X and Y be any two centered Gaussian random vectors in a separable Banach space B with norm $\|\cdot\|$. We use $|\cdot|_{\mu(X)}$ to denote the inner product norm induced on H_μ by $\mu = \mathcal{L}(X)$. Then for any $\lambda > 0$ and $\epsilon > 0$,

$$\mathbb{P}(\|X+Y\| \leq \epsilon) \geq \mathbb{P}(\|X\| \leq \lambda\epsilon) \cdot \mathbb{E}\exp\{-2^{-1}\lambda^2 |Y|^2_{\mu(X)}\}.$$

Next we provide the local time counterpart of this inequality, which is crucial in our estimates. Suppose that the process $X(\mathbf{t})$, $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$, where $\mathbf{T} = (T_1, \ldots, T_p) \in (\mathbb{R}_+)^p$, can be viewed as a Gaussian random vector in a separable Banach space B such that the evaluations $x \mapsto x(\mathbf{t})$ are measurable (say $B = C([\mathbf{0}, \mathbf{T}]; \mathbb{R}^d)$, for concreteness). Let $\mathcal{H}(X)$ denote the reproducing kernel Hilbert space (RKHS) of $X(\mathbf{t})$, $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$ equipped with the norm $\|\cdot\|$. Now we will make a crucial assumption that the independent process $Y(\mathbf{t})$, $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$ has almost all paths in $\mathcal{H}(X)$.

Proposition 3.2 *In the above setting, under the assumptions of Proposition 3.1, we have*

$$\mathbb{E}\left[L_{X+Y}(A,0)^m\right] \geq \mathbb{E}e^{-\frac{1}{2}\|Y\|^2}\mathbb{E}\left[L_X(A,0)^m\right], \tag{3.8}$$

for every $A \subset [\mathbf{0}, \mathbf{T}]$ and $m \in \mathbb{N}$.

Proof: Applying Lemma 3.6(ii), for $g(x) = \prod_{k=1}^m h_\epsilon(x(\mathbf{s}_k))$, $x \in B$, we get

$$\begin{aligned}\mathbb{E}\left[L_{X+Y}(A,0,\epsilon)^m\right] = & \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \, \mathbb{E}\prod_{k=1}^m h_\epsilon\big(X(\mathbf{s}_k) + Y(\mathbf{s}_k)\big) \\ \geq & \mathbb{E}e^{-\frac{1}{2}\|Y\|^2} \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \, \mathbb{E}\prod_{k=1}^m h_\epsilon\big(X(\mathbf{s}_k))\big) \\ = & \mathbb{E}e^{-\frac{1}{2}\|Y\|^2}\mathbb{E}\left[L_X(A,0,\epsilon)^m\right].\end{aligned}$$

Applying (3.5) for both processes, X and $X+Y$, we get (3.8). $\square$

3.2 The remainder in the decomposition of $B^H(t)$

Assume that $H \in (0,1/2) \cup (1/2,1)$ and recall the decomposition (1.6),

$$c_H^{-1} B^H(t) = W^H(t) + Z^H(t), \quad t \geq 0$$

where the remainder process $Z^H(t)$ can be written as

$$Z^H(t) = \int_0^\infty \{(t+s)^{H-1/2} - s^{H-1/2}\}\, d\bar{B}(s), \tag{3.9}$$

with $\bar{B}(s) := B(-s)$, $s \geq 0$. Clearly, $Z^H(t)$ is a self-similar process with index H and the processes $W^H(t)$ and $Z^H(t)$ are independent. With the aim to use bound (3.8), we examine whether paths of $Z^H(t)$ are in the RKHS of $W^H(t)$, considered as a Gaussian random vector in $C([0,T];\mathbb{R}^d)$.

Proposition 3.3 *Process* $\left\{Z^H(t)\right\}_{t>0}$ *has* C^∞*-sample paths. Moreover,*

$$Z^H(t) = I_{0+}^{H+1/2} V^H(t), \tag{3.10}$$

where I_{0+}^{α} *is defined in (A6) and* $V^H(t)$ *is a* $\left(-\frac{1}{2}\right)$*-self-similar Gaussian process given by*

$$V^H(t) = \frac{-1}{\Gamma(\frac{1}{2}-H)} \int_0^\infty \frac{t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}}{t+s} \, d\bar{B}(s). \tag{3.11}$$

Proof: For every $n \in \mathbb{N}$ and $t > 0$

$$\frac{\partial^n}{\partial t^n} Z^H(t) = (H - \frac{1}{2}) \cdots (H - \frac{2n-1}{2}) \int_0^\infty (t+s)^{H-(2n+1)/2} \, d\bar{B}(s)$$

is well defined n^{th}-derivative process with continuous paths on $(0,\infty)$.

For the second part of the proposition, put

$$K(t,s) = \frac{-t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}}{\Gamma(\frac{1}{2}-H)(t+s)}, \quad t, s > 0. \tag{3.12}$$

Notice that for every $t > 0$, $\int_0^\infty K(t,s)^2 \, ds < \infty$ and that for every $s > 0$

$$I_{0+}^{H+1/2} \left[K(\cdot, s)\right](t) = (t+s)^{H-1/2} - s^{H-1/2}.$$

Interchanging the order of stochastic and deterministic integrations we get (3.10). The self-similarity of order $\left(-\frac{1}{2}\right)$ follows directly from (3.11). $\square$

Lemma 3.4 *Almost surely* $\left\{V^H(t)\right\}_{t\in(0,T]} \notin L_2[0,T]$*, for every* $T > 0$*. Therefore, almost surely, sample paths of* $\left\{Z^H(t)\right\}_{t\in[0,T]}$ *are not in the RKHS of* $\left\{W^H(t)\right\}_{t\in[0,T]}$*.*

Proof: Since

$$\int_0^T \mathbb{E}\left[\left(V^H(t)\right)^2\right] \, dt = C \int_0^T t^{-1} \, dt = \infty,$$

the lemma follows by the integrability a Gaussian seminorm and the zero-one law. $\square$

The next proposition is fundamental to our technique relating the local times of $B^H(t)$ and $W^H(t)$.

Proposition 3.5 *For any $a > 0$ there is a Gaussian process $\{Z_a^H(t)\}_{t\geq 0}$ such that*

(i) $Z_a^H(t) = Z^H(t)$ *for all* $t \geq a$;

(ii) Almost all sample paths of $\{Z_a^H(t)\}_{t\in[0,T]}$ *belong to the RKHS of* $\{W^H(t)\}_{t\in[0,T]}$, *for any* $T > 0$.

Proof: First consider $H \in (0, \frac{1}{2})$, so that $m = \lceil H + 1/2 \rceil = 1$. Define

$$Z_a^H(t) = \begin{cases} \frac{t}{a} Z^H(a), & 0 \leq t \leq a \\ Z^H(t), & t > a. \end{cases}$$

By Corollary A4 it is enough to verify that $Z_a^H(t)$ has paths in $AC_2^1[0,T]$ and $Z_a^H(0) = 0$. But this is obvious by Proposition 3.3.

Now we consider $H \in (\frac{1}{2}, 1)$, so that $m = \lceil H + 1/2 \rceil = 2$. Define

$$Z_a^H(t) = \begin{cases} \big(3Z^H(a) - a\dot{Z}^H(a)\big)(t/a)^2 + \big(-2Z^H(a) + a\dot{Z}^H(a)\big)(t/a)^3, & 0 \leq t \leq a \\ Z^H(t), & t > a \end{cases}$$

where $\dot{Z}^H(t) := \dfrac{\partial}{\partial t} Z^H(t)$. By Corollary A4 it is enough to verify that $Z_a^H(t)$ has paths in $AC_2^2[0,T]$, $Z_a^H(0) = 0$ and $\dot{Z}_a^H(0) = 0$. The continuity of $Z_a^H(t)$ and $\dot{Z}_a^H(t)$ at $t = a$ follows by a direct verification and the rest of the claim by Proposition 3.3. □

3.3 Technical lemmas

The following auxiliary results and formulas are used in the proofs of main theorems. They are given here for a convenient reference.

Lemma 3.6 *Let μ be a centered Gaussian measure in a separable Banach space B. Let $g : B \mapsto \mathbb{R}_+$ be a measurable function. Then*

(i) if $\{x \in B : g(x) \geq t\}$ *is symmetric and convex for every* $t > 0$, *then for every* $y \in B$

$$\int_B g(x+y)\,\mu(dx) \leq \int_B g(x)\,\mu(dx);$$

(ii) if g is symmetric ($g(-x) = g(x)$, $x \in B$), then for every y in the RKHS $\mathcal{H}_\mu$ of μ

$$\int_B g(x+y)\,\mu(dx) \geq \exp\left\{-\frac{1}{2}\|y\|_\mu^2\right\} \int_B g(x)\,\mu(dx),$$

where $\|y\|_\mu$ denotes the norm in $\mathcal{H}_\mu$.

Proof: Part *(i)* follows from Anderson's inequality

$$\begin{aligned}\int_B g(x+y)\,\mu(dx) &= \int_0^\infty \mu\{x\in B:\ g(x+y)\ge t\}\,dt\\ &\le \int_0^\infty \mu\{x\in B:\ g(x)\ge t\}\,dt = \int_B g(x)\,\mu(dx).\end{aligned}$$

Part *(ii)* uses Cameron-Martin formula and the convexity of exponential function

$$\begin{aligned}\int_B g(x+y)\,\mu(dx) &= \int_B g(x)\exp\left\{\langle x,y\rangle_\mu - \frac{1}{2}\|y\|_\mu^2\right\}\,\mu(dx)\\ &= \frac{1}{2}\int_B g(x)\exp\left\{\langle x,y\rangle_\mu - \frac{1}{2}\|y\|_\mu^2\right\}\,\mu(dx)\\ &\quad + \frac{1}{2}\int_B g(x)\exp\left\{-\langle x,y\rangle_\mu - \frac{1}{2}\|y\|_\mu^2\right\}\,\mu(dx)\\ &\ge \exp\left\{-\frac{1}{2}\|y\|_\mu^2\right\}\int_B g(x)\,\mu(dx).\end{aligned}$$

$\square$

The next lemma is well-known and goes back at least to 1950s in equivalent forms, see Anderson [1], p42, Berman [5], p293, [6], p71. The basic fact is that conditional distribution of X_k given all the $X_i, 1\le i<k$ is a univariate Gaussian distribution with (conditional) mean $\mathbb{E}(X_k|X_1,\ldots,X_{k-1})$ and (conditional) variance

$$\det(\mathrm{Cov}(X_1,\ldots,X_k))/\det(\mathrm{Cov}(X_1,\ldots,X_{k-1}))$$

for $1\le k\le m$. For the completeness we provide an equivalent geometric argument for the validity of this lemma, see Appendix, Lemma A5.

Lemma 3.7 *Let $(X_1,\ldots,X_m)$ be a mean-zero Gaussian random vector. Then*

$$\det(\mathrm{Cov}(X_1,\ldots,X_m)) = \mathrm{Var}(X_1)\mathrm{Var}(X_2\,|\,X_1)\cdots\mathrm{Var}(X_m\,|\,X_{m-1},\ldots,X_1).$$

Let $B^H(t)$ be given by its moving average representation (1.4). By the deconvolution formula of Pipiras and Taqqu [37] we also have

$$B(t) = c_H^*\int_{-\infty}^t \left((t-s)_+^{1/2-H} - (-s)_+^{1/2-H}\right)\,dB^H(s), \tag{3.13}$$

where $c_H^* = \{c_H\Gamma(H+1/2)\Gamma(3/2-H)\}^{-1}$ and the integral with respect to $B^H(t)$ is well-defined in the L^2-sense. It follows from (1.4) and (3.13) that for every $t\in\mathbb{R}$

$$\mathcal{F}_t := \sigma\{B^H(s);\ \ -\infty<s\le t\} = \sigma\{B(s);\ \ -\infty<s\le t\}, \tag{3.14}$$

where the second equality holds modulo sets of probability zero. Then for every $s < t$

$$\mathbb{E}(B^H(t)\,|\,\mathcal{F}_s) = c_H \int_{-\infty}^{s} \Big((t-u)^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\Big)\, dB(u). \tag{3.15}$$

If $d = 1$, then for every $s < t$

$$\begin{aligned}\mathrm{Var}\,(B^H(t)\,|\,\mathcal{F}_s) &= \mathbb{E}\Big\{\Big[B^H(t) - \mathbb{E}(B^H(t)\,|\,\mathcal{F}_s)\Big]^2 \,|\, \mathcal{F}_s\Big\}\\ &= \mathbb{E}\Big\{\int_s^t (t-u)^{H-\frac{1}{2}}\, dB(u)\,|\,\mathcal{F}_s\Big\}\\ &= c_H^2 \int_s^t (t-u)^{2H-1}\, du = \frac{c_H^2}{2H}(t-s)^{2H}\,. \end{aligned} \tag{3.16}$$

For the reader's convenience we also quote the following lemma due to König and Mörters, [26, Lemma2.3].

Lemma 3.8 *Let $Y \geq 0$ be a random variable and let $\gamma > 0$. If*

$$\lim_{m\to\infty} \frac{1}{m} \log \frac{1}{(m!)^\gamma} \mathbb{E}Y^m = \kappa \tag{3.17}$$

for some $\kappa \in \mathbb{R}$, then

$$\lim_{y\to\infty} \frac{1}{y^{1/\gamma}} \log \mathbb{P}\{Y \geq y\} = -\gamma e^{-\kappa/\gamma}. \tag{3.18}$$

4 Large deviations for local times

4.1 Proof of Theorem 2.1 – superadditivity argument

In the light of Lemma 3.8, it is enough to show that the limit in (3.17) exists for $Y = L_1^0(B^H)$ and for $\gamma = Hd$. We will prove it by a superadditivity argument. Let τ be an exponential time independent of $B^H(t)$. We will first show that for any integer $m, n \geq 1$,

$$\mathbb{E}\Big[L_\tau^0(B^H)^{m+n}\Big] \geq \binom{m+n}{m} \mathbb{E}\Big[L_\tau^0(B^H)^m\Big]\mathbb{E}\Big[L_\tau^0(B^H)^n\Big] \tag{4.1}$$

Let $t > 0$ be fixed. Notice that by Theorem 7.1, the Gaussian process $B^H(t)$ satisfies the condition (3.2) posted in Lemma 3.1. By (3.4), therefore,

$$
\begin{aligned}
&\mathbb{E}\Big[L_t^0(B^H)^m\Big]\\
&=\frac{1}{(2\pi)^{md}}\int_{[0,t]^m}ds_1\cdots ds_m\int_{(\mathbb{R}^d)^m}d\lambda_1\cdots d\lambda_m\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\cdot B^H(s_k)\Big)\Big\}\\
&=\frac{1}{(2\pi)^{md}}\int_{[0,t]^m}ds_1\cdots ds_m\Big[\int_{\mathbb{R}^m}d\lambda_1\cdots d\lambda_m\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k B_0^H(s_k)\Big)\Big\}\Big]^d
\end{aligned}
$$

where $B_0^H(t)$ is 1-dimensional fractional Brownian motion.

By integration with respect to Gaussian measures

$$
\begin{aligned}
&\int_{\mathbb{R}^m}d\lambda_1\cdots d\lambda_m\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k B_0^H(s_k)\Big)\Big\}\\
&=(2\pi)^{m/2}\det\Big\{\mathrm{Cov}\Big(B_0^H(s_1),\cdots,B_0^H(s_m)\Big)\Big\}^{-1/2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\Big[L_t^0(B^H)^m\Big]\\
&=\frac{1}{(2\pi)^{md/2}}\int_{[0,t]^m}ds_1\cdots ds_m\det\Big\{\mathrm{Cov}\Big(B_0^H(s_1),\cdots,B_0^H(s_m)\Big)\Big\}^{-d/2} \qquad (4.2)\\
&=\frac{m!}{(2\pi)^{md/2}}\int_{[0,t]_<^m}ds_1\cdots ds_m\det\Big\{\mathrm{Cov}\Big(B_0^H(s_1),\cdots,B_0^H(s_m)\Big)\Big\}^{-d/2}.
\end{aligned}
$$

In (4.2) and elsewhere, for any $A\subset\mathbb{R}^+$ and an integer $m\geq 1$, we define

$$
A_<^m=\big\{(s_1,\cdots,s_m)\in A^m;\ \ s_1<\cdots<s_m\big\}.
$$

Put

$$
\mathcal{A}(s_1,\cdots,s_k)=\sigma\Big\{B_0^H(s_1),\cdots,B_0^H(s_k)\Big\},\qquad k=1,\cdots,m,
$$

and $\mathcal{A}(s_1,\cdots,s_k)=\{\emptyset,\Omega\}$ when $k=0$. By Lemma 3.7,

$$
\begin{aligned}
&\mathbb{E}\Big[L_t^0(B^H)^m\Big] \qquad (4.3)\\
&=\frac{m!}{(2\pi)^{md/2}}\int_{[0,t]_<^m}ds_1\cdots ds_m\prod_{k=1}^m\mathrm{Var}\Big(B_0^H(s_k)|B_0^H(s_1),\cdots,B_0^H(s_{k-1})\Big)^{-d/2}.
\end{aligned}
$$

We are ready to establish (4.1). Put

$$
\varphi_m(s_1,\cdots,s_m)=\prod_{k=1}^m\mathrm{Var}\Big(B_0^H(s_k)|B_0^H(s_1),\cdots,B_0^H(s_{k-1})\Big)^{-d/2}
$$

and let $m, n \geq 1$ be integers. Then, for any $s_1 < \cdots < s_{n+m}$ and $n+1 \leq k \leq n+m$,

$$\begin{aligned}
&\mathrm{Var}\Big(B_0^H(s_k)|B_0^H(s_1),\cdots,B_0^H(s_{k-1})\Big)\\
&\quad= \mathrm{Var}\Big(B_0^H(s_k)-B_0^H(s_n)|B_0^H(s_1),\cdots,B_0^H(s_{k-1})\Big)\\
&\quad= \mathrm{Var}\Big(B_0^H(s_k)-B_0^H(s_n)|B_0^H(s_1),\cdots,B_0^H(s_n),B_0^H(s_{n+1})-B_0^H(s_n),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdots,B_0^H(s_{k-1})-B_0^H(s_n)\Big)\\
&\quad\leq \mathrm{Var}\Big(B_0^H(s_k)-B_0^H(s_n)|B_0^H(s_{n+1})-B_0^H(s_n),\cdots,B_0^H(s_{k-1})-B_0^H(s_n)\Big)\\
&\quad= \mathrm{Var}\Big(B_0^H(s_k-s_n)|B_0^H(s_{n+1}-s_n),\cdots,B_0^H(s_{k-1}-s_n)\Big),
\end{aligned}$$

where the last step follows from the stationarity of increments. Thus

$$\varphi_{n+m}(s_1,\cdots,s_{n+m}) \geq \varphi_n(s_1,\cdots,s_n)\varphi_m(s_{n+1}-s_n,\cdots,s_{n+m}-s_n).$$

Notice that from (4.2)

$$\begin{aligned}
\mathbb{E}\Big[L_\tau^0(B^H)^m\Big] &= \frac{m!}{(2\pi)^{md/2}}\mathbb{E}\int_{[0,\tau]_<^m} ds_1\cdots ds_m\,\varphi_m(s_1,\cdots,s_m)\\
&= \frac{m!}{(2\pi)^{md/2}}\mathbb{E}\int_{s_1<\cdots<s_m} 1_{s_m<\tau}ds_1\cdots ds_m\,\varphi_m(s_1,\cdots,s_m)\\
&= \frac{m!}{(2\pi)^{md/2}}\int_{(\mathbb{R}^+)_<^m} ds_1\cdots ds_m\,\varphi_m(s_1,\cdots,s_m)e^{-s_m}\,. \qquad (4.4)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}\Big[L_\tau^0(B^H)^{n+m}\Big] &= \frac{(n+m)!}{(2\pi)^{(n+m)d/2}}\int_{(\mathbb{R}^+)_<^{n+m}} ds_1\cdots ds_{n+m}\,\varphi_{n+m}(s_1,\cdots,s_{n+m})e^{-s_{n+m}}\\
&\geq \frac{(n+m)!}{(2\pi)^{(n+m)d/2}}\int_{(\mathbb{R}^+)_<^{n+m}} ds_1\cdots ds_{n+m}\,\varphi_n(s_1,\cdots,s_n)e^{-s_n}\\
&\qquad\qquad\times\varphi_m(s_{n+1}-s_n,\cdots,s_{n+m}-s_n)e^{-(s_{n+m}-s_n)}\\
&= \frac{(n+m)!}{(2\pi)^{(n+m)d/2}}\int_{(\mathbb{R}^+)_<^{n}} ds_1\cdots ds_n\,\varphi_n(s_1,\cdots,s_n)e^{-s_n}\\
&\qquad\qquad\times\int_{(\mathbb{R}^+)_<^m} dt_1\cdots dt_m\,\varphi_m(t_1,\cdots,t_m)e^{-t_m}\\
&= \binom{n+m}{m}\mathbb{E}\Big[L_\tau^0(B^H)^n\Big]\mathbb{E}\Big[L_\tau^0(B^H)^m\Big]\,.
\end{aligned}$$

We proved relation (4.1) that says that the sequence $m \mapsto \log\frac{1}{m!}\mathbb{E}\Big[L_\tau^0(B^H)^m\Big]$ is super-additive. By Fekete's lemma the limit

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{m!}\mathbb{E}\Big[L_\tau^0(B^H)^m\Big]=\sup_{m\geq 1}\frac{1}{m}\log\frac{1}{m!}\mathbb{E}\Big[L_\tau^0(B^H)^m\Big]=\log L\,, \tag{4.5}$$

exists, possibly as an extended number.

By the scaling property (1.1),

$$\begin{aligned}\mathbb{E}\Big[L_\tau^0(B^H)^m\Big]&=\mathbb{E}\Big[\tau^{(1-Hd)m}\Big]\,\mathbb{E}\Big[L_1^0(B^H)^m\Big]\\&=\Gamma\left(1+(1-Hd)m\right)\,\mathbb{E}\Big[L_1^0(B^H)^m\Big]\,.\end{aligned}$$

From (4.5) and Stirling's formula we get

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^{Hd}}\mathbb{E}\Big[L_1^0(B^H)^m\Big]=\log\Big\{(1-Hd)^{-(1-Hd)}L\Big\}. \tag{4.6}$$

Applying Lemma 3.8 we establish (2.1) with

$$\theta(H,d)=Hd(1-Hd)^{-1+1/Hd}L^{-1/Hd}\,. \tag{4.7}$$

To obtain (2.2) and complete the proof it is enough to show that

$$(2\pi)^{-d/2}\Gamma(1-Hd)\leq L\leq\Big(H^{-1}\pi c_H^2\Big)^{-d/2}\Gamma(1-Hd)\,. \tag{4.8}$$

By (4.1)

$$\frac{1}{m!}\mathbb{E}\Big[L_\tau^0(B^H)^m\Big]\geq\big\{\mathbb{E}L_\tau^0(B^H)\big\}^m=\big\{(2\pi)^{-d/2}\Gamma(1-Hd)\big\}^m\,,$$

where the equality comes from (4.4) (for $m=1$). This proves the lower bound in (4.8).

To prove the upper bound, we first notice that

$$\begin{aligned}\mathrm{Var}\Big(B_0^H(s_k)\,|\,B^H(s_1),\cdots,B^H(s_{k-1})\Big)&\geq\mathrm{Var}\Big(B_0^H(s_k)|\,B_0(s),s\leq s_{k-1}\Big)\\&=\frac{c_H^2}{2H}(s_k-s_{k-1})^{2H},\end{aligned} \tag{4.9}$$

where we used (3.16). Hence the function φ defined above satisfies

$$\varphi_m(s_1,\cdots,s_m)\leq\big(2H/c_H^2\big)^{md/2}\prod_{k=1}^m(s_k-s_{k-1})^{-Hd},$$

and by (4.4),

$$\mathbb{E}\Big[L_\tau^0(B^H)^m\Big] \le \left(2H/c_H^2\right)^{md/2} m! \int_{(\mathbb{R}^+)^m_<} ds_1\cdots ds_m \prod_{k=1}^m (s_k - s_{k-1})^{-Hd} e^{-s_m} \tag{4.10}$$

$$= \left(2H/c_H^2\right)^{md/2} m! \left\{\int_0^\infty t^{-Hd} e^{-t}\, dt\right\}^m = \left(2H/c_H^2\right)^{md/2} m!\Gamma(1-Hd)^m .$$

This establishes (4.8) and completes the proof. □

4.2 Proof of Theorem 2.2 – comparison argument

First we note that

$$L_t^0\left(c_H^{-1}B^H\right) = c_H^d L_t^0\left(B^H\right). \tag{4.11}$$

Thus, from the decomposition (1.6) and (3.6) for every $m \in \mathbb{N}$,

$$c_H^{md}\mathbb{E}\left[L_1^0\left(B^H\right)^m\right] \le \mathbb{E}\left[L_1^0\left(W^H\right)^m\right]. \tag{4.12}$$

To prove a reverse inequality (up to a multiplicative constant) we use notation (3.1). Fix $a \in (0,1)$ and let let $Z_a^H(t)$, $t \ge 0$ be the process specified in Proposition 3.5 that is also independent of $W^H(t)$, $t \ge 0$. We have

$$c_H^d L_1^0\left(B^H\right) = L_{c_H^{-1}B^H}([0,1],0) \ge L_{c_H^{-1}B^H}([a,1],0) = L_{W^H+Z_a^H}([a,1],0).$$

Thus, by (3.8) we get

$$\begin{aligned} c_H^{md}\mathbb{E}\left[L_1^0\left(B^H\right)^m\right] \ge& \mathbb{E}\left[L_{W^H+Z_a^H}([a,1],0)^m\right] \ge K_a\mathbb{E}\left[L_{W^H}([a,1],0)^m\right] \\ =& K_a\mathbb{E}\left[\left(L_1^0(W^H) - L_a^0(W^H)\right)^m\right] \\ \ge& K_a\left\{\mathbb{E}\left[L_1^0(W^H)^m\right]^{1/m} - \mathbb{E}\left[L_a^0(W^H)^m\right]^{1/m}\right\}^m \\ =& K_a\left(1-a^{1-Hd}\right)^m \mathbb{E}\left[L_1^0(W^H)^m\right], \end{aligned}$$

where the last equality uses self-similarity (1.3) and $K_a = \mathbb{E}\exp\left\{-\frac{1}{2}\|Z_a^H\|^2\right\}$. Here $\|Z_a^H\| < \infty$ a.s. is the RKHS norm associated with $\{W^H(t)\}_{t\in[0,1]}$ and computed for paths of $\{Z_a^H(t)\}_{t\in[0,1]}$. This together with (4.12) yields

$$c_H^{md}\mathbb{E}\left[L_1^0\left(B^H\right)^m\right] \le \mathbb{E}\left[L_1^0\left(W^H\right)^m\right] \le K_a^{-1}\left(1-a^{1-Hd}\right)^{-m} c_H^{md}\mathbb{E}\left[L_1^0\left(B^H\right)^m\right].$$

Applying the limit as in (4.6) to both sides and then passing $a \to 0$ gives

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^{Hd}}\mathbb{E}\Big[L_1^0(W^H)^m\Big] = \log\Big\{c_H^d(1-Hd)^{-(1-Hd)}L\Big\}.$$

Therefore, by Lemma 3.8 the limit in (2.5) exists and $\tilde\theta(H,d) = c_H^{-1/H}\theta(H,d)$ by (4.7). □

5 Large deviations for intersection local times

5.1 Proof of Theorem 2.3 – subadditivity argument

Let $\tilde{\alpha}^H_\epsilon(A)$ be defined analogously to (1.9) by

$$\tilde{\alpha}^H_\epsilon(A) = \int_{\mathbb{R}^d}\int_A \prod_{j=1}^p p_\epsilon\big(W^H_j(s_j) - x\big)\, ds_1\cdots ds_p\, dx,$$

where p_ϵ is as in (1.10). We will first prove the subadditivity property: for every $m, n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\Big[\tilde{\alpha}^H_\epsilon\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^{m+n}\Big] \\
&\quad\le \binom{m+n}{m}^p \mathbb{E}\Big[\tilde{\alpha}^H_\epsilon\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^{m}\Big]\,\mathbb{E}\Big[\tilde{\alpha}^H_\epsilon\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^{n}\Big],
\end{aligned}\tag{5.1}$$

where $\tau_1,\ldots,\tau_p$ are iid exponential random variables with mean 1 and independent of $W^H_1(t),\ldots,W^H_p(t)$. Indeed, since

$$\tilde{\alpha}^H_\epsilon\big([0,t_1]\times\cdots\times[0,t_p]\big)^m = \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \prod_{j=1}^p\prod_{k=1}^m \int_0^{t_j} p_\epsilon\big(W^H_j(s_{j,k}) - x_k\big)\, ds_{j,k},$$

we can write

$$\mathbb{E}\left[\tilde{\alpha}^H_\epsilon\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^{m+n}\right] = \int_{(\mathbb{R}^d)^{m+n}} dx_1\cdots dx_{m+n}\,\xi(x_1,\ldots,x_{m+n})^p, \tag{5.2}$$

where

$$\xi(x_1,\ldots,x_{m+n}) = \int_0^\infty dt\, e^{-t}\int_{[0,t]^{m+n}} ds_1\cdots ds_{m+n}\,\mathbb{E}\prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_k\big).$$

Let

$$D_t = \big\{(s_1,\ldots,s_{m+n}) \in [0,t]^{m+n} :\, \min\{s_1,\ldots,s_m\} \le \min\{s_{m+1},\ldots,s_{m+n}\}\big\}\,.$$

There are exactly $\binom{m+n}{m}$ permutations σ_i of $\{1,\ldots,m+n\}$ such that $\bigcup_i \sigma_i^{-1}D_t = [0,t]^{m+n}$ and $\sigma_i^{-1}D_t$ are disjoint modulo sets of measure zero (here, $\sigma(s_1,\ldots,s_{m+n}) :=$

$(s_{\sigma(1)}, \dots, s_{\sigma(m+n)}))$. Therefore,

$$\begin{aligned}
&\int_{[0,t]^{m+n}} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_k\big) \\
&\qquad = \sum_i \int_{\sigma_i^{-1} D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_k\big) \\
&\qquad = \sum_i \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_{\sigma_i(k)}\big),
\end{aligned}$$

which gives by Hölder inequality

$$\begin{aligned}
\xi(x_1, \dots, x_{m+n})^p &= \left\{ \sum_i \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_{\sigma_i(k)}\big) \right\}^p \\
&\le \binom{m+n}{m}^{p-1} \sum_i \left\{ \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_{\sigma_i(k)}\big) \right\}^p .
\end{aligned}$$

Substituting into (5.2) yields

$$\begin{aligned}
\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1] \times \cdots \times [0,\tau_p]\big)^{m+n}\Big] &\le \binom{m+n}{m}^{p-1} \sum_i \int_{(\mathbb{R}^d)^{m+n}} dx_1 \cdots dx_{m+n} \\
&\quad \times \left\{ \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_{\sigma_i(k)}\big) \right\}^p \\
= \binom{m+n}{m}^p \int_{(\mathbb{R}^d)^{m+n}} dx_1 \cdots dx_{m+n} &\left\{ \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_k\big) \right\}^p .
\end{aligned}$$

Since the last integrand can be written as

$$\begin{aligned}
\left\{ \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\epsilon\big(W^H(s_k) - x_k\big) \right\}^p &= \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p e^{-(t_1 + \cdots + t_p)} \\
\times \int_{D_{t_1} \times \cdots \times D_{t_p}} \Big(\prod_{j=1}^p ds_{j,1} \cdots ds_{j,m+n} \Big)\, &\mathbb{E} \prod_{k=1}^{m+n} \prod_{j=1}^p p_\epsilon\big(W_j^H(s_{j,k}) - x_k\big),
\end{aligned}$$

after integrating with respect to $x_1, \dots, x_{m+n}$ we get

$$\begin{aligned}
\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1] \times \cdots \times [0,\tau_p]\big)^{m+n}\Big] &\le \binom{m+n}{m}^p \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p e^{-(t_1 + \cdots + t_p)} \qquad (5.3) \\
\times \int_{D_{t_1} \times \cdots \times D_{t_p}} \Big(\prod_{j=1}^p ds_{j,1} \cdots ds_{j,m+n} \Big)\, &\mathbb{E} \prod_{k=1}^{m+n} g_\epsilon\big(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k})\big),
\end{aligned}$$

where

$$g_\epsilon(y_1,\cdots,y_p):=\int_{\mathbb{R}^d}\prod_{j=1}^p p_\epsilon(y_j-x)\,dx \tag{5.4}$$
$$=(2\pi\epsilon)^{-dp/2}\int_{\mathbb{R}^d}e^{-\left(|x|^2-2x\cdot\overline{y}+p^{-1}\sum_{i=1}^p|y_i|^2\right)p/(2\epsilon)}$$
$$=(2\pi\epsilon)^{-d(p-1)/2}p^{-d/2}\exp\Big\{-\frac{1}{2\epsilon}\sum_{j=1}^p|y_j-\overline{y}|^2\Big\},$$

and $\overline{y}:=p^{-1}\sum_{i=1}^p y_i$ for $y_1,\ldots,y_p\in\mathbb{R}^d$. Moreover,

$$\int_{D_{t_1}\times\cdots\times D_{t_p}}\Big(\prod_{j=1}^p ds_{j,1}\cdots ds_{j,m+n}\Big)\,\mathbb{E}\prod_{k=1}^{m+n}g_\epsilon\big(W_1^H(s_{1,k}),\ldots,W_p^H(s_{p,k})\big) \tag{5.5}$$
$$=\int_{[\mathbf{0},\mathbf{t}]^m}\Big(\prod_{j=1}^p ds_{j,1}\cdots ds_{j,m}\Big)\int_{[\mathbf{0},\mathbf{t}-\mathbf{s}^*]^n}\Big(\prod_{j=1}^p ds_{j,m+1}\cdots ds_{j,m+n}\Big)$$
$$\times\,\mathbb{E}\prod_{k=1}^m g_\epsilon\big(W_1^H(s_{1,k}),\ldots,W_p^H(s_{p,k})\big)\prod_{k=m+1}^{m+n}g_\epsilon\big(W_1^H(s_1^*+s_{1,k}),\cdots,W_p^H(s_p^*+s_{p,k})\big),$$

where

$$\mathbf{t}=(t_1,\cdots,t_p),\qquad \mathbf{s}^*=(s_1^*,\cdots,s_p^*),$$

and

$$s_j^*=\max\{s_{j,k}:\,1\le k\le m\}.$$

Assuming that $W_j^H(t)$ are given by (1.2) with independent Brownian motions $B_j(t)$, define $\mathcal{F}_{\mathbf{s}^*}=\sigma\big\{B_j(u_j):\,u_j\le s_j^*,\ j=1,\ldots,p\big\}$. Put also

$$Y_j(s_j^*,s)=\int_{s_j^*}^{s_j^*+s}(s_j^*+s-u)^{H-\frac12}dB_j(u)\ \text{and}\ Z(s_j^*,s)=\int_0^{s_j^*}(s_j^*+s-u)^{H-\frac12}dB_j(u),$$

so that $W_j(s_j^*+s)=Y_j(s_j^*,s)+Z_j(s_j^*,s)$. The last expectation can be written as

$$\mathbb{E}\Big\{\prod_{k=1}^m g_\epsilon\big(W_1^H(s_{1,k}),\ldots,W_p^H(s_{p,k})\big)$$
$$\times\,\mathbb{E}\Big[\prod_{k=m+1}^{m+n}g_\epsilon\big(Y_1^H(s_1^*,s_{1,k})+Z_1^H(s_1^*,s_{1,k}),\cdots,Y_p^H(s_p^*,s_{p,k})+Z_p^H(s_p^*,s_{p,k})\big)\Big|\mathcal{F}_{s^*}\Big]\Big\}$$
$$\le\mathbb{E}\Big[\prod_{k=1}^m g_\epsilon\big(W_1^H(s_{1,k}),\ldots,W_p^H(s_{p,k})\big)\Big]\mathbb{E}\Big[\prod_{k=m+1}^{m+n}g_\epsilon\big(Y_1^H(s_1^*,s_{1,k}),\cdots,Y_p^H(s_p^*,s_{p,k})\big)\Big]$$
$$=\mathbb{E}\Big[\prod_{k=1}^m g_\epsilon\big(W_1^H(s_{1,k}),\ldots,W_p^H(s_{p,k})\big)\Big]\mathbb{E}\Big[\prod_{k=m+1}^{m+n}g_\epsilon\big(W_1^H(s_{1,k}),\cdots,W_p^H(s_{p,k})\big)\Big],$$

where the inequality follows from Lemma 3.6(i) (see the evaluation of g_ϵ in (5.4) and the positive quadratic form associated with it) and the last equality from that

$$\big(Y_1(s_1^*, s_{1,k}), \dots, Y_p(s_p^*, s_{p,k})\big) \overset{d}{=} \big(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k})\big).$$

Combining the above bound with (5.5) and then with (5.3) we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^{m+n}\Big] \le \binom{m+n}{m}^p \int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p e^{-(t_1+\cdots+t_p)}\\
&\quad\times \int_{[\mathbf{0},\mathbf{t}]^m} \Big(\prod_{j=1}^p ds_{j,1}\cdots ds_{j,m}\Big)\mathbb{E}\prod_{k=1}^m g_\epsilon\big(W_1^H(s_{1,k}),\dots,W_p^H(s_{p,k})\big)\\
&\quad\times \int_{[\mathbf{0},\mathbf{t}-\mathbf{s}^*]^n} \Big(\prod_{j=1}^p ds_{j,m+1}\cdots ds_{j,m+n}\Big)\mathbb{E}\prod_{k=m+1}^{m+n} g_\epsilon\big(W_1^H(s_{1,k}),\cdots,W_p^H(s_{p,k})\big)\\
&\quad= \binom{m+n}{m}^p \int_{(\mathbb{R}_+)^m} \Big(\prod_{j=1}^p ds_{j,1}\cdots ds_{j,m}\Big)\mathbb{E}\prod_{k=1}^m g_\epsilon\big(W_1^H(s_{1,k}),\dots,W_p^H(s_{p,k})\big)\\
&\qquad\times e^{-(s_1^*+\cdots+s_p^*)}\int_{[\mathbf{s}^*,\infty]^p} dt_1\cdots dt_p e^{-[(t_1-s_1^*)+\cdots+(t_p-s_p^*)]}\\
&\qquad\times \int_{[\mathbf{0},\mathbf{t}-\mathbf{s}^*]^n} \Big(\prod_{j=1}^p ds_{j,m+1}\cdots ds_{j,m+n}\Big)\mathbb{E}\prod_{k=m+1}^{m+n} g_\epsilon\big(W_1^H(s_{1,k}),\cdots,W_p^H(s_{p,k})\big)\\
&\quad= \binom{m+n}{m}^p \mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]\,\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^n\Big],
\end{aligned}$$

where in the last equality we use

$$e^{-(s_1^*+\cdots+s_p^*)} = \int_{(\mathbb{R}_+)^p} e^{-(t_1+\cdots+t_p)}\prod_{k=1}^m \mathbf{1}_{[\mathbf{s}^*,\mathbf{t}]}(s_{1,k},\dots,s_{p,k})\, dt_1\cdots dt_p$$

and the definition of g_ϵ in (5.4). The subadditivity (5.1) is thus proved for any $\epsilon > 0$.

Now we would like to take $\epsilon \to 0$ on the both sides of (5.1) in an attempt to establish

$$\begin{aligned}
&\mathbb{E}\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^{m+n} \qquad (5.6)\\
&\le \binom{m+n}{n}^p \mathbb{E}\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\,\mathbb{E}\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^n.
\end{aligned}$$

To this end we need to show that for any $m \ge 1$, $\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)$ is indeed in $\mathcal{L}^m(\Omega,\mathcal{A},\mathbb{P})$ and

$$\lim_{\epsilon\to 0^+}\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big] = \mathbb{E}\Big[\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]. \qquad (5.7)$$

Indeed, using (5.1) repeatedly we have that

$$\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]\le (m!)^p\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big].$$

Notice that

$$\begin{aligned}&\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]=\int_{\mathbb{R}^d}\bigg[\int_0^\infty e^{-t}\mathbb{E}p_\epsilon\big(W^H(t)-x\big)dt\bigg]^p\\&=\int_{\mathbb{R}^d}\bigg[\int_0^\infty e^{-t}\int_{\mathbb{R}^d}p_\epsilon(y-x)p_{t^*}(y)\,dy\bigg]^p dx\end{aligned}$$

where $t^*=(2H)^{-1}t^{2H}$ and the last step follows from the easy-to-check fact that $W^H(t)\sim N(0,(2H)^{-1}t^{2H}I_d)$. By Jensen inequality, the right hand side is less than or equal to

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_\epsilon(y-x)\bigg[\int_0^\infty e^{-t}\int_{\mathbb{R}^d}p_{t^*}(y)\,dy\bigg]^p dydx=\int_{\mathbb{R}^d}\bigg[\int_0^\infty e^{-t}\int_{\mathbb{R}^d}p_{t^*}(y)\,dy\bigg]^p dy\\&=\int_0^\infty\!\!\cdots\!\int_0^\infty dt_1\cdots dt_p e^{-(t_1+\cdots+t_p)}\int_{\mathbb{R}^d}\prod_{j=1}^p p_{t_j^*}(x)dx\\&=\Big(H/\pi\Big)^{d(p-1)/2}\int_0^\infty\!\!\cdots\!\int_0^\infty e^{-(t_1+\cdots+t_p)}\Big(\sum_{j=1}^p\prod_{1\le k\ne j\le p}t_k^{2H}\Big)^{-d/2}dt_1\cdots dt_p\end{aligned}$$

where the last step follows from a routine Gaussian integration.

By arithmetic-geometric mean inequality,

$$\frac{1}{p}\sum_{j=1}^p\prod_{1\le k\ne j\le p}t_k^{2H}\ge\prod_{j=1}^p\prod_{1\le k\ne j\le p}t_k^{2H/p}=\prod_{j=1}^p t_k^{2H(p-1)/p}.$$

So we have

$$\begin{aligned}\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]&\le\Big(H/\pi\Big)^{d(p-1)/2}p^{-d/2}\bigg(\int_0^\infty t^{-Hd(p-1)/p}e^{-t}dt\bigg)^p\\&=\Big(H/\pi\Big)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p.\end{aligned}$$

Summarizing our computation, we obtain

$$(m!)^{-p}\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]\le\bigg(\Big(H/\pi\Big)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p\bigg)^m. \tag{5.8}$$

By Theorem 7.1, the process

$$X^H(t_1,\cdots,t_p)=\Big(W_1^H(t_1)-W_2^H(t_2),\cdots,W_{p-1}^H(t_{p-1})-W_p^H(t_p)\Big)$$

satisfies the condition (3.2) with $A=[\mathbf{0},\mathbf{t}]=[0,t_1]\times\cdots[0,t_p]$ for any $t_1,\cdots,t_p\geq 0$ and

$$\begin{aligned}&\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\\&=\int_{[\mathbf{0},\mathbf{t}]}h_\epsilon\big(W_1^H(s_1)-W_2^H(s_2),\cdots,W_{p-1}^H(s_{p-1})-W_p^H(s_p)\big)ds_1\cdots ds_p\end{aligned}$$

where

$$h(x_1,\cdots,x_{p-1})=\int_{\mathbb{R}^d}p_1(-x)\prod_{j=1}^{p-1}p_1\Big(\sum_{k=j}^{p-1}x_k-x\Big)dx$$

is a non-degenerate normal density on $\mathbb{R}^{d(p-1)}$. By Lemma 3.1, $\tilde{\alpha}_\epsilon^H\big([0,t_1]\times\cdots\times[0,t_p]\big)\in\mathcal{L}^m(\Omega,\mathcal{A},\mathbb{P})$ and

$$\lim_{\epsilon\to 0^+}\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big]=\mathbb{E}\Big[\tilde{\alpha}^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big]. \tag{5.9}$$

In addition, by the representation (3.7) one can see that for any $\epsilon'<\epsilon$,

$$\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big]\leq\mathbb{E}\Big[\tilde{\alpha}_{\epsilon'}^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big].$$

Thus, (5.7) follows from monotonic convergence theorem and the identities

$$\begin{aligned}&\tilde{\alpha}_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\\&=\int_{(\mathbb{R}^+)^p}e^{-(t_1+\cdots+t_p)}\mathbb{E}\Big[\tilde{\alpha}_\epsilon^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big]dt_1\cdots dt_p\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}&\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\\&=\int_{(\mathbb{R}^+)^p}e^{-(t_1+\cdots+t_p)}\mathbb{E}\Big[\tilde{\alpha}^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big]dt_1\cdots dt_p.\end{aligned} \tag{5.11}$$

Further, by (5.8) we obtain the bound

$$(m!)^{-p}\mathbb{E}\Big[\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]\leq\bigg(\Big(H/\pi\Big)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p\bigg)^m. \tag{5.12}$$

The inequality (5.6) implies that the sequence $m\mapsto\log(m!)^{-p}\mathbb{E}\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m$ is sub-additive. Hence the limit

$$\lim_{m\to\infty}\frac{1}{m}\log(m!)^{-p}\mathbb{E}\tilde{\alpha}^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m=c(d,H,p) \tag{5.13}$$

exists, possibly as an extended number. Further, by (5.12)

$$c(d,H,p)\le \log\left\{\Big(H/\pi\Big)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p\right\}. \tag{5.14}$$

Now we will deduce the moments behavior of $\tilde\alpha^H\big([0,1]^p\big)$.

Notice that $\tau_*=\min\{\tau_1,\cdots,\tau_p\}$ is an exponential time with parameter p.

$$\begin{aligned}\mathbb{E}\tilde\alpha^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m &\ge \mathbb{E}\tilde\alpha^H\big([0,\tau_*]^p\big)^m\\ &=\mathbb{E}\tau_*^{(p-Hd(p-1))m}\mathbb{E}\tilde\alpha^H\big([0,1]^p\big)^m\\ &=p^{-(p-Hd(p-1))m}\Gamma\big(1+(p-Hd(p-1))m\big)\mathbb{E}\tilde\alpha^H\big([0,1]^p\big)^m.\end{aligned}$$

By Stirling's formula,

$$\begin{aligned}&\limsup_{m\to\infty}\frac{1}{m}\log\big\{(m!)^{-Hd(p-1)}\mathbb{E}\tilde\alpha^H\big([0,1]^p\big)^m\big\}\\ &\qquad\qquad\le c(d,H,p)+(p-Hd(p-1))\log(1-Hd/p^*).\end{aligned}$$

On the other hand, for every $t_1,\dots,t_p>0$,

$$\begin{aligned}&\mathbb{E}\tilde\alpha_\epsilon^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\\ &=\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\prod_{j=1}^p\int_{[0,t_j]^m}ds_1\cdots ds_m\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W^H(s_k)-x_k\big)\\ &\le\prod_{j=1}^p\left\{\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\left(\int_{[0,t_j]^m}ds_1\cdots ds_m\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W^H(s_k)-x_k\big)\right)^p\right\}^{1/p}\\ &=\prod_{j=1}^p\big\{\mathbb{E}\tilde\alpha_\epsilon^H\big([0,t_j]^p\big)^m\big\}^{1/p}.\end{aligned}$$

Letting $\epsilon\to0$, from (5.9) we get

$$\mathbb{E}\tilde\alpha^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\le\prod_{j=1}^p\big\{\mathbb{E}\tilde\alpha^H\big([0,t_j]^p\big)^m\big\}^{1/p}=\mathbb{E}\tilde\alpha\big([0,1]^p\big)^m\cdot\prod_{j=1}^p t_j^{1-Hd/p^*},$$

where the last equality uses self-similarity (1.12). Hence

$$\begin{aligned}&\mathbb{E}\tilde\alpha^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m \qquad (5.15)\\ &\quad=\int_{(\mathbb{R}_+)^p}dt_1\cdots dt_p\,e^{-(t_1+\cdots+t_p)}\mathbb{E}\tilde\alpha^H\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\\ &\quad\le\mathbb{E}\tilde\alpha^m\big([0,1]^p\big)^m\int_{(\mathbb{R}_+)^p}dt_1\cdots dt_p\,e^{-(t_1+\cdots+t_p)}(t_1\cdots t_p)^{m(1-Hd(p-1)/p)}\\ &\quad=\mathbb{E}\tilde\alpha^H\big([0,1]^p\big)^m\Gamma\Big(1+m(1-Hd(p-1)/p)\Big)^p.\end{aligned}$$

By Stirling's formula again,

$$\liminf_{m\to\infty}\frac{1}{m}\log\left\{(m!)^{-Hd(p-1)}\mathbb{E}\tilde{\alpha}^H\big([0,1]^p\big)^m\right\}$$
$$\geq c(d,H,p)+(p-Hd(p-1))\log(1-Hd(p-1)/p).$$

We have shown that

$$\lim_{m\to\infty}\frac{1}{m}\log\left\{(m!)^{-Hd(p-1)}\mathbb{E}\tilde{\alpha}^H\big([0,1]^p\big)^m\right\}=C(d,H,p), \tag{5.16}$$

where by (5.14),

$$\begin{aligned} C(d,H,p)&=c(d,H,p)+(p-Hd(p-1))\log(1-Hd(p-1)/p) \\ &\leq \log\left\{\Big(H/\pi\Big)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p(1-Hd/p^*)^{p-Hd(p-1)}\right\}. \end{aligned} \tag{5.17}$$

On the other hand, let $\bar{\alpha}(A)$ be the intersection local time generated by $c_H^{-1}B_1^H(t),\cdots,c_H^{-1}B_p^H(t)$. We have that

$$\bar{\alpha}(A)=c_H^{d(p-1)}\alpha(A),\qquad A\subset(\mathbb{R}^+)^p. \tag{5.18}$$

In view of the decomposition (1.6), by Lemma 3.1 we have that

$$\mathbb{E}\Big[\tilde{\alpha}_H\big([0,1]^p\big)^m\Big]\geq\mathbb{E}\Big[\bar{\alpha}_H\big([0,1]^p\big)^m\Big]=c_H^{d(p-1)m}\mathbb{E}\Big[\alpha_H\big([0,1]^p\big)^m\Big]. \tag{5.19}$$

It follows from (5.25) below that

$$\begin{aligned} C(d,H,p)\geq\log\bigg\{&c_H^{d(p-1)}\bigg((p^*)^{\frac{d}{2p^*}}(2\pi)^{-\frac{d}{2p^*}}\int_0^\infty\big(1+t^{2H}\big)^{-d/2}e^{-t}dt\bigg)^p \\ &\times(1-Hd(p-1)/p)^{p-Hd(p-1)}\bigg\}. \end{aligned} \tag{5.20}$$

Applying Lemma 3.8 leads the first conclusion (2.7) of our theorem with

$$\tilde{K}(H,d,p)=Hd(p-1)\exp\left\{-\frac{C(H,d,p)}{Hd(p-1)}\right\}$$

and therefore the bounds given in (2.8) follows from (5.17) and (5.20). □

5.2 Proof of Theorem 2.4 – comparison argument

In connection to (5.16), we first show that

$$\lim_{m\to\infty}\frac{1}{m}\log\left\{(m!)^{-Hd(p-1)}\mathbb{E}\alpha^H\big([0,1]^p\big)^m\right\}=C(d,H,p)-d(p-1)\log c_H, \tag{5.21}$$

The upper bound follows immediately from (5.16) and the comparison (5.19). To establish the lower bound, we once again consider the intersection local time $\bar{\alpha}^H(A)$ generated by the normalized fractional Brownian motions

$$\bar{B}_1^H(t) = c_H^{-1} B_1^H(t), \cdots, \bar{B}_p^H(t) = c_H^{-1} B_p^H(t).$$

For any $\epsilon > 0$, define

$$\bar{\alpha}_\epsilon^H(A) = \int_{\mathbb{R}^d} \int_A \prod_{j=1}^p p_\epsilon\big(\bar{B}_j^H(s_j) - x\big)\, ds_1 \cdots ds_p\, dx, \tag{5.22}$$

Let $0 < \delta < 1$ be a small but fixed number. Notice

$$\begin{aligned}
&\mathbb{E}\bar{\alpha}_\epsilon\big([0,1]^p\big)^m \geq \mathbb{E}\alpha_\epsilon\big([\delta,1]^p\big)^m \\
&= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \prod_{j=1}^p \mathbb{E} \int_\delta^1 p_\epsilon\big(\bar{B}_j(s_k) - x_k\big) \\
&= \int_{([\delta,1]^p)^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \mathbb{E} \prod_{k=1}^m g_\epsilon\Big(\bar{B}_1^H(s_{1,k}), \cdots, \bar{B}_p^H(s_{p,k})\Big)
\end{aligned}$$

where $g_\epsilon(x_1, \cdots, x_p)$ is defined by (5.4) and we adopt the notation $\mathbf{s_k} = (s_{1,k}, \cdots s_{p,k})$.

Consider $\big(W_1^H(t_1), \cdots, W_p^H(t_p)\big)$ ($\mathbf{t} = (t_1, \cdots, t_p) \in [0,1]^p$) as a Gaussian random variable taking values in the Banach space $\otimes_{j=1}^p C\big\{[0,1]^p, \mathbb{R}^d\big\}$. Then the reproducing kernel Hilbert space of $\Big(W_1^H(t_1), \cdots, W_p^H(t_p)\Big)$ is $\widetilde{H}_W = \otimes_{j=1}^p H_W$. For each $\big(f_1(t_1) \cdots, f_p(t)\big) \in \widetilde{H}_W$

$$\big|\big|\big(f_1(t_1) \cdots, f_p(t)\big)\big|\big|_{\widetilde{H}_W}^2 = \sum_{j=1}^p ||f_j||_{H_W}^2$$

where $||\cdot||_{H_W}$ is the reproducing kernel Hilbert norm of H_W.

Let $Z_{\delta,1}^H(t), \cdots, Z_{\delta,p}^H(t)$ be the processes constructed in Lemma 3.5 (with $a = \delta$) by $Z_1^H(t), \cdots, Z_p^H(t)$, respectively. For each $(\mathbf{s}_1, \cdots, \mathbf{s}_m) \in [\delta, 1]^p)^m$ by the decomposition (1.4) we have

$$\begin{aligned}
&\mathbb{E} \prod_{k=1}^m g_\epsilon\Big(\bar{B}_1^H(s_{1,k}), \cdots, \bar{B}_p^H(s_{p,k})\Big) \\
&= \mathbb{E} \prod_{k=1}^m g_\epsilon\Big(W_1^H(s_{1,k}) + Z_1^H(s_{1,k}), \cdots, W_p^H(s_{p,k}) + Z_p^H(s_{p,k})\Big) \\
&= \mathbb{E} \prod_{k=1}^m g_\epsilon\Big(W_1^H(s_{1,k}) + Z_{\delta,1}^H(s_{1,k}), \cdots, W_p^H(s_{p,k}) + Z_{\delta,p}^H(s_{p,k})\Big)
\end{aligned}$$

Fixed $(\mathbf{s}_1, \cdots, \mathbf{s}_m) \in [\delta, 1]^p)^m$. Applying Lemma 3.6(ii) to the functional $g(f_1, \cdots, f_p)$ on $\otimes_{j=1}^p C\big\{[0,1]^p, \mathbb{R}^d\big\}$ defined by

$$g(f_1, \cdots, f_p) \equiv \prod_{k=1}^m g_\epsilon\Big(f_1(s_{1,k}), \cdots, f_p(s_{p,k})\Big) \quad (f_1, \cdots, f_p) \in \otimes_{j=1}^p C\big\{[0,1]^p, \mathbb{R}^d\big\},$$

then the right hand side is greater than

$$\begin{aligned}
&\bigg(\mathbb{E}\exp\Big\{-\frac{1}{2}||Z_\delta^H||_{H_W}^2\Big\}\bigg)^p \mathbb{E} g\big(W_1^H, \cdots, W_p^H\big) \\
&= \bigg(\mathbb{E}\exp\Big\{-\frac{1}{2}||Z_\delta^H||_{H_W}^2\Big\}\bigg)^p \mathbb{E}\prod_{k=1}^m g_\epsilon\Big(W_1^H(s_{1,k}), \cdots, W_p^H(s_{p,k})\Big).
\end{aligned}$$

Summarizing our estimate, we have

$$\begin{aligned}
&\mathbb{E}\bar{\alpha}_\epsilon\big([0,1]^p\big)^m \\
&\geq \bigg(\mathbb{E}\exp\Big\{-\frac{1}{2}||Z_\delta^H||_{H_W}^2\Big\}\bigg)^p \int_{([\delta,1]^p)^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \mathbb{E}\prod_{k=1}^m g_\epsilon\Big(W_1^H(s_{1,k}), \cdots, W_p^H(s_{p,k})\Big) \\
&= \bigg(\mathbb{E}\exp\Big\{-\frac{1}{2}||Z_\delta^H||_{H_W}^2\Big\}\bigg)^p \mathbb{E}\tilde{\alpha}_\epsilon\big([\delta, 1]^p\big)^m.
\end{aligned}$$

By Lemma 3.1, letting $\epsilon \to 0^+$ on both sides yields

$$\mathbb{E}\bar{\alpha}\big([0,1]^p\big)^m \geq \bigg(\mathbb{E}\exp\Big\{-\frac{1}{2}||Z_\delta^H||_{H_W}^2\Big\}\bigg)^p \mathbb{E}\tilde{\alpha}_\epsilon\big([\delta, 1]^p\big)^m.$$

In view of (5.18),

$$\begin{aligned}
&\liminf_{m\to\infty} \frac{1}{m}\log(m!)^{-Hd(p-1)}\mathbb{E}\Big[\alpha^H\big([0,1]^p\big)^m\Big] \\
&\geq d(p-1)\log c_H + \liminf_{m\to\infty}\frac{1}{m}\log(m!)^{Hd(p-1)}\mathbb{E}\Big[\tilde{\alpha}^H\big([\delta,1]^p\big)^m\Big].
\end{aligned} \tag{5.23}$$

To establish the lower bound for (5.21), therefore, it remains to show that

$$\liminf_{\delta\to 0^+}\liminf_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^{Hd(p-1)}}\mathbb{E}\Big[\tilde{\alpha}^H\big([\delta,1]^p\big)^m\Big] \geq C(H,d,p). \tag{5.24}$$

Write

$$\tilde{\alpha}\big([0,1]^p\big) = \tilde{\alpha}\big([\delta,1]\times[0,1]^{p-1}\big) + \tilde{\alpha}\big([0,\delta]\times[0,1]^{p-1}\big).$$

By triangular inequality,

$$\begin{aligned}
&\bigg\{\mathbb{E}\Big[\tilde{\alpha}\big([0,1]^p\big)^m\Big]\bigg\}^{1/m} \\
&\leq \bigg\{\mathbb{E}\Big[\tilde{\alpha}\big([\delta,1]\times[0,1]^{p-1}\big)^m\Big]\bigg\}^{1/m} + \bigg\{\mathbb{E}\Big[\tilde{\alpha}\big([0,\delta]\times[0,1]^{p-1}\big)^m\Big]\bigg\}^{1/m}.
\end{aligned}$$

Given $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\tilde\alpha_\epsilon\big([\delta,1]\times[0,1]^{p-1}\big)^m\Big]\\
&=\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\Big[\int_{[\delta,1]^m}\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W^H(s_k)-x_k\big)ds_1\cdots ds_m\Big]\\
&\qquad\times\Big[\int_{[0,1]^m}\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W^H(s_k)-x_k\big)ds_1\cdots ds_m\Big]^{p-1}\\
&\le\Big\{\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\Big[\int_{[\delta,1]^m}\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W^H(s_k)-x_k\big)ds_1\cdots ds_m\Big]^p\Big\}^{1/p}\\
&\times\Big\{\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\Big[\int_{[0,1]^m}\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W^H(s_k)-x_k\big)ds_1\cdots ds_m\Big]^p\Big\}^{(p-1)/p}\\
&=\Big\{\mathbb{E}\Big[\tilde\alpha_\epsilon\big([\delta,1]^p\big)^m\Big]\Big\}^{1/p}\Big\{\mathbb{E}\Big[\tilde\alpha_\epsilon\big([0,1]^p\big)^m\Big]\Big\}^{(p-1)/p}.
\end{aligned}$$

Letting $\epsilon\to 0^+$ yields

$$\mathbb{E}\Big[\tilde\alpha\big([\delta,1]\times[0,1]^{p-1}\big)^m\Big]\le\Big\{\mathbb{E}\Big[\tilde\alpha\big([\delta,1]^p\big)^m\Big]\Big\}^{1/p}\Big\{\mathbb{E}\Big[\tilde\alpha\big([0,1]^p\big)^m\Big]\Big\}^{(p-1)/p}.$$

Similarly,

$$\mathbb{E}\Big[\tilde\alpha\big([0,\delta]\times[0,1]^{p-1}\big)^m\Big]\le\Big\{\mathbb{E}\Big[\tilde\alpha\big([0,\delta]^p\big)^m\Big]\Big\}^{1/p}\Big\{\mathbb{E}\Big[\tilde\alpha\big([0,1]^p\big)^m\Big]\Big\}^{(p-1)/p}.$$

So we have

$$\Big\{\mathbb{E}\Big[\tilde\alpha\big([0,1]^p\big)^m\Big]\Big\}^{1/mp}\le\Big\{\mathbb{E}\Big[\tilde\alpha\big([\delta,1]^p\big)^m\Big]\Big\}^{1/mp}+\Big\{\mathbb{E}\Big[\tilde\alpha\big([0,\delta]^p\big)^m\Big]\Big\}^{1/mp}.$$

By scaling,

$$\mathbb{E}\Big[\tilde\alpha\big([0,\delta]^p\big)^m\Big]=\delta^{(p-Hd(p-1))m}\mathbb{E}\Big[\tilde\alpha\big([0,1]^p\big)^m\Big].$$

Thus

$$\mathbb{E}\Big[\tilde\alpha\big([\delta,1]^p\big)^m\Big]\ge\Big[1-\delta^{1-Hd(p-1)/p}\Big]^{mp}\mathbb{E}\Big[\tilde\alpha\big([0,1]^p\big)^m\Big].$$

Therefore, (5.24) follows from (5.16).

To bound the limit in (5.21) from below, we claim that

$$\begin{aligned}
&\lim_{m\to\infty}\frac{1}{m}\log\big\{(m!)^{-Hd(p-1)}\mathbb{E}\alpha^H\big([0,1]^p\big)^m\big\} &(5.25)\\
&\ge p\log\Big\{(1-Hd/p^*)^{-(1-Hd/p^*)}(p^*)^{\frac{d}{2p^*}}(2\pi)^{-\frac{d}{2p^*}}\int_0^\infty\big(1+t^{2H}\big)^{-d/2}e^{-t}dt\Big\}.
\end{aligned}$$

Let $\tau_1, \cdots, \tau_p$ be i.i.d. exponential times independent of $B_1^H(t), \cdots, B_p^H(t)$. Given $\epsilon > 0$

$$\mathbb{E}\Big[\alpha_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big] = \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\, Q_\epsilon^p(x_1,\cdots,x_m)$$

where

$$Q_\epsilon(x_1,\cdots,x_m) = \int_0^\infty e^{-t}\bigg[\int_{[0,t]^m} ds_1\cdots ds_m \mathbb{E}\prod_{k=1}^m p_\epsilon\big(B^H(s_k)-x_k\big)\bigg]dt\,.$$

Let $f(x_1,\cdots,x_m)$ be a rapidly decreasing function on $(\mathbb{R}^d)^m$ such that

$$\int_{(\mathbb{R}^d)^m} |f(x_1,\cdots,x_m)|^{p^*} dx_1\cdots dx_m = 1\,.$$

By Hölder inequality,

$$\begin{aligned}
&\Big\{\mathbb{E}\Big[\alpha_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^m\Big\}^{1/p}\\
&\geq \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m f(x_1,\cdots,x_m) Q_\epsilon(x_1,\cdots,x_m)\\
&= \int_0^\infty e^{-t}\int_{[0,t]^m}\bigg[\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m f(x_1,\cdots,x_m) H_{\mathbf{s},\epsilon}(x_1,\cdots,x_m)\bigg] ds_1\cdots ds_m\,dt,
\end{aligned}$$

where

$$H_{\mathbf{s},\epsilon}(x_1,\cdots,x_m) = \mathbb{E}\prod_{k=1}^m p_\epsilon\big(B^H(s_k)-x_k\big) \qquad \mathbf{s}=(s_1,\cdots,s_m).$$

Consider the Fourier transform

$$\widehat{f}(\lambda_1,\cdots,\lambda_m) = \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m f(x_1,\cdots,x_m)\exp\bigg\{i\sum_{k=1}^m \lambda_k\cdot x_k\bigg\}.$$

It is easy to see that

$$\widehat{H}_{\mathbf{s},\epsilon}(\lambda_1,\cdots,\lambda_m) = \exp\bigg\{-\frac{\epsilon}{2}\sum_{k=1}^m|\lambda_k|^2 - \frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}\lambda_k\cdot B^H(\lambda_k)\Big)\bigg\}.$$

By Parseval identity,

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m f(x_1,\cdots,x_m) H_{\mathbf{s},\epsilon}(x_1,\cdots,x_m)\\
&= \frac{1}{(2\pi)^{md}}\int_{(\mathbb{R}^d)^m} d\lambda_1\cdots d\lambda_m \widehat{f}(\lambda_1,\cdots,\lambda_m)\\
&\times\exp\bigg\{-\frac{\epsilon}{2}\sum_{k=1}^m|\lambda_k|^2 - \frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\cdot B^H(s_k)\Big)\bigg\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big\{\mathbb{E}\Big[\alpha_\epsilon^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^m\Big\}^{1/p}\\
&\ge \frac{1}{(2\pi)^{md}}\int_0^\infty e^{-t}dt\int_{[0,t]^m} ds_1\cdots ds_m\Big[\int_{(\mathbb{R}^d)^m} d\lambda_1\cdots d\lambda_m\\
&\times \widehat{f}(\lambda_1,\cdots,\lambda_m)\exp\Big\{-\frac{\epsilon}{2}\sum_{k=1}^m|\lambda_k|^2-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\cdot B^H(s_k)\Big)\Big\}\Big].
\end{aligned}$$

We now let $\epsilon\to 0^+$ on the both hand sides. Noticing that the left hand side falls into an obvious similarity to (5.7),

$$\begin{aligned}
&\Big\{\mathbb{E}\Big[\alpha^H\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^m\Big\}^{1/p}\\
&\ge \frac{1}{(2\pi)^{md}}\int_0^\infty e^{-t}dt\int_{[0,t]^m} ds_1\cdots ds_m\Big[\int_{(\mathbb{R}^d)^m} d\lambda_1\cdots d\lambda_m\\
&\times \widehat{f}(\lambda_1,\cdots,\lambda_m)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\cdot B^H(s_k)\Big)\Big\}\Big].
\end{aligned}\tag{5.26}$$

We now specify the function $f(x_1,\cdots,x_m)$ as

$$f(x_1,\cdots,x_m)=C^m\prod_{k=1}^m p_1(x_k)$$

where

$$C=(p^*)^{\frac{d}{2p^*}}(2\pi)^{\frac{d(p^*-1)}{2p^*}}.$$

We have

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^m} d\lambda_1\cdots d\lambda_m\widehat{f}(\lambda_1,\cdots,\lambda_m)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\cdot B^H(\lambda_k)\Big)\Big\}\\
&=C^m\Big[\int_{\mathbb{R}^m} d\lambda_1\cdots d\lambda_m\exp\Big\{-\frac{1}{2}\sum_{k=1}^m\lambda_k^2-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_kB_0^H(s_k)\Big)\Big\}\Big]^d
\end{aligned}$$

where $B_0^H(t)$ is an 1-dimensional fractional Brownian motion.

Let $\xi_1,\cdots\xi_m$ be i.i.d. standard normal random variable independent of $B_0^H(t)$. Write

$$\eta_k=\xi_k+B_0^H(s_k)\qquad k=1,\cdots,m.$$

We have

$$\frac{1}{2}\sum_{k=1}^m\lambda_k^2+\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_kB_0^H(s_k)\Big)=\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\eta_k\Big).$$

By Gaussian integration,

$$\int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \exp\Big\{-\frac{\sigma^2}{2}\sum_{k=1}^m \lambda_k^2 - \frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda_k B_0^H(s_k)\Big)\Big\}$$
$$= (2\pi)^{m/2}\det\Big\{\mathrm{Cov}\big(\eta_1, \cdots, \eta_m\big)\Big\}^{-1/2}.$$

When $s_1 < \cdots < s_m$, by Lemma 3.7,

$$\det\Big\{\mathrm{Cov}\big(\eta_1, \cdots, \eta_m\big)\Big\} = \prod_{k=1}^m \mathrm{Var}\Big(\eta_k \vert \eta_1, \cdots, \eta_{k-1}\Big)$$
$$= \prod_{k=1}^m \Big\{1 + \mathrm{Var}\Big(B_0^H(s_k) \vert B_0^H(s_1), \cdots, B_0^H(s_{k-1})\Big)\Big\}$$
$$\le \prod_{k=1}^m \Big\{1 + (s_k - s_{k-1})^{2H}\Big\}$$

where the last step follow from the computation

$$\mathrm{Var}\Big(B_0^H(s_k) \vert B_0^H(s_1), \cdots, B_0^H(s_{k-1})\Big)$$
$$= \mathrm{Var}\Big(B_0^H(s_k) - B_0^H(s_{k-1}) \vert B_0^H(s_1), \cdots, B_0^H(s_{k-1})\Big)$$
$$\le \mathrm{Var}\Big(B_0^H(s_k) - B_0^H(s_{k-1})\Big) = (s_k - s_{k-1})^{2H}.$$

Summarizing our argument since (5.26), we obtain

$$\Big\{\mathbb{E}\Big[\alpha^H\big([0,\tau_1] \times \cdots \times [0,\tau_p]\big)\Big]^m\Big\}^{1/p}$$
$$\ge m!\big(C(2\pi)^{-d/2}\big)^m \int_0^\infty e^{-t}dt \int_{[0,t]_<^m} ds_1 \cdots ds_m \prod_{k=1}^m \Big\{1 + (s_k - s_{k-1})^{2H}\Big\}^{-d/2}$$
$$= m!\big(C(2\pi)^{-d/2}\big)^m \Big[\int_0^\infty \big(1 + t^{2H}\big)^{-d/2} e^{-t}dt\Big]^m.$$

Or,

$$\mathbb{E}\Big[\alpha^H\big([0,\tau_1] \times \cdots \times [0,\tau_p]\big)\Big]^m \tag{5.27}$$
$$\ge (m!)^p \big(C(2\pi)^{-d/2}\big)^{mp} \Big[\int_0^\infty \big(1 + t^{2H}\big)^{-d/2} e^{-t}dt\Big]^{pm}.$$

On the other hand, with obvious similarity to (5.15)

$$\mathbb{E}\Big[\alpha^H\big([0,\tau_1] \times \cdots \times [0,\tau_p]\big)\Big]^m \le \mathbb{E}\Big[\alpha^H\big([0,1]^p\big)\Big]^m \Big\{\Gamma\big(1 + m(1 - Hd(p-1)/p)\big)\Big\}^p.$$

Hence, (5.25) follows from (5.27) and Stirling formula.

By (5.17) and (5.25), the limit given in (5.21) is finite. By Lemma 3.8, the large deviation given in (2.9) holds with

$$
\begin{aligned}
K(H,d,p) &= Hd(p-1)\exp\Big\{-\frac{C(H,d,p)-d(p-1)\log c_H}{Hd(p-1)}\Big\}\\
&= c_H^{1/H}Hd(p-1)\exp\Big\{-\frac{C(H,d,p)}{Hd(p-1)}\Big\} = c_H^{1/H}\tilde{K}(H,d,p).
\end{aligned}
$$

□

6 The law of the iterated logarithm

We will prove Theorem 2.5 in this section. Due to the similarity of arguments, we will only establish (2.17). By the self-similarity property (1.12), the large deviation limit of Theorem 2.3 can be rewritten as

$$
\begin{aligned}
&\lim_{t\to\infty}(\log\log t)^{-1}\log\mathbb{P}\Big\{\tilde{\alpha}^H\big([0,t]^p\big)\geq \lambda t^{p-Hd(p-1)}(\log\log t)^{Hd(p-1)}\Big\}\\
&= -\tilde{K}(H,d,p)\lambda^{p^*/Hdp} \qquad (\lambda>0).
\end{aligned} \tag{6.1}
$$

Therefore, the upper bound

$$
\limsup_{t\to\infty} t^{Hd(p-1)-p}(\log\log t)^{-Hd(p-1)}\widetilde{\alpha}^H\big([0,t]^p\big)\leq \widetilde{K}(H,d,p)^{-Hd(p-1)} \qquad a.s
$$

is a consequence of the standard argument using Borel-Cantelli lemma.

To show the lower bound, we proceed in several steps. First let $N>1$ be a large but fixed number and write $t_n = N^n$ $(n=1,2,\cdots)$. For each n, let $(\mathbb{H}_n, \|\cdot\|_{\mathbb{H}_n})$ be the reproducing kernel Hilbert space generated by $W^H(\cdot)$ when viewed as a Gaussian random variable in $C\big([0,t_{n+1}];\mathbb{R}^d\big)$. Define the d-dimensional process

$$
Q_n^H(t) = \int_0^{t_n}(t+u)^{H-1/2}dB(u) \qquad t\geq 0,
$$

where $B(u)$ is a standard d-dimensional Brownian motion. Now we define the following modifications of $Q_n^H(t)$ (cf. the proof of Proposition 3.5).

When $H\in(0,1/2)$, we put

$$
G_n^H(t) = \begin{cases} \frac{t}{t_n}Q_n^H(t_n), & 0\leq t\leq t_n\\ Q_n^H(t), & t>t_n. \end{cases}
$$

When $H \in (\frac{1}{2}, 1)$, $G_n^H(t)$ is defined by

$$G_n^H(t) = \begin{cases} \Big(3Q_n^H(t_n) - t_n\dot{Q}_n^H(t_n)\Big)(t/t_n)^2 + \Big(-2Q_n^H(t_n) + t_n\dot{Q}_n^H(t_n)\Big)(t/t_n)^3, & 0 \le t \le t_n \\ Q_n^H(t), & t > t_n. \end{cases}$$

Lemma 6.1 *Almost surely $\{G_n^H(t)\}_{t\in[0,t_{n+1}]} \subset \mathbb{H}_n$, for every $n \ge 1$. Furthermore,*

$$\sup_n \mathbb{E}\|G_n^H\|_{\mathbb{H}_n}^2 < \infty. \tag{6.2}$$

Proof: Obviously, it suffices to consider the case $d = 1$. By the same argument as in Proposition 3.5, we infer that $\{G_n^H(t)\}_{t\in[0,t_{n+1}]} \subset \mathbb{H}_n$ almost surely. If $H \in (0, 1/2)$, then $m = \lceil H + 1/2 \rceil = 1$ and by Corollary A4

$$\begin{aligned}\|G_n^H\|_{\mathbb{H}_n}^2 &= \frac{1}{\Gamma(H+1/2)^2} \int_0^{t_{n+1}} |I_{0+}^{1-(H+1/2)} \dot{G}_n^H(t)|^2 \, dt \\ &= \frac{1}{\Gamma(H+1/2)^2 \Gamma(1/2-H)^2} \int_0^{t_{n+1}} \left| \int_0^t (t-s)^{-(H+1/2)} \dot{G}_n^H(s)\, ds \right|^2 dt. \end{aligned} \tag{6.3}$$

For $t \ge t_n$,

$$\begin{aligned}\int_0^t (t-s)^{-(H+1/2)} \dot{G}_n^H(s)\, ds = t_n^{-1} Q^H(t_n) \int_0^{t_n} (t-s)^{-(H+1/2)}\, ds \\ + (H-1/2) \int_0^{t_n} \left[\int_{t_n}^t (t-s)^{-(H+1/2)} (s+u)^{H-3/2}\, ds \right] dB(u).\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}\left| \int_0^t (t-s)^{-(H+1/2)} \dot{G}_n^H(s) ds \right|^2 &\le 2t_n^{-2} \mathbb{E}\big[Q_n^H(t_n)\big]^2 \left\{ \int_0^{t_n} (t-s)^{-(H+1/2)} ds \right\}^2 \\ &\quad + (2H-1)\mathbb{E}\left\{ \int_0^{t_n} \left[\int_{t_n}^t (t-s)^{-(H+1/2)} (s+u)^{H-3/2} ds \right] dB(u) \right\}^2 \\ &\le C\left\{ t_n^{2H-2} t^{1-2H} + t^{-1} \right\} \end{aligned} \tag{6.4}$$

where C is a constant depending only on H ($C > 0$ will be allowed to be different at different places). Indeed,

$$\mathbb{E}\big[Q_n^H(t_n)\big]^2 = \int_0^{t_n} (t_n + u)^{2H-1}\, du = \frac{2^{2H} - 1}{2H}\, t_n^{2H}, \tag{6.5}$$

$$\int_0^{t_n} (t-s)^{-(H+1/2)} ds \le \frac{2}{1-2H}\, t^{1/2-H},$$

and

$$\mathbb{E}\Big\{\int_0^{t_n}\Big[\int_{t_n}^t (t-s)^{-(H+1/2)}(s+u)^{H-3/2}\,ds\Big]\,dB(u)\Big\}^2$$
$$\begin{aligned}
&=\int_0^{t_n}\Big[\int_{t_n}^t (t-s)^{-(H+1/2)}(s+u)^{H-3/2}\,ds\Big]^2\,du\\
&\le\int_0^{\infty}\Big[\int_0^t (t-s)^{-(H+1/2)}(s+u)^{H-3/2}\,ds\Big]^2\,du\\
&=\Big(\frac{2}{1-2H}\Big)^2\int_0^\infty \frac{t^{1-2H}u^{2H-1}}{(t+u)^2}\,du = C\,\frac{1}{t},
\end{aligned}$$

where in the last two steps we used the identities

$$\int_0^t (t-s)^{-(H+1/2)}(s+u)^{H-3/2}\,ds=\frac{2}{1-2H}\frac{t^{1/2-H}u^{H-1/2}}{t+u}$$

and

$$\int_0^\infty \frac{t^{1-2H}u^{2H-1}}{(t+u)^2}\,du=\pi(1-2H)\csc(2\pi H)\,\frac{1}{t}.$$

Combining these estimates we get (6.4).

When $0\le t\le t_n$,

$$\int_0^t (t-s)^{-(H+1/2)}\dot{G}_n^H(s)\,ds=\frac{2}{1-2H}\,t_n^{-1}Q_n^H(t_n)\,t^{1/2-H}.$$

Taking expectation in (6.3) and using bound (6.4) together with the above equality and (6.5) we get

$$\begin{aligned}
\mathbb{E}\|G_n^H\|^2_{\mathbb{H}_n}&\le C+Ct_n^{2H-2}\int_{t_n}^{t_{n+1}} t^{1-2H}\,dt+C\int_{t_n}^{t_{n+1}}\frac{1}{t}\,dt\\
&=C+C\Big[(t_{n+1}/t_n)^{2-2H}-1\Big]+C\log(t_{n+1}/t_n)\\
&\le CN^{2-2H},
\end{aligned}$$

because $t_{n+1}/t_n=N\ge 2$ is fixed. The proof is complete in the case $H\in(0,1/2)$.

If $H\in(1/2,1)$, then $m=\lceil H+1/2\rceil=2$ and by Corollary A4

$$\begin{aligned}
\|G_n^H\|^2_{\mathbb{H}_n}&=\frac{1}{\Gamma(H+1/2)^2}\int_0^{t_{n+1}}|I_{0+}^{2-(H+1/2)}\ddot{G}_n^H(t)|^2\,dt\\
&=\frac{1}{\Gamma(H+1/2)^2\Gamma(3/2-H)^2}\int_0^{t_{n+1}}\Big|\int_0^t (t-s)^{1/2-H}\ddot{G}_n^H(s)\,ds\Big|^2\,dt. \qquad (6.6)
\end{aligned}$$

Put

$$\xi_n=\Big(3Q_n^H(t_n)-t_n\dot{Q}_n^H(t_n)\Big)\,t_n^{-2},\qquad \eta_n=\Big(-2Q_n^H(t_n)+t_n\dot{Q}_n^H(t_n)\Big)\,t_n^{-3},$$

so that $G_n^H(t)=\xi_n t^2+\eta_n t^3$ when $0\le t\le t_n$.

Then, for $0\le t\le t_n$,

$$\int_0^t (t-s)^{1/2-H}\ddot G_n^H(s)\,ds = 2\xi_n\int_0^t (t-s)^{1/2-H}\,ds + 6\eta_n\int_0^t (t-s)^{1/2-H}s\,ds,$$

so that

$$\mathbb{E}\Big|\int_0^t (t-s)^{1/2-H}\ddot G_n^H(s)\,ds\Big|^2 \le C\left[\mathbb{E}\xi_n^2\,t^{3-2H}+\mathbb{E}\eta_n^2\,t^{5-2H}\right].$$

Since

$$\mathbb{E}\xi_n^2 \le C\left[t_n^{-4}\,\mathbb{E}Q_n^H(t_n)^2+t_n^{-2}\,\mathbb{E}\dot Q_n^H(t_n)^2\right]=C\,t_n^{2H-4}$$

and

$$\mathbb{E}\eta_n^2 \le C\left[t_n^{-6}\,\mathbb{E}Q_n^H(t_n)^2+t_n^{-4}\,\mathbb{E}\dot Q_n^H(t_n)^2\right]=C\,t_n^{2H-6},$$

we get for $0\le t\le t_n$,

$$\mathbb{E}\Big|\int_0^t (t-s)^{1/2-H}\ddot G_n^H(s)\,ds\Big|^2 \le C\left[t_n^{2H-4}\,t^{3-2H}+t_n^{2H-6}\,t^{5-2H}\right]. \tag{6.7}$$

If $t\ge t_n$, then

$$\begin{aligned}\int_0^t (t-s)^{1/2-H}\ddot G_n^H(s)\,ds &= \int_0^{t_n} (t-s)^{1/2-H}\ddot G_n^H(s)\,ds\\ &\quad + (H-1/2)(H-3/2)\int_0^{t_n}\left[\int_{t_n}^t (t-s)^{1/2-H}(s+u)^{H-5/2}\,ds\right]dB(u).\end{aligned}$$

Similarly as above,

$$\begin{aligned}&\mathbb{E}\Big|\int_0^{t_n} (t-s)^{1/2-H}\ddot G_n^H(s)\,ds\Big|^2\\ &\qquad\le C\left[\mathbb{E}\xi_n^2\left(\int_0^{t_n}(t-s)^{1/2-H}\,ds\right)^2+\mathbb{E}\eta_n^2\left(\int_0^{t_n}(t-s)^{1/2-H}s\,ds\right)^2\right]\\ &\qquad\le C\left[t_n^{2H-4}\,t^{3-2H}+t_n^{2H-6}\,t^{5-2H}\right],\end{aligned} \tag{6.8}$$

which is the same kind estimate as (6.7). Then, as $t_n\le t\le t_{n+1}$,

$$\begin{aligned}&\mathbb{E}\left\{\int_0^{t_n}\left[\int_{t_n}^t (t-s)^{1/2-H}(s+u)^{H-5/2}\,ds\right]dB(u)\right\}^2\\ &\qquad=\int_0^{t_n}\left[\int_{t_n}^t (t-s)^{1/2-H}(s+u)^{H-5/2}\,ds\right]^2du\\ &\qquad=\left(\frac{2}{3-2H}\right)^2\int_0^{t_n}\frac{(t-t_n)^{3-2H}(u+t_n)^{2H-3}}{(t+u)^2}\,du\\ &\qquad\le\left(\frac{2}{3-2H}\right)^2\left(\frac{t}{t_n}-1\right)^{3-2H}\frac{t_n}{t^2}\le C\,(N-1)^{3-2H}\;\frac{1}{t},\end{aligned} \tag{6.9}$$

where the first inequality comes from the mean value theorem. Combining (6.6)–(6.9) we obtain

$$\begin{aligned}\mathbb{E}\|G_n^H\|^2_{\mathbb{H}_n} &\le C\int_0^{t_n}\left[t_n^{2H-4}\,t^{3-2H}+t_n^{2H-6}\,t^{5-2H}\right]\,dt\\ &+C\int_{t_n}^{t_{n+1}}\left[t_n^{2H-4}\,t^{3-2H}+t_n^{2H-6}\,t^{5-2H}\right]\,dt+C\,(N-1)^{3-2H}\int_{t_n}^{t_{n+1}}\frac{1}{t}\,dt\\ &\le C\,(t_{n+1}/t_n-1)^{4-2H}+C\,(t_{n+1}/t_n-1)^{6-2H}+C\,(N-1)^{3-2H}\log(t_{n+1}/t_n)\\ &\le C(N-1)^{6-2H}.\end{aligned}$$

This bound, independent of n, concludes the proof. □

Define the sigma field

$$\mathcal{F}_t=\sigma\Big\{\big(B_1(s),\cdots,B_p(s)\big);\quad s\le t\Big\}.$$

We claim that for any $\lambda<\widetilde{K}(H,d,p)^{-Hd(p-1)}$, one can take N sufficiently large, so that

$$\sum_n\mathbb{P}\Big\{\tilde\alpha^H\big([2t_n,t_{n+1}]^p\big)\ge\lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\Big|\mathcal{F}_{t_n}\Big\}=\infty\quad a.s.\tag{6.10}$$

Let $\epsilon>0$ be fixed and write

$$\begin{aligned}\tilde\alpha^H_\epsilon\big([2t_n,t_{n+1}]^p\big)&=\int_{[2t_n,t_{n+1}]^p}ds_1\cdots ds_p\,g_\epsilon\big(W_1^H(s_1),\cdots,W_p^H(s_p)\big)\\ &=\int_{[t_n,t_{n+1}-t_n]^p}ds_1\cdots ds_p\,g_\epsilon\big(W_1^H(t_n+s_1),\cdots,W_p^H(t_n+s_p)\big)\\ &=\int_{[t_n,t_{n+1}-t_n]^p}ds_1\cdots ds_p\,g_\epsilon\big(Y_1^H(s_1)+Z_1^H(s_1),\cdots,Y_p^H(s_p)+Z_p^H(s_p)\big),\end{aligned}$$

where $g_\epsilon(x_1,\cdots,x_p)$ is given in (5.4) and

$$Y_j^H(t)=\int_{t_n}^{t_n+t}(t_n+t-s)^{H-1/2}dB_j(s),\qquad Z_j^H(t)=\int_0^{t_n}(t_n+t-s)^{H-1/2}dB_j(s)$$

$(j=1,\cdots,p)$.

Consider a symmetric set $A\subset\otimes_{j=1}^pC\Big\{[0,t_{n+1}],\mathbb{R}^d\Big\}$ defined by

$$A=\Bigg\{(f_1,\cdots,f_p)\in\otimes_{j=1}^pC\Big\{[0,t_{n+1}],\mathbb{R}^d\Big\};$$
$$\int_{[t_n,t_{n+1}-t_n]^p}ds_1\cdots ds_p\,g_\epsilon\big(f_1(s_1),\cdots,f_p(s_p)\big)\ge\lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\Bigg\}.$$

For any $(f_1, \cdots, f_p) \in \otimes_{j=1}^p \mathbb{H}_n$, applying Lemma 3.6-(ii) to the indicator of A leads to

$$\mathbb{P}\Big\{\big(W_1^H + f_1, \cdots, W_p^H + f_p\big) \in A\Big\} \geq \exp\Big\{-\frac{1}{2}\sum_{j=1}^p \|f\|_{H_n}^2\Big\}\mathbb{P}\Big\{\big(W_1^H, \cdots, W_p\big) \in A\Big\},$$

if $f_1, \cdots, f_p \in \mathbb{H}_n$.

Notice that

$$\Big\{Z^H(t);\quad t_n \leq t \leq t_{n+1}\Big\} \stackrel{d}{=} \Big\{Q_n^H(t);\quad t_n \leq t \leq t_{n+1}\Big\} = \Big\{G_n^H(t);\quad t_n \leq t \leq t_{n+1}\Big\}$$

$$\Big\{Y^H(t);\quad t_n \leq t \leq t_{n+1}\Big\} \stackrel{d}{=} \Big\{W^H(t);\quad t_n \leq t \leq t_{n+1}\Big\}$$

and $Y^H(t)$ and $Z^H(t)$ are in dependent. By Lemma 6.1,

$$\begin{aligned}&\mathbb{P}\Big\{\big(Y_1^H + Z_1^H, \cdots, Y_p^H + Z_p^H\big) \in A\Big|\mathcal{F}_{t_n}\Big\}\\ &\geq \exp\Big\{-\frac{1}{2}\sum_{j=1}^p \|G_{n,j}^H\|_{\mathbb{H}_n}^2\Big\}\mathbb{P}\Big\{\big(W_1^H, \cdots, W_p\big) \in A\Big\},\end{aligned}$$

or

$$\begin{aligned}&\mathbb{P}\Big\{\tilde{\alpha}_\epsilon^H\big([2t_n, t_{n+1}]^p\big) \geq \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\Big|\mathcal{F}_{t_n}\Big\}\\ &\geq \exp\Big\{-\frac{1}{2}\sum_{j=1}^p \|G_{n,j}^H\|_{\mathbb{H}_n}^2\Big\}\mathbb{P}\Big\{\tilde{\alpha}_\epsilon^H\big([t_n, t_{n+1}-t_n]^p\big) \geq \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\Big\}.\end{aligned}$$

Letting $\epsilon \to 0^+$ on the both sides yields

$$\begin{aligned}&\mathbb{P}\Big\{\tilde{\alpha}^H\big([2t_n, t_{n+1}]^p\big) \geq \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\Big|\mathcal{F}_{t_n}\Big\}\\ &\geq \exp\Big\{-\frac{1}{2}\sum_{j=1}^p \|G_{n,j}^H\|_{\mathbb{H}_n}^2\Big\}\mathbb{P}\Big\{\tilde{\alpha}^H\big([t_n, t_{n+1}-t_n]^p\big) \geq \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\Big\}.\end{aligned}$$

By (6.1) and by an argument similar to the one used for (5.24), for $\lambda < \widetilde{K}(H,d,p)^{-Hd(p-1)}$ and any small $\delta > 0$, one can take N sufficiently large so that

$$\begin{aligned}&\mathbb{P}\Big\{\tilde{\alpha}^H\big([t_n, t_{n+1}-t_n]^p\big) \geq \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\Big\}\\ &\geq \exp\Big\{-(1-\delta)\log\log t_{n+1}\Big\} = (n\log N)^{-1+\delta}\end{aligned}$$

for large n.

To establish (6.10), therefore, it suffices to show that for any $\epsilon, \delta > 0$,

$$\sum_n \frac{1}{n^{1-\delta}}\mathbf{1}\Big\{\sum_{j=1}^p \|G_{n,j}^H\|_{\mathbb{H}_n}^2 \leq \epsilon\log\log t_{n+1}\Big\} = \infty \quad a.s. \tag{6.11}$$

Indeed, by Lemma 6.1 G_n^H can be viewed as a Gaussian sequence taking values in H_n. By the Gaussian tail estimate, see [29], p.59, there is $u = u(\epsilon) > 0$ such that for large n

$$\mathbb{P}\Big\{\sum_{j=1}^p \|G_{n,j}^H\|_{\mathbb{H}_n}^2 \geq \epsilon \log\log t_{n+1}\Big\} \leq \frac{1}{n^u}\,.$$

Then for $0 < \delta < u$,

$$\sum_n \frac{1}{n^{1-\delta}} \mathbf{1}\Big\{\sum_{j=1}^p \|G_{n,j}^H\|_{\mathbb{H}_n}^2 \geq \epsilon \log\log t_{n+1}\Big\} < \infty \quad a.s.$$

which leads to (6.11).

By Corollary 5.29, p. 96 in [7], (6.11) implies that

$$\limsup_{n\to\infty} t_{n+1}^{Hd(p-1)-p} (\log\log t_{n+1})^{-Hd(p-1)} \tilde{\alpha}^H\big([2t_n, t_{n+1}]^p\big) \geq \lambda \quad a.s.$$

which leads to

$$\limsup_{t\to\infty} t^{Hd(p-1)-p} (\log\log t^{-Hd(p-1)} \tilde{\alpha}^H\big([0,t]^p\big) \geq \lambda \quad a.s.$$

Letting $\lambda \to \widetilde{K}(H,d,p)^{-Hd(p-1)}$ on the right hand side leads to the lower bound as claimed. $\square$

7 Local times of Gaussian fields

We begin with mentioning the work of Geman, Horowitz and Rosen ([19]) on the condition for the existence and continuity of the local times of the Gaussian fields, see also recent work of Wu and Xiao [42]. Let $X(\mathbf{t})$ ($\mathbf{t} \in (\mathbb{R}^+)^p$) be a mean zero Gaussian field taking values in $\mathbb{R}^d$ such that there is a $\gamma > 0$ such that for any $t > 0$ and $m = 1, 2, \cdots$,

$$\begin{aligned}&\int_{([0,t]^p)^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \\ &\times \Big(\prod_{k=1}^m |\lambda_k|^\gamma\Big) \exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\Big)\Big\} < \infty.\end{aligned} \tag{7.1}$$

Geman, Horowitz and Rosen (Theorem (2.8) in [19]) proved that the occupation time

$$\mu_{\mathbf{t}}(B) = \int_{[\mathbf{0},\mathbf{t}]} \mathbf{1}_{\{X(\mathbf{s})\in B\}}\, d\mathbf{s} \quad B \subset \mathbb{R}^d$$

is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Further, the correspondent density function formally written as

$$\alpha\big([\mathbf{0},\mathbf{t}],x\big)=\int_{[\mathbf{0},\mathbf{t}]}\delta_x\big(X(\mathbf{s})\big)\,d\mathbf{s}$$

is jointly continuous in $(\mathbf{t},x)$. For fixed x, the distribution function $\alpha\big([\mathbf{0},\mathbf{t}],x\big)$ ($\mathbf{t}\in(\mathbb{R}^+)^P$) generates a (random) measure $\alpha(A,x)$ ($A\subset(\mathbb{R}^+)^P$) on $(\mathbb{R}^+)^P$ which is called the local time of $X(\mathbf{t})$.

In this paper, the result of Geman, Horowitz and Rosen is applied to the following four Gaussian fields:

1. The d-dimensional fractional Brownian motion $X_1(t)=B^H(t)$.
2. The d-dimensional Riemann-Liouville process $X_2(t)=W^H(t)$.
3. The $d(p-1)$-dimension Gaussian field
$$X_3(t_1,\cdots,t_p)=\Big(B_1^H(t_1)-B_2^H(t_2),\cdots,B_{p-1}^H(t_{p-1})-B_p^H(t_p)\Big).$$
4. The $d(p-1)$-dimension Gaussian field
$$X_4(t_1,\cdots,t_p)=\Big(W_1^H(t_1)-W_2^H(t_2),\cdots,W_{p-1}^H(t_{p-1})-W_p^H(t_p)\Big).$$

Theorem 7.1 *Under $Hd<1$, $X_1(t)$ and $X_2(t)$ satisfy the condition (7.1); under $Hd<p^*$, $X_3(\mathbf{t})$ and $X_4(\mathbf{t})$ satisfy the condition (7.1). Consequently, X_1, X_2, X_3 and X_4 have continuous (jointly in time and space variables) local times.*

Proof: Due to similarity we only verify (7.1) for X_3, which becomes

$$\int_{([0,t]^p)^m}d\mathbf{s}_1\cdots d\mathbf{s}_m\int_{(\mathbb{R}^{d(p-1)})^m}d\tilde{\lambda}_1\cdots d\tilde{\lambda}_m \tag{7.2}$$
$$\times\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\tilde{\lambda}_k\cdot X(\mathbf{s}_k)\Big)\Big\}\prod_{k=1}^m|\tilde{\lambda}_k|^\gamma<\infty$$

where we use the notation

$$\mathbf{s}_k=(t_{k,1},\cdots,t_{k,p})\ \text{ and }\ \tilde{\lambda}_k=(\lambda_{k,1},\cdots,\lambda_{k,p-1})\,.$$

Notice that

$$\mathrm{Var}\Big(\sum_{k=1}^m\tilde{\lambda}_k\cdot X(\mathbf{s}_k)\Big)=\sum_{j=1}^p\mathrm{Var}\Big(\sum_{k=1}^m(\lambda_{k,j}-\lambda_{k,j-1})\cdot B^H(s_{k,j})\Big)$$

with the convention $\lambda_{k,0} = \lambda_{k,p} = 0$. By suitable substitution, a bound

$$|\tilde{\lambda}_k| \le C \prod_{j=1}^p \max\{1, |\lambda_{k,j} - \lambda_{k,j-1}|\}$$

we have

$$\int_{(\mathbb{R}^{d(p-1)})^m} d\tilde{\lambda}_1 \cdots d\tilde{\lambda}_m \exp\left\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \tilde{\lambda}_k \cdot X(\mathbf{s}_k) \Big) \right\} \prod_{k=1}^m |\tilde{\lambda}_k|^\gamma$$
$$\le C \int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{j=1}^p H_j(\bar{\lambda}_j)$$

where

$$H_j(\bar{\lambda}_j) = \Big(\prod_{k=1}^m \max\{1, |\lambda_{k,j}|^\gamma\} \Big) \exp\left\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda_{k,j} \cdot B^H(s_{k,j}) \Big) \right\}$$

for $\bar{\lambda}_j = (\lambda_{1,j}, \cdots, \lambda_{m,j})$ $(1 \le j \le p-1)$ and $\bar{\lambda}_p = -(\bar{\lambda}_1 + \cdots + \bar{\lambda}_{p-1})$.
Write

$$\prod_{j=1}^p H_j(\bar{\lambda}_j) = \prod_{j=1}^p \prod_{1 \le k \ne j \le p} H_k(\bar{\lambda}_k)^{1/(p-1)}.$$

By Hölder inequality

$$\int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{j=1}^p H_j(\bar{\lambda}_j)$$
$$\le \prod_{j=1}^p \left\{ \int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{1 \le k \ne j \le p} H_k(\bar{\lambda}_k)^{p^*} \right\}^{1/p}.$$

When $j = p$,

$$\int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{1 \le k < p} H_k(\bar{\lambda}_k)^{p^*} = \prod_{k=1}^{p-1} \int_{\mathbb{R}^{md}} H_k(\bar{\lambda})^{p^*} d\bar{\lambda}\,.$$

As for $1 \le j \le p-1$, recall that $\bar{\lambda}_p = -(\bar{\lambda}_1 + \cdots + \bar{\lambda}_{p-1})$. By translation invariance,

$$\int_{\mathbb{R}^{md}} H_p(\bar{\lambda}_p)^{p^*} d\bar{\lambda}_j = \int_{\mathbb{R}^{md}} H_p(\bar{\lambda})^{p^*} d\bar{\lambda}.$$

By Fubini theorem, for fixed j,

$$\int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{1 \le k \ne j \le p} H_k(\bar{\lambda}_k)^{p^*} = \prod_{1 \le k \ne j \le p} \int_{\mathbb{R}^{md}} H_k(\bar{\lambda})^{p^*} d\bar{\lambda}.$$

Summarize our argument,

$$\begin{aligned}
&\int_{([0,t]^p)^m} d\mathbf{s}_1\cdots d\mathbf{s}_m \int_{(\mathbb{R}^{d(p-1)})^m} d\bar\lambda_1\cdots d\bar\lambda_m\\
&\times \exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \bar\lambda_k\cdot X(\mathbf{s}_k)\Big)\Big\}\prod_{k=1}^m |\bar\lambda_k|^\gamma\\
&\le C\bigg\{\int_{[0,t]^m} ds_1\cdots ds_m\bigg[\int_{(\mathbb{R}^d)^m} d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m \max\{1,|\lambda_k|^{p^*\gamma}\}\Big)\\
&\times \exp\Big\{-\frac{p^*}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda_k\cdot B^H(s_k)\Big)\Big\}\bigg]^{1/p^*}\bigg\}^p.
\end{aligned}$$

Hence all we need is to find $\gamma>0$ such that

$$\begin{aligned}
&\int_{[0,t]^m} ds_1\cdots ds_m\bigg[\int_{(\mathbb{R}^d)^m} d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m |\lambda_k|^\gamma\Big)\\
&\times \exp\Big\{-\frac{p^*}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda_k\cdot B^H(s_k)\Big)\Big\}\bigg]^{1/p^*}<\infty
\end{aligned}\tag{7.3}$$

for all $m=1,2\cdots$. Further separating variable and substituting variable, the above is reduced to

$$\begin{aligned}
&\int_{[0,t]^m} ds_1\cdots ds_m\bigg[\int_{\mathbb{R}^m} d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m |\lambda_k|^\gamma\Big)\\
&\times \exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m \lambda_k B_0^H(s_k)\Big)\Big\}\bigg]^{d/p^*}<\infty.
\end{aligned}\tag{7.4}$$

By (4.9), for any $s_1<\cdots<s_k$,

$$\begin{aligned}
&\mathrm{Var}\big(B_0^H(s_k)-B_0^H(s_{k-1})|B_0^H(s_1),\cdots,B_0^H(s_{k-1})\big)\\
&\ge\frac{1}{2H}(s_k-s_{k-1})^{2H}=\frac{1}{2H}\mathrm{Var}\big(B_0^H(s_k)-B_0^H(s_{k-1})\big).
\end{aligned}$$

This property is generalized into the notion known as local non-determinism. By Lemma 2.3 in Berman [6], there is constant $c_m>0$ such that for any $\lambda_1,\cdots,\lambda_m\in\mathbb{R}$ and any $s_1<\cdots<s_m$

$$\mathrm{Var}\Big(\sum_{k=1}^m \lambda_k\big(B_0^H(s_k)-B_0^H(s_{k-1})\big)\Big)\ge c_m\sum_{k=1}^m (s_k-s_{k-1})^{2H}\lambda_k^2.$$

Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^m} d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m|\lambda_k|^\gamma\Big)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k B_0^H(s_k)\Big)\Big\}\\
&=\int_{\mathbb{R}^m} d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m|\lambda_k-\lambda_{k-1}|^\gamma\Big)\\
&\times\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\big(B_0^H(s_k)-B_0^H(s_{k-1})\big)\Big)\Big\}\\
&\le\int_{\mathbb{R}^m} d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m|\lambda_k-\lambda_{k-1}|^\gamma\Big)\exp\Big\{-c_m\sum_{k=1}^m(s_k-s_{k-1})^{2H}\lambda_k^2\Big\}.
\end{aligned}$$

Using triangle inequality (for which we take $\gamma\le 1$)

$$\prod_{k=1}^m|\lambda_k-\lambda_{k-1}|^\gamma\le\prod_{k=1}^m\big(|\lambda_k|^\gamma+|\lambda_{k-1}|^\gamma\big)=\sum_{j_1,\cdots,j_m}\prod_{k=1}^m|\lambda_k|^{\delta_{j_k}},$$

where $\delta_{j_k}=0$, γ or 2γ. Notice that

$$\prod_{k=1}^m|\lambda_k|^{\delta_{j_k}}\le\prod_{k=1}^m(1\vee|\lambda_k|)^{\delta_{j_k}}\le\prod_{k=1}^m(1\vee|\lambda_k|)^{2\gamma}.$$

Notice the number of the terms in the previous summation is at most 2^m. Thus,

$$\prod_{k=1}^m|\lambda_k-\lambda_{k-1}|^\gamma\le 2^m\prod_{k=1}^m(1\vee|\lambda_k|)^{2\gamma}.$$

In this way, the problem is reduced to finding $\gamma>0$ such that

$$\begin{aligned}
\int_{[0,t]^m_<} ds_1\cdots ds_m\Big[\int_{\mathbb{R}^m} d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m|\lambda_k|^\gamma\Big)&\\
\times\exp\Big\{-c_m\sum_{k=1}^m(s_k-s_{k-1})^{2H}\lambda_k^2\Big\}\Big]^{d/p^*}&<\infty. \qquad (7.5)
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_{\mathbb{R}^m} d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m|\lambda_k|^\gamma\Big)\exp\Big\{-c_m\sum_{k=1}^m(s_k-s_{k-1})^{2H}\lambda_k^2\Big\}&\\
=\prod_{k=1}^m\int_{-\infty}^\infty|\lambda|^\gamma e^{-c_m(s_k-s_{k-1})^{2H}\lambda^2}d\lambda&\\
=\Big\{\int_{-\infty}^\infty|\lambda|^\gamma e^{-c_m\lambda^2}d\lambda\Big\}^m\prod_{k=1}^m(s_k-s_{k-1})^{-(1+\gamma)H}&.
\end{aligned}$$

Therefore, we need to find $\gamma > 0$ such that

$$\int_{[0,t]^m_<} ds_1 \cdots ds_m \prod_{k=1}^{m} (s_k - s_{k-1})^{-(1+\gamma)Hd/p^*} < \infty.$$

This is always possible because $Hd < p^*$, so that there is a $\gamma > 0$ such that

$$(1+\gamma)Hd < p^*$$

and we finish the proof. $\square$

8 Appendix

The constant

Lemma A1 *Let $\{B^H(t)\}_{t\in\mathbb{R}}$ be a standard fractional Brownian motion given by*

$$B^H(t) = c_H \int_{-\infty}^{t} \left((t-s)^{H-1/2} - (-s)_+^{H-1/2} \right) dB(s), \tag{A1}$$

where $\{B(t)\}_{t\in\mathbb{R}}$ is a standard Brownian motion. Then

$$c_H = \sqrt{2H}\, 2^H B\,(1-H, H+1/2)^{-1/2}\,, \tag{A2}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$ is the usual beta function.

Proof. Since $\mathrm{Var}(B^H(1)) = 1$ we get

$$c_H = \left\{ \int_0^\infty \left((1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx + \frac{1}{2H} \right\}^{-1/2}. \tag{A3}$$

Put

$$I = \int_0^\infty \left((1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx\,.$$

Then

$$\begin{aligned}
I &= \lim_{\mu\to 0^+} \int_0^\infty \left((1+x)^{H-1/2} - x^{H-1/2} \right)^2 e^{-\mu x}\,dx \\
&= \lim_{\mu\to 0^+} \left\{ (e^\mu + 1)\mu^{-2H}\Gamma(2H) - e^\mu \mu^{-2H}\gamma(2H,\mu) - 2\int_0^\infty (1+x)^{H-1/2} x^{H-1/2} e^{-\mu x}\,dx \right\} \\
&= -\frac{1}{2H} + \lim_{\mu\to 0^+} \left\{ 2e^{\mu/2}\mu^{-2H}\Gamma(2H) - 2\int_0^\infty (1+x)^{H-1/2} x^{H-1/2} e^{-\mu x}\,dx \right\} \\
&= -\frac{1}{2H} + \lim_{\mu\to 0^+} \left\{ 2e^{\mu/2}\mu^{-2H}\Gamma(2H) - \frac{2}{\sqrt{\pi}} e^{\mu/2}\Gamma\big(H+\frac{1}{2}\big)\,\mu^{-H} K_{-H}\big(\frac{\mu}{2}\big) \right\},
\end{aligned}$$

where $\gamma(z,x)$ and $K_\nu(z)$ are the incomplete gamma function and modified Bessel function of the second kind, respectively. The third equality uses the facts that $e^\mu \mu^{-2H}\gamma(2H,\mu) = \frac{1}{2H} + \mathrm{o}(1)$, and that $(e^\mu+1)\mu^{-2H} = 2e^{\mu/2}\mu^{-2H} + o(1)$ for $H<1$, as $\mu \to 0$. The forth equality applies formula 3.3838 in [20].

Using the duplication formula

$$\Gamma(2H) = \frac{2^{2H-1}}{\sqrt{\pi}}\Gamma(H)\Gamma\big(H+\frac{1}{2}\big),$$

see 8.3351 [20], we get

$$I = -\frac{1}{2H} + \frac{1}{\sqrt{\pi}}\,\Gamma\big(H+\frac{1}{2}\big)\lim_{\mu\to 0^+}\Big\{\mu^{-2H}2^{2H}\Gamma(H) - 2\mu^{-H}\,K_H\big(\frac{\mu}{2}\big)\Big\}. \tag{A4}$$

Notice that

$$\mu^{-2H}2^{2H}\Gamma(H) = \int_0^\infty x^{H-1}e^{-\frac{\mu^2}{4}x}\,dx,$$

and by the identity

$$K_\nu(z) = \frac{1}{2}\big(\frac{z}{2}\big)^\nu \int_0^\infty t^{-\nu-1}e^{-t-\frac{z^2}{4t}}\,dt,$$

see 3.4326 in [20], we also get

$$2\mu^{-H}\,K_H\big(\frac{\mu}{2}\big) = \int_0^\infty x^{H-1}e^{-\frac{\mu^2}{4}x-\frac{1}{4x}}\,dx.$$

Hence, as $\mu \to 0$,

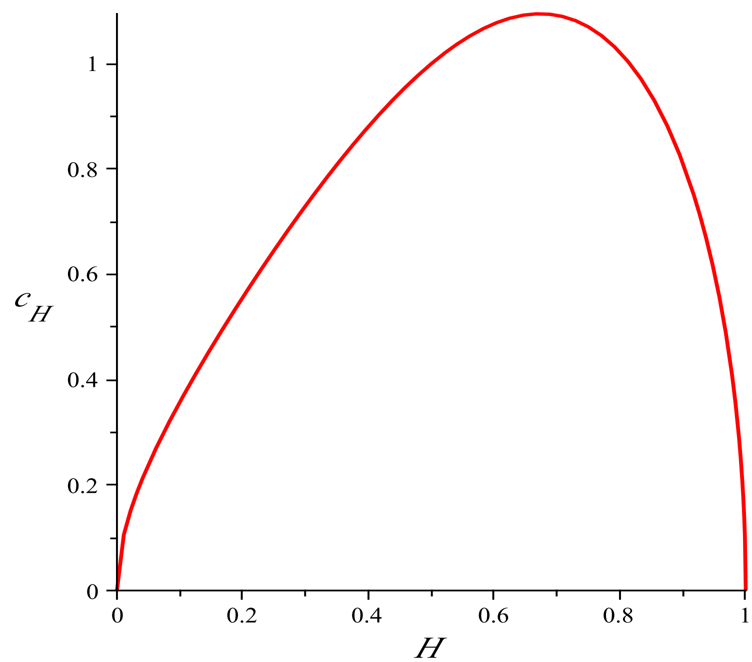
$$\begin{aligned}\mu^{-2H}2^{2H}\Gamma(H) - 2\mu^{-H}\,K_H\big(\frac{\mu}{2}\big) &= \int_0^\infty x^{H-1}e^{-\frac{\mu^2}{4}x}(1-e^{-\frac{1}{4x}})\,dx \\ &\to \int_0^\infty x^{H-1}(1-e^{-\frac{1}{4x}})\,dx = 4^{-H}H^{-1}\Gamma(1-H).\end{aligned}$$

The last expression comes from change of variable and integration by parts. Substituting this into (A4) we conclude that

$$I = -\frac{1}{2H} + \frac{\Gamma(1-H)\Gamma\big(H+\frac{1}{2}\big)}{\sqrt{\pi}\,4^H H} = -\frac{1}{2H} + \frac{B\big(1-H, H+\frac{1}{2}\big)}{2H4^H}. \tag{A5}$$

The last equality follows from well-known formula $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ (see, e.g., 8.3841 [20]). Combining (A5) with (A3) yields the formula (A2) for c_H. □

Below we give a plot of c_H as a function of $H \in (0,1)$.



The RKHS of a Riemann–Liouville process

Consider the Riemann–Liouville process $W^H(t)$ with index $H > 0$ defined by (1.2). The reproducing kernel Hilbert space (RKHS) $\mathbb{H}$ of $\{W^H(t)\}_{t\in[0,T]}$ follows standard theory of RKHS, see [31] and [4]. A convenient form for us can be found in van der Vaart and van Zanten [41, Lemma 10.2] as $\mathbb{H} = I_{0+}^{H+1/2}(L_2[0,T])$, where

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0+}^{t} (t-s)^{\alpha-1} f(s)\, ds\,, \quad t \in [0,T] \tag{A6}$$

is the Riemann–Liouville fractional integral of order $\alpha > 0$; for $\alpha = 0$, $I_{0+}^0 f := f$. In this section we address the question, given $f \in C([0,T])$, how to verify that f is in $\mathbb{H}$? The results are used in Sections 3.2.

Let $AC^m[0,T]$ denote the space of functions f which have continuous derivatives up to order $m-1$ on $[0,T]$ with $f^{(m-1)}$ absolutely continuous on $[0,T]$, where $m \in \mathbb{N}$. Put

$$AC_2^m[a,b] = \left\{ f \in AC^m[a,b] : \int_a^b |f^{(m)}(x)|^2\, dx < \infty \right\}. \tag{A7}$$

Proposition A2 *Let $H > 0$ and let m be the smallest integer greater than or equal to $H + 1/2$. For $f \in C[0,T]$, put $f_H := I_{0+}^{m-(H+1/2)} f$. The RKHS of the process $\{W^H(t)\}_{t\in[0,T]}$, viewed as a random element in $C[0,T]$, is a Hilbert space*

$$\mathbb{H} = \left\{ f \in C[0,T] :\ f_H \in AC_2^m[0,T] \text{ and } f_H^{(k)}(0) = 0,\ \text{for } k = 0, \ldots, m-1 \right\}.$$

The RKHS-norm of f is given by

$$\|f\|_{\mathbb{H}} = \frac{1}{\Gamma(H+1/2)} \left(\int_0^T |f_H^{(m)}(t)|^2 \, dt \right)^{1/2} . \tag{A8}$$

Proof: By [41, Lemma 10.2] we have $\mathbb{H} = I_{0+}^{H+1/2}(L_2[0,T])$ and

$$\|I_{0+}^{H+1/2} g\|_{\mathbb{H}} = \Gamma(H+1/2)^{-1} \cdot \|g\|_{L_2[0,T]} , \quad g \in L_2[0,T]. \tag{A9}$$

Let $f \in \mathbb{H}$, so that $f = I_{0+}^{H+1/2} g$ for some $g \in L_2[0,T]$. By the semigroup property of $\{I_{0+}^{\alpha} : \alpha \geq 0\}$ (see [40, Theorem 2.5],

$$f_H = I_{0+}^{m-(H+1/2)} I_{0+}^{H+1/2} g = I_{0+}^{m} g.$$

Hence $f_H \in AC_2^m[0,T]$ and $f_H^{(k)}(0) = 0$ for $k = 0, \ldots, m-1$. Moreover, $f_H^{(m)} = g \in L_2[0,T]$. Clearly (A8) follows from (A9).

Conversely, if for a continuous function f we have that $f_H \in AC_2^m[0,T]$ with $f_H^{(k)}(0) = 0$ for $k = 0, \ldots, m-1$, then

$$f_H(t) = \sum_{k=0}^{m-1} \frac{f_H^{(k)}(0)}{k!} t^k + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f_H^{(m)}(s) \, ds = I^m f_H^{(m)}(t),$$

where $f_H^{(m)} \in L_2[0,T]$. Again by the semigroup property,

$$0 = I_{0+}^{m-(H+1/2)} f - I_{0+}^{m} f_H^{(m)} = I_{0+}^{m-(H+1/2)} \left(f - I^{H+1/2} f_H^{(m)} \right) .$$

Since the operator $I_{0+}^{m-(H+1/2)} : L^2[0,T] \mapsto L_2[0,T]$ is injective (see [40, Theorem 13.1]), we get $f = I_{0+}^{H+1/2} f_H^{(m)}$. □

Remark A3 *Proposition A2 also covers the well-known cases of a Brownian motion and k-times integrated Brownian motion, $k = 0, 1, \ldots$. In these cases $H + 1/2 = k + 1$ is a positive integer, so that $f_H = I_{0+}^{0} f = f$. Consequently, $f \in \mathbb{H}$ if and only if $f \in AC_2^{k+1}[0,T]$ and $f(0) = \cdots = f^{(k)}(0) = 0$.*

The following is a simple sufficient condition for function to belong to the RKHS of $\{W^H(t)\}_{t \in [0,T]}$.

Corollary A4 *Let $m = \lceil H + 1/2 \rceil$ be as in Proposition A2. Then any function f in $AC_2^m[0,T]$, with $f(0) = \cdots = f^{(m-1)}(0) = 0$, belongs to $\mathbb{H}$ and*

$$\|f\|_{\mathbb{H}} = \frac{1}{\Gamma(H+1/2)} \left(\int_0^T |I_{0+}^{m-(H+1/2)} f^{(m)}(t)|^2 \, dt \right)^{1/2} .$$

Proof: We can write $f = I_{0+}^m f^{(m)}$, where $f^{(m)} \in L_2[0,T]$. Then we have

$$f_H = I_{0+}^{m-(H+1/2)} f = I_{0+}^{m-(H+1/2)} I_{0+}^m f^{(m)} = I_{0+}^m \left(I_{0+}^{m-(H+1/2)} f^{(m)} \right).$$

Hence $f_H^{(k)}(0) = 0$ for $k = 0, \ldots, m-1$ and $f_H^{(m)} = I_{0+}^{m-(H+1/2)} f^{(m)} \in L^2[0,T]$. The formula for the norm is a consequence of the last equality and (A8). □

Determinant of a Gaussian covariance

Lemma A5 *Let $x_1, \ldots, x_n$ be vectors in a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and let $C = [\langle x_i, x_j \rangle]_{1 \le i,j \le n}$. Then*

$$\det(C) = \|x_1\|^2 \, \|x_2 - \mathrm{proj}_{x_1}(x_2)\|^2 \cdots \|x_n - \mathrm{proj}_{x_1,\ldots,x_{n-1}}(x_n)\|^2$$

where $\mathrm{proj}_{x_1,\ldots,x_{i-1}}(x_i)$ denotes the orthogonal projection of x_i onto the linear space spanned by $x_1, \ldots, x_{i-1}$.

Proof. Notice that vectors $y_i = x_i - \mathrm{proj}_{x_1,\ldots,x_{i-1}}(x_i)$, $i = 1, \ldots, n$, are orthogonal and

$$x_i = a_{i1} y_1 + \cdots + a_{in} y_n$$

for some $a_{ij} \in \mathbb{R}$ with $a_{ii} = 1$ and $a_{ij} = 0$ for $j > i$. Since

$$\langle x_i, x_j \rangle = \sum_{k=1}^n a_{ik} a_{jk} \|y_k\|^2,$$

we have $C = AA^\top$, where $A = [a_{ij}\|y_j\|]_{1 \le i,j \le n}$ is a lower triangular matrix with $\|y_i\|$'s on the diagonal. Hence

$$\det(C) = \det(A)^2 = \prod_{i=1}^n \|y_i\|^2.$$

□

References

[1] Anderson, T.W. *An introduction to multivariate statistical analysis.* Wiley Publications in Statistics, 1958.

[2] Asselah, A. and Castell, F. (2007) Self-intersection local times for random walk, and random walk in random scenery in dimension $d \geq 5$. *Probab. Theor. Rel. Fields* **138** 1-32.

[3] Bass, R. F. and Chen, X. Self intersection local time: critical exponent, large deviations and law of the iterated logarithm. *Ann. Probab.* **32** (2004) 3221–3247.

[4] Berlinet, A. and Thomas-Agnan, C. *Reproducing kernel Hilbert spaces in probability and statistics*. Kluwer Academic, 2004.

[5] Berman, S. M. (1969). Local times and sample function properties of stationary Gaussian processes. *Trans. Amer. Math. Soc.* **137** 277–299.

[6] Berman, S. M. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** 69-94.

[7] Breiman, L. *Probability*. Addison-Wesley Massachusetts, 1968.

[8] Chen, X. (2004). Exponential asymptotics and law of the iterated logarithm for intersection local times of random walks. *Ann. Probab.* **32** 3248–3300.

[9] Chen, X. *Random Walk Intersections: Large Deviations and Related Topics.* Mathematical Surveys and Monographs, AMS (to appear).

[10] Chen, X. and Li, W.V. (2003). Quadratic functionals and small ball probabilities for the m-fold integrated Brownian motion, *Annals of Probability*, **31**, 1052-1077.

[11] Chen, X. and Li, W.V. (2004). Large and moderate deviations for intersection local times. *Probab. Theor. Rel. Fields* **128** 213–254.

[12] Chen, X. and Li, W.V. and Rosen, J. (2005). Large deviations for local times of stable processes and stable random walks in 1 dimension, *Electronic Journal of Probability*, **10**, Paper no. 16, 577-608.

[13] Davies, P.L. (1976). Tail behaviour for positive random variables with entire characteristic functions and completely regular growth, *Z. Ang. Math. Mech.* **56** T334–336.

[14] Dembo, A. and Zeitouni, O. *Large Deviations Techniques and Applications. (2nd ed.),* Springer, New York, 1998.

[15] Donsker, M. D. and Varadhan, S. R. S. (1981). The polaron problem and large deviations. *New stochastic methods in physics. Phys. Rep.* **77** 235-237.

[16] Fernández, R., Fröhlich, J. and Sokal, A. D. *Random Walks, Critical Phenomena, and Triviality in quantum field theory.* Springer, New York, 1992.

[17] Fleischmann, K., Mörters, P. and Wachtel, V. (2008). Moderate deviations for random walk in random scenery. *Stoch. Proc. Appl.* **118** 1768-1802.

[18] Gantert, N., König, W. and Shi, Z. (2007). Annealed deviations of random walk in random scenery. *Ann. Inst. H. Poincaré* **43** 47-76.

[19] Geman, D., Horowitz, J. and Rosen, J. (1984). A local time analysis of intersections of Brownian paths in the plane. *Ann. Probab.* **12** 86-107.

[20] Gradshteyn, I.S. and Ryzhik, I.M. *Table of Integrals, Series, and Products. Sixth Ed.* Academic Press, 2000.

[21] Hamana, Y. and Kesten, H. (2001). A large-deviation result for the range of random walk and for the Wiener sausage. *Probab. Theory Related Fields* **120** 183-208.

[22] van der Hoftad, R., König, W. and Mörters, P. (2006). The universality classes in the parabolic Anderson model. *Comm. Math. Phys.* **267** 307-353.

[23] den Hollander, F.*Random Polymers.* Lecture Notes in Mathematics **1974** Springer, Heidelberg, 2009.

[24] Hu, Y., Nualart, D. (2005). Renormalized self-intersection local time for fractional Brownian motion. *Ann. Probab.* **33** 948–983.

[25] Hu, Y., Nualart, D. and Song, J. (2008). Integral representation of renormalized self-intersection local time. *Journal of Functional Analysis* **255** 2507-2532.

[26] König, W. and Mörters, P. (2002). Brownian intersection local times: Upper tail asymptotics and thick points. *Ann Probab.* **30** 1605-1656.

[27] Lawler, G. F. *Intersections of Random Walks, Probability and Its applications.* Birkhäuser Boston, 1991.

[28] Le Gall, J-F. (1986). Propriétés d'intersection des marches aléatoires. I. Convergence vers le temps local d'intersection. *Comm. Math. Phys.* **104**, 471-507.

[29] Ledoux, M. and Talagrand, M. *Probability on Banach Spaces*, Springer, Berlin, 1991.

[30] Li, W.V. and Linde, W. (1998). Existence of small ball constants for fractional Brownian motions. *C.R. Acad. Sci. Paris*, **326** , 1329-1334.

[31] Li, W.V. and Linde, W. (1999). Approximation, metric entropy and small ball estimates for Gaussian measures. *Ann. Probab.* **27**, 1556-1578.

[32] Li, W.V. and Shao, Q.M. (2001). Gaussian processes: inequalities, small ball probabilities and applications. Handbook of Statistics, Vol. **19**, *Stochastic processes: Theory and methods*, Edited by C.R. Rao and D. Shanbhag, 533-598, Elsevier.

[33] Madras, N. and Slade, G. (1993). *The Self-avoiding Walk.* Birkhäuser, Boston.

[34] Mandelbrot, B. and Van Ness, J, (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** 422–437.

[35] Marcus, M. B. and Rosen, J. (1997). Laws of the iterated logarithm for intersections of random walks on Z^4. *Ann. Inst. H. Poincaré Probab. Statist.* **33** 37–63.

[36] Nualart, D. and Ortiz-Latorre, S. (2007). Intersection local time for two independent fractional Brownian motions. *J. Theor. Probability* **20** 759-757.

[37] Pipiras, V. and Taqqu, M.S. (2002). Deconvolution of fractional Brownian motion. *J. Time Ser. Anal.* **23** 487-501.

[38] Revuz, D. and Yor, M. *Continuous martingales and Brownian motion. Third edition* Springer-Verlag, 1999.

[39] Rosen, J. (1987). The intersection local time of fractional Brownian motion in the plane. *J. Multivariate Anal.* **23** 37-46.

[40] Samko, S. G., Kilbas, A. A. and Marichev, O. I. Fractional Integrals and Derivatives. Gordon and Breach Science Publishers, Yverdon, 1993.

[41] van der Vaart, A.W. and van Zanten, J.H. (2008). Reproducing kernel Hilbert spaces of Gaussian priors. *IMS Collections Pushing the Limits of Contemporary Statistics*: Contributions in Honor of Jayanta K. Ghosh Vol. 3(2008) 200–222.

[42] Wu, D. and Xiao, Y. (2009) Regularity of intersection local times of fractional Brownian motions. To appear in J. Theor. Probab.

[43] Xiao, Y. (2006). Properties of local nondeterminism of Gaussian and stable random fields and their applications. *nn. Fac. Sci. Toulouse Math.* **XV** 157-193.

[44] Xiao, Y. (2007). Strong local nondeterminism of Gaussian random fields and its applications. *Asymptotic Theory in Probability and Statistics with Applications* (T.-L. Lai, Q.-M. Shao and L. Qian, eds) 136–176, Higher Education Press, Beijing.

Xia Chen
Department of Mathematics
University of Tennessee
Knoxville TN 37996, USA
xchen@math.utk.edu

Wenbo V. Li
Department of Mathematical Sciences
University of Delaware
Newark DE 19716, USA
wli@math.udel.edu

Jan Rosiński
Department of Mathematics
University of Tennessee
Knoxville TN 37996, USA
rosinski@math.utk.edu

Qi-Man Shao
Department of Mathematics
Hong Kong University of Science and Technology
Hong Kong, China
maqmshao@ust.hk