Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 323 (2006) 1373–1382
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# An analysis of the last round matching problem

Wenbo V. Li [a,1], Fengshan Liu [b,2], Xiquan Shi [b,*,3]

[a] Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA
[b] Department of Applied Mathematics and Theoretical Physics, Delaware State University, Dover, DE 19901, USA

Received 7 June 2005

Available online 27 December 2005

Submitted by J. Rosinski

**Abstract**

The probability distribution of the number of players in the last round of a matching problem is analyzed and the existence of the limiting distribution is proved by using convolution method.


*Keywords:* Montmort random variable; Multi-round matching problem; Convolution

## 1. Introduction

The matching problem or the "Hats Problem" goes back to at least 1713 when it was proposed by the French mathematician Pierre de Montmort in his book [9] on games of gambling and chance, *Essay d'Analyse sur les Jeux de Hazard* (see also [11]). Although it has many formulations, the most common one is as follows:

*Suppose that each of n men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. What is the probability that exactly k of the men select their own hats? And what is the expected number of people that select their own hats?*

* Corresponding author.
*E-mail addresses:* wli@math.udel.edu (W.V. Li), fliu@desu.edu (F. Liu), xshi@desu.edu (X. Shi).
1 Supported in part by NSF grant DMS-0505805.
2 Supported in part by ARO (DAAD 19-03-1-0375).
3 Supported in part by ARO (DAAD 19-03-1-0375).


doi:10.1016/j.jmaa.2005.11.024

There are several approaches such as the inclusion–exclusion principle, recurrence relations, binomial inversion, to find the distribution of the so called Montmort random variable $X_n$, the number of matches with $n$ men. These approaches can be found, for example, in [10, pp. 125–127] and [4]. See more details in the next section. A celebrated paper of Kaplansky [7] first showed that $X_n$ has an approximate Poisson distribution. Takács [12] gave an extensive history for this single round version of the matching problem. The problem of a multi-round matching is as follows:

*Suppose that those choosing their own hats depart, while the others put their selected hats in the center of the room, mixed them up, and then reselect. Also, suppose that this process continues until each individual has his own hat. Assume we start with n men. What is the expected number $\tau_n$ of rounds that are necessary?*

Detailed analysis can be found in [10, p. 110], see also Section 2 for more detail. The purpose of this note is to analyze the following questions in the multi-round matching problem:

*What is the expected number of men, $L_n$, on the last round? And what is the limiting distribution of $L_n$ as $n \to \infty$?*

Note that the last round is the round that all remaining players obtained their own hats. This requires us to determine the distribution of the stopped outcome and hence it can be viewed as conditioning on the future. There are many problems of interests that involve the moments (outcomes) of the last events at a stopping time, and they are in general hard problems even in determination of the expected values (see [13, p. 162], for a case of independent random walks).

We will show in Section 3 that the limiting distribution of $L_n$ exists and the expectation $\mathbb{E}(L_n)$ converges to a constant $l \approx 2.26264703816$. The exact value of the constant $l$ is still unknown. Let $p_n$ be the probability that any particular individual man (among $n$ men) is in the last round. Then we have $p_n = \mathbb{E}(L_n)/n$ by taking the expectation on the relation $L_n = \sum_{i=1}^n 1_{A_i}$, where $A_i$ is the event that the $i$th man is in the last round. Thus $p_n = \mathbb{E}(L_n)/n \sim l/n$ as $n \to \infty$ is the asymptotic probability of winning (a winner is defined to be one in the last round), and the fair pay off for a winner is $1/p_n \sim n/l \approx n/2.262647$ unit if each man pays a unit to play.

The remaining of the paper is organized as follows. In Section 2, we provide mathematical formulation of the problem together with some remarkable properties of the Montmort random variable $X_n$, and we give the recursive relations for the expectation $\mathbb{E}(L_n)$ and estimate the distribution $q_{n,m} = \mathbb{P}(L_n = m)$. We prove our main result in Section 3 by using convolution method. In Section 4, we provide a table for the distribution of $L_n$ for $2 \leqslant n \leqslant 15$ and numerical approximations of the limiting values. Related problems, questions and remarks are collected at the end of Section 4.

## 2. Recursive relations

Mathematically, we are dealing with random permutation of $n$ elements and the Montmort random variable $X_n$ is the number of fixed elements in a random permutation. Its distribution is well known and is given by

$$p_{n,k} = \mathbb{P}(X_n = k) = \frac{1}{k!}\sum_{i=0}^{n-k}(-1)^i\frac{1}{i!}, \quad k = 0, 1, \ldots, n. \tag{2.1}$$

Feller [3, p. 231] showed that both the mean and the variance of $X_n$ are equal to one for all $n \geqslant 2$. He also derived the Poisson approximation of rate $\lambda = 1$ for $X_n$. Some rather remarkable properties of $X_n$ were derived in [1], see also [5,6]. It is shown that the $k$th moment $\mathbb{E}(X_n^k)$ is the number of ways of putting $k$ different things into at most $\min(k,n)$ indistinguishable cells, with no cell empty, i.e.,

$$\mathbb{E}\big(X_n^k\big) = \sum_{i=1}^{\min(k,n)} S(k,i)$$

where $S(k,i)$ is the Stirling number of the second kind, i.e., the number of ways of partitioning a set of $k$ elements into $i$ nonempty sets (i.e., set blocks). In particular, every moment of every $X_n$ is a positive integer and $\mathbb{E}(X_n^k)$ are the same for all $k$, $1 \leqslant k \leqslant n$, with $n$ fixed. Also, it is easy to see that the $k$-factorial moment of $X_n$ is one for $1 \leqslant k \leqslant n$ and zero for $k > n$, i.e.,

$$\mathbb{E}\big(X_n(X_n-1)\cdots(X_n-k+1)\big) = \begin{cases} 1 & \text{for } 1 \leqslant k \leqslant n, \\ 0 & \text{for } k > n. \end{cases}$$

More detailed and recent study on generalized matching, based on the Stein's method of exchangeable pairs, is given in [2]. In particular, we know

$$\big\|\mathcal{L}(X_n) - \mathcal{L}\big(\text{Poisson}(1)\big)\big\|_{\text{TV}} \leqslant \frac{2^n}{n!}$$

where $\mathcal{L}(\cdot)$ is the probability law for the random variable and $\|\cdot\|_{\text{TV}}$ is the total variation norm for measure. For the multi-round matching problem, the expected number of rounds $\tau_n$ is $\mathbb{E}(\tau_n) = n$, assuming we start with $n$ men, see [10, p. 110].

Next we derive recursive relations for the expectation $\mathbb{E}(L_n)$ and the distribution $q_{n,m} = \mathbb{P}(L_n = m)$, where $L_n$ is the number of people in the last round in the multi-round matching problem, assuming we start with $n$ men. Let $l_n = \mathbb{E}(L_n)$ and $X_n$ be the number of matchings in the first round. Then we have by using conditional expectation,

$$l_n = \sum_{k=0}^{n} \mathbb{E}(L_n \mid X_n = k) \cdot \mathbb{P}(X_n = k) = l_n p_{n,0} + n p_{n,n} + \sum_{k=1}^{n-1} l_{n-k} \cdot p_{n,k}$$

which can be rewritten as

$$l_n(1 - p_{n,0}) = n p_{n,n} + \sum_{k=1}^{n-1} l_k p_{n,n-k}.$$

From (2.1) we have the recursive relation

$$l_n \cdot \sum_{i=1}^{n} (-1)^{i+1} \frac{1}{i!} = \frac{1}{(n-1)!} + \sum_{k=1}^{n-1} \frac{l_k}{(n-k)!} \sum_{i=0}^{k} (-1)^i \frac{1}{i!} \tag{2.2}$$

with $l_0 = 0$, $l_1 = 1$, $l_2 = 2$, $l_3 = 9/4$, $l_4 = 34/15$, $l_5 = 43/19$, $l_6 = 3912/1729$. It is easy to show that $l_n$ converges to $l = 2.2626470381671\ldots$ very quickly. See Section 4 for more discussion.

Next we turn to the distribution function of $L_n$. Let $q_{n,m} = \mathbb{P}(L_n = m)$, $2 \leqslant m \leqslant n$, be the distribution function of $L_n$. Then by conditioning on $X_n$, the number of matchings in the first round, we have for $2 \leqslant m < n$,

$$q_{n,m} = \sum_{k=0}^{n-m} \mathbb{P}(L_n = m \mid X_n = k) \cdot \mathbb{P}(X_n = k) = q_{n,m} \cdot p_{n,0} + \sum_{k=1}^{n-m} q_{n-k,m} \cdot p_{n,k}$$

and

$$\begin{aligned} q_{n,n} &= \mathbb{P}(L_n = n \mid X_n = 0) \cdot \mathbb{P}(X_n = 0) + \mathbb{P}(L_n = n \mid X_n = n) \cdot \mathbb{P}(X_n = n) \\ &= q_{n,n} \cdot p_{n,0} + \frac{1}{n!}. \end{aligned}$$

Thus we obtain the recursive relation

$$\begin{aligned} q_{n,m} &= (1 - p_{n,0})^{-1} \sum_{k=m}^{n-1} q_{k,m} \cdot p_{n,n-k}, \quad 2 \leqslant m < n, \\ q_{n,n} &= \frac{1}{(1 - p_{n,0})n!} = \frac{1}{n! \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i!}} \end{aligned} \tag{2.3}$$

with $q_{2,2} = 1$, $q_{3,2} = 3/4$, $q_{3,3} = 1/4$, $q_{4,2} = 4/5$, $q_{4,3} = 2/15$, $q_{4,4} = 1/15$.

Finally we can state our main results.

**Theorem 2.1.** *The random variables $L_n$ converges in distribution, i.e.,*

$$\lim_{n\to\infty} q_{n,m} = q_m > 0, \quad m \geqslant 2,$$

*and* $\lim_{n\to\infty} \mathbb{E}(L_n) = l = \sum_{m=2}^{\infty} m q_m$.

## 3. Convolution limit

We first show the following lemma for the existence of a limit of convolutions.

**Lemma 3.1.** *Assume the sequence $h_n$ converges to a finite real number $h$ as $n \to \infty$ and the series $\sum_{i=1}^{\infty} |g_i|$ converges. Then the convolution sequence $\sum_{i=1}^{n} g_i h_{n-i}$ converges.*

**Proof.** For any given $\varepsilon > 0$, there exists a positive integer $N$ such that for any $m > n \geqslant N$, $|h_n - h| \leqslant \varepsilon$, $|h_m - h_n| \leqslant \varepsilon$, $\sum_{i=n+1}^{m} |g_i| \leqslant \varepsilon$. Hence for any $m > n \geqslant 2N$,

$$\begin{aligned} &\left| \sum_{i=1}^{m} g_i h_{m-i} - \sum_{i=1}^{n} g_i h_{n-i} \right| \\ &\quad \leqslant \sum_{i=1}^{n-N} |g_i| \cdot |h_{m-i} - h_{n-i}| + \sum_{i=n-N+1}^{n} |g_i| \cdot |h_{m-i} - h_{n-i}| + \sum_{i=n+1}^{m} |g_i| \cdot |h_{m-i}| \\ &\quad \leqslant \varepsilon \cdot \sum_{i=1}^{n-N} |g_i| + 2M \sum_{i=n-N+1}^{n} |g_i| + M \sum_{i=n+1}^{m} |g_i| \leqslant \varepsilon \cdot \sum_{i=1}^{\infty} |g_i| + 3M\varepsilon \end{aligned} \tag{3.1}$$

where $M = \max_{i \geqslant 0} |h_i| < \infty$. □

**Proof of Theorem 2.1.** Let $l_n$ be recursively defined as in (2.2). We first show $l_n$ converges. The goal is to rewrite $l_n$ as a convolution. We start with writing our recursive equation (2.2) into a convolution form with error terms. Denote the tail of the Taylor series expansion of the function $e^{-x}$ at $x = 1$ by

$$a_n = \sum_{i=n+1}^{\infty} (-1)^i \frac{1}{i!} = \int_0^1 \frac{(-1)^{n+1}(1-t)^n}{n!} e^{-t}\, dt.$$

Then for any integer $n \geqslant 1$,

$$|a_n| \leqslant \frac{1}{(n+1)!}. \tag{3.2}$$

Using the Taylor expansion $e^{-1} = \sum_{i=0}^{\infty}(-1)^i \frac{1}{i!}$, we can rewrite (2.2) as

$$l_n\left(1 - e^{-1} + a_n\right) = \frac{1}{(n-1)!} + \sum_{k=1}^{n-1} \frac{l_k}{(n-k)!} \cdot \left(e^{-1} - a_k\right)$$

which gives us

$$l_n = \frac{1}{e-1} \sum_{k=1}^{n-1} \frac{l_k}{(n-k)!} + g_n, \quad n \geqslant 2, \tag{3.3}$$

where

$$g_n = \frac{e}{e-1}\left(\frac{1}{(n-1)!} - \sum_{k=1}^{n} \frac{a_k l_k}{(n-k)!}\right).$$

Next we show $|g_n|$ is very small for large $n$. Obviously, we know $l_k \leqslant k$ for any $k$ since the number of people on the last round cannot be more than the total number of people we started with. Using (3.2), it holds for any $n \geqslant 1$,

$$\begin{aligned}
\left(1 - e^{-1}\right)|g_n| &\leqslant \frac{1}{(n-1)!} + \sum_{k=1}^{n} \frac{k}{(k+1)!(n-k)!} \\
&= \frac{1}{(n-1)!} + \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} - \frac{1}{(n+1)!}\sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \frac{1}{(n-1)!} + \frac{2^n}{n!} - \frac{2^{n+1}-1}{(n+1)!}.
\end{aligned} \tag{3.4}$$

Therefore, we have

$$\sum_{n=1}^{\infty} |g_n| < \infty. \tag{3.5}$$

To write $l_n$ explicitly as a convolution, we use the generating function $\sum_{n=1}^{\infty} l_n x^n$ which converges for $|x| < 1$ due to the fact that $l_n \leqslant n$. From (3.3), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} l_n x^n &= \frac{1}{e-1} \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n-1} \frac{l_k}{(n-k)!} + \sum_{n=1}^{\infty} g_n x^n \\
&= \frac{1}{e-1} \sum_{k=1}^{\infty} l_k \sum_{n=k+1}^{\infty} \frac{1}{(n-k)!} x^n + \sum_{n=1}^{\infty} g_n x^n \\
&= \frac{e^x - 1}{e-1} \sum_{k=1}^{\infty} l_k x^k + \sum_{n=1}^{\infty} g_n x^n.
\end{aligned}$$

Solving for the generating function $\sum_{n=1}^{\infty} l_n x^n$ from above, we obtain

$$\sum_{n=1}^{\infty} l_n x^n = \frac{1-e^{-1}}{1-e^{x-1}} \sum_{n=1}^{\infty} g_n x^n = \left(1-e^{-1}\right) \sum_{k=0}^{\infty} e^{k(x-1)} \sum_{j=1}^{\infty} g_j x^j$$
$$= \left(1-e^{-1}\right) \sum_{k=0}^{\infty} e^{-k} \sum_{i=0}^{\infty} \frac{x^i k^i}{i!} \sum_{j=1}^{\infty} g_j x^j$$
$$= \left(1-e^{-1}\right) \sum_{i=0}^{\infty} h_i x^i \sum_{j=1}^{\infty} g_j x^j = \left(1-e^{-1}\right) \sum_{n=1}^{\infty} \sum_{i=1}^{n} g_i h_{n-i} x^n,$$

where

$$h_n = \frac{1}{n!} \sum_{k=0}^{\infty} k^n e^{-k}. \tag{3.6}$$

Thus, we obtain

$$l_n = \left(1-e^{-1}\right) \sum_{i=1}^{n} g_i h_{n-i}, \tag{3.7}$$

which is in a convolution form.

To show the convergence of the sequence $h_n$ defined in (3.6), we note that $f(x) = x^n e^{-x}$ is strictly increasing in the interval $[0, n]$ and strictly decreasing in the interval $[n, \infty)$, for any integer $n \geqslant 1$. Thus using Stirling's formula for $n^n e^{-n}$, we have for $n$ large,

$$\sum_{k=0}^{\infty} k^n e^{-k} = \sum_{k=0}^{n-1} k^n e^{-k} + n^n e^{-n} + \sum_{k=n+1}^{\infty} k^n e^{-k}$$
$$\leqslant \int_0^{\infty} x^n e^{-x}\, dx + \frac{n!}{\sqrt{2n\pi}} = n!\left(1 + (2n\pi)^{-1/2}\right). \tag{3.8}$$

Similarly, we have

$$\sum_{k=0}^{\infty} k^n e^{-k} = \sum_{k=0}^{n} k^n e^{-k} - n^n e^{-n} + \sum_{k=n}^{\infty} k^n e^{-k}$$
$$\geqslant \int_0^{\infty} x^n e^{-x}\, dx - \frac{n!}{\sqrt{2n\pi}} = n!\left(1 - (2n\pi)^{-1/2}\right). \tag{3.9}$$

Combining the above estimates together, we see

$$\lim_{n\to\infty} h_n = 1. \tag{3.10}$$

And thus the limit of $l_n$ exists by Lemma 3.1, (3.7), (3.5) and (3.10).

Next we follow the same idea as above to show the existence of the limit of $q_{n,m}$ as $n \to \infty$ for fixed $m \geqslant 2$. The recursive relation (2.3) can be rewritten as

$$q_{n,m} = (e-1)^{-1} \sum_{k=m}^{n-1} \frac{q_{k,m}}{(n-k)!} + g_{n,m}, \quad 2 \leqslant m \leqslant n, \tag{3.11}$$

where

$$g_{n,m} = -\left(1 - e^{-1}\right)^{-1} \sum_{k=m}^{n} \frac{a_k q_{k,m}}{(n-k)!}.$$

Using (3.2) and $q_{n,m} \leqslant 1$, it holds for $n \geqslant m \geqslant 2$,

$$\left(1 - e^{-1}\right)|g_{n,m}| \leqslant \sum_{k=m}^{n} \frac{1}{(k+1)!(n-k)!} \leqslant \sum_{k=0}^{n+1} \frac{1}{k!(n+1-k)!} = \frac{2^{n+1}}{(n+1)!}.$$

Therefore, for any fixed $m \geqslant 2$, the series $\sum_{n=2}^{\infty} |g_{n,m}|$ converges. To write $q_{n,m}$ explicitly as a convolution for fixed $m \geqslant 2$, we use the generating function $\sum_{n=m}^{\infty} q_{n,m} x^n$ which converges for $|x| < 1$ since $q_{n,m} \leqslant 1$. From (3.11), we have

$$\begin{aligned}
\sum_{n=m}^{\infty} q_{n,m} x^n &= \frac{1}{e-1} \sum_{n=m}^{\infty} x^n \sum_{k=m}^{n-1} \frac{q_{k,m}}{(n-k)!} + \sum_{n=m}^{\infty} g_{n,m} x^n \\
&= \frac{1}{e-1} \sum_{k=m}^{\infty} \sum_{n=k+1}^{\infty} \frac{q_{k,m}}{(n-k)!} x^n + \sum_{n=m}^{\infty} g_{n,m} x^n \\
&= \frac{e^x - 1}{e-1} \sum_{k=m}^{\infty} q_{k,m} x^k + \sum_{n=m}^{\infty} g_{n,m} x^n.
\end{aligned}$$

Solving for the generating function $\sum_{n=m}^{\infty} q_{n,m} x^n$ and then expanding it similar to the arguments used for $l_n$, we obtain

$$\sum_{n=m}^{\infty} q_{n,m} x^n = \frac{1 - e^{-1}}{1 - e^{x-1}} \sum_{n=m}^{\infty} g_{n,m} x^n = \left(1 - e^{-1}\right) \sum_{n=m}^{\infty} \sum_{i=m}^{n} g_{i,m} h_{n-i} x^n,$$

where $h_i$ is defined in (3.6). Thus, we obtain for $n \geqslant m \geqslant 2$,

$$q_{n,m} = \left(1 - e^{-1}\right) \sum_{i=m}^{n} g_{i,m} h_{n-i}.$$

Therefore, similar to the argument for $l_n$, $q_{n,m}$ has a limit as $n \to \infty$ for each $m \geqslant 2$. □

## 4. Remarks and related problems

Our argument of showing the existence of the limit can also provide an estimate of order $n^{-1/2}$ on the speed of convergence. We only consider the speed of $l_n = \mathbb{E}(L_n)$ to its limit $l$ here since the other cases are similar.

**Proposition 4.1.** $|l_n - l| = O(n^{-1/2})$.

**Proof.** By taking $m \to \infty$ in the estimates (3.1) and using the basic facts (3.4), (3.8) and (3.9), we have

$$\begin{aligned}
|l_n - l| &= \left| \sum_{i=1}^{n} g_i h_{n-i} - l \right| \\
&\leqslant \sum_{i=1}^{n-N} |g_i||h_{n-i} - 1| + \sum_{i=n-N+1}^{n} |g_i||h_{n-i} - 1| + M \sum_{i=n+1}^{\infty} |g_i| \\
&\leqslant \max_{N \leqslant k \leqslant n-1} |h_k - 1| \cdot \sum_{i=1}^{n-N} |g_i| + e^{-1} \sum_{i=n-N+1}^{n} |g_i| + \left(1 + e^{-1}\right) \sum_{i=n+1}^{\infty} |g_i| \\
&\leqslant (2N\pi)^{-1/2} \cdot \sum_{i=1}^{\infty} \frac{2^i}{i!} + \left(1 + 2e^{-1}\right) \sum_{i=n-N+1}^{\infty} \frac{2^i}{i!} \\
&\leqslant \left(e^2 - 1\right)(2N\pi)^{-1/2} + \left(1 + 2e^{-1}\right) \frac{2^{n-N}}{(n-N)!}
\end{aligned}$$

where $M = \max_{i \geqslant 0} h_i \leqslant 1 + e^{-1}$. Taking $N = n - \log n$, we see that

$$|l_n - l| = O\left(n^{-1/2}\right). \qquad \square$$

Furthermore, from the computation for $l_n$, see also Table 1 for $q_{n,m}$, we conjecture that the convergence is at geometric rate.

Next we mention a martingale associated with the problem. Let $R_k$ be the number of men after the $k$th round. Then $R_0 = n$ and $\mathbb{E}(R_{k+1} \mid R_k) = R_k - 1$. Thus

$$M_k(n) = R_k - (n - k)$$

is a martingale. Let

$$\tau_n = \inf\{k\colon\ R_k = 0\}$$

be the number of rounds played. Then optimal stopping theorem implies $\mathbb{E}(M_\tau) = \mathbb{E}(M_0) = n$, i.e., $\mathbb{E}(\tau_n) = n$. Note that the martingale $M_k(n)$ is in triangular arrays. It is nature to consider a functional central limit theorem and the techniques used in Li and Pritchard [8] are helpful.

Table 1
Table of $q_{n,m}$

| $n$ | $m = 2$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | | | | | | |
| 3 | 0.75 | 0.25 | | | | | |
| 4 | 0.8 | 0.133333 | 0.0666667 | | | | |
| 5 | 0.802632 | 0.144737 | 0.0394737 | 0.0131579 | | | |
| 6 | 0.801966 | 0.145518 | 0.0426836 | 0.00763447 | 0.0021978 | | |
| 7 | 0.801877 | 0.145377 | 0.0428913 | 0.00826109 | 0.00127964 | 0.000313873 | |
| 8 | 0.801888 | 0.145353 | 0.0428509 | 0.00830217 | 0.00138452 | 0.000182657 | 3.92357e−005 |
| 9 | 0.801890 | 0.145355 | 0.0428442 | 0.00829432 | 0.00139138 | 0.00019763 | 2.28344e−005 |
| 10 | 0.801890 | 0.145356 | 0.0428448 | 0.00829301 | 0.00139006 | 0.00019861 | 2.47063e−005 |
| 11 | 0.801890 | 0.145356 | 0.0428450 | 0.00829313 | 0.00138985 | 0.000198422 | 2.48288e−005 |
| 12 | 0.801890 | 0.145356 | 0.0428450 | 0.00829316 | 0.00138987 | 0.000198391 | 2.48053e−005 |
| 13 | 0.801890 | 0.145356 | 0.0428450 | 0.00829316 | 0.00138987 | 0.000198394 | 2.48014e−005 |
| 14 | 0.801890 | 0.145356 | 0.0428450 | 0.00829316 | 0.00138987 | 0.000198395 | 2.48017e−005 |
| 15 | 0.801890 | 0.145356 | 0.0428450 | 0.00829316 | 0.00138987 | 0.000198395 | 2.48018e−005 |

It would be also useful to find other martingales associated with the problem. Other problems along this line are central limit theorems for $\tau_n$, upper and lower tail probabilities for $\tau_n$, etc.

There are also many other generalizations of the one round matching, see Feller [3, p. 102] for several multi-round matchings. All of them can be generalized to the last round problem also and our convolution method seems apply to these problems.

Finally, we mention a sort of complementary problem, the so called *Christmas gift problem*. Consider the Christmas party game where each person from 1 to $n$ brings a gift. Slips with numbers $1, 2, \ldots, n$ are placed in a hat. Everyone randomly selects a slip at the same time and then receives the gift brought by the person corresponding to the number. However, if someone draws her/his own number, she/he puts the slip back into the hat, and redraw with those remain. Also, suppose that this process continues until each individual has a gift from someone else or there is only one slip in the hat. What is the probability $Q_n$ that there is only one slip remains in the hat? Let $E_n$ be the event that there is only one slip in the hat (the case that the game cannot finish). Then we have the following recursive formula:

$$Q_n = \mathbb{P}(E_n) = \sum_{k=0}^{n} \mathbb{P}(E_n \mid X_n = k) p_{n,k} = \sum_{k=1}^{n} Q_k p_{n,k} \tag{4.1}$$

with $Q_1 = 1$, $Q_2 = 0$, $Q_3 = 3/5$, $Q_4 = 6/23$. It is easy to see the limit

$$Q := \lim_{n\to\infty} Q_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{Q_k}{k!}\left(e^{-1} - a_{n-k}\right) = e^{-1} \sum_{k=1}^{\infty} \frac{Q_k}{k!},$$

and the convergence rate

$$|Q_n - Q| \leqslant \frac{2^{n+1}+1}{(n+1)!}$$

since

$$\left|\sum_{k=1}^{n} \frac{Q_k a_{n-k}}{k!}\right| \leqslant \sum_{k=1}^{n} \frac{1}{(n-k+1)!k!} \leqslant \frac{2^{n+1}}{(n+1)!}.$$

Note also that $Q_n$ in (4.1) satisfies the generating functions relation

$$\sum_{n=1}^{\infty} Q_n x^n = (1-x)^{-1} e^{-x} \sum_{n=1}^{\infty} \frac{Q_n}{n!} x^n.$$

## Acknowledgments

The authors sincerely thank the referees for their valuable suggestions and comments.

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