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RECENT DEVELOPMENTS ON LOWER TAIL PROBABILITIES FOR GAUSSIAN PROCESSES

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This paper surveys briefly some recent developments on lower tail probabilities for real valued Gaussian processes. Connections and applications to various problems are discussed. A new and simplified argument is given and it is of independent interest.

Keywords: Lower tail probability; Gaussian processes.

1. Introduction

Let $X = (X_t)_{t\in S}$ be a real valued Gaussian process indexed by S, that is, each finite linear combination $\sum_i \alpha_i X_{t_i}$, $t_i \in S$, is a real valued Gaussian (normally distributed) random variable. Assume that it is separable and has mean zero. The distribution of the Gaussian process X is therefore completely determined by its covariance function $\mathbb{E}(X_s X_t)$, $s, t \in S$. The lower tail probability for the Gaussian process studies the behaviour of

$$\mathbb{P}\left(\sup_{t\in S}(X_t - X_{t_0}) \le x\right) \quad \text{as } x \to 0 \tag{1.1}$$

with $t_0 \in S$ fixed, while the small ball probability (or small deviation) and the large deviation for the Gaussian process study the behaviour of

$$\mathbb{P}\left(\sup_{t\in S}|X_t - X_{t_0}| \le x\right) \quad \text{as } x \to 0$$

and

$$\mathbb{P}\left(\sup_{t\in S}(X_t - X_{t_0}) \geq \lambda\right) \quad \text{as } \lambda \to \infty$$

respectively. It is well known that large deviation results play a fundamental role in studying the upper limits of Gaussian processes, such as the Strassen type of law of the iterated logarithm. The theory of large deviation has been well developed, see, for example, Leadbetter, Lindgren and Rootzen (1983), Ledoux and Talagrand (1991), Ledouux (1996) and Bogachev (1998) for Gaussian processes, Varadhan (1984) and Dembo and Zeitouni (1998) for the general theory of large deviations. The small ball probability is a key step in studying the lower limits of the Gaussian processes. It has been found that the small ball estimate has close connections with various approximation quantities of compact sets and operators. We refer to the survey of Li and Shao (2001a) for recent developments and various applications in this direction.

There are various motivations for the study of lower tail probability other than its own importance. The study is related to the following different problems among many others: (i) the most visited sites of symmetric stable processes (Bass, Eisenbaum and Shi, 2000); (ii) Brownian pursuit problem (Bramson and Griffeath, 1991; Kesten, 1992); (iii) random polynomials (Dembo, Poonen, Shao and Zeitouni, 2001); (iv) the first passage time for the Slepian process (Slepian, 1961; Shepp, 1971); (v) the first exit time of integrated Brownian motion (McKean, 1963; Groeneboom, Jongbloed and Wellner, 1999); and (vi) the zero-crossings of Gaussian noise (Wong, 1966; 1970).

The main aim of this paper is to review recent results on the lower tail probabilities for Gaussian processes. Connections and applications to various problems are also discussed. A new and simplified argument is given in Theorem 3.1 and it is of independent interest. It should be emphasized that the lower tail probability in (1.1) is closely related to the one-sided (fixed) level crossing probability

$$\mathbb{P}\left(\sup_{t\in\lambda S}(X_t - X_{t_0}) \leq a\right) \quad \text{as } \lambda \to \infty \tag{1.2}$$

which can sometimes be viewed as the upper tail behavior

$$\mathbb{P}(\tau > \lambda) \quad \text{as } \lambda \to \infty \tag{1.3}$$

for the first hitting/passage/exit/coupling time τ determined by S. In fact, under appropriate scaling condition on X_t and S, (1.1) and (1.2) are equivalent (see Secs. 3 and 4 for examples). This is why useful techniques for (1.1) also work well for (1.2) and vice versa. When X_t is a Markov process, there are systemetic studies for (1.3) and many analytic tools are available. When X_t is Gaussian, there are very few general methods. Slepian's lemma (inequality) is one of the most powerful tools and several other methods are developed very recently in Li and Shao (2001b,c) (see Secs. 2 and 3 for details). We hope that readers can use recent results summarized here in their own works and contribute to this exciting area of research. There is also a need for a systematic study of various techniques and applications which are spread over different areas.

Finally, we mention that there are only a handful of different Gaussian processes for which the precise distribution of the supremum over finite intervals is known. In all these ten or fewer cases, the derivation of the distribution of the supremum is based *not* on general Gaussian techniques but either on the simplistic nature of the process or on Markov methods when it is also a Markov process. On the other hand, these cases can be used to obtain bounds for related processes via Slepian's lemma and/or the comparison inequality at the end of Sec. 3.

Throughout this paper, $\log(x) = \ln(\max(x, e))$ for $x \in \mathbb{R}^1$ and ln is the natural logarithm.

2. Lower Tail Probabilities for General Gaussian Processes

Let $X = (X_t)_{t\in S}$ be a real valued Gaussian random process indexed by S with mean zero. Define an L^2-metric induced by the process X as

$$d(s,t) = (\mathbb{E}\,|X_s - X_t|^2)^{1/2}, \quad s,t \in S.$$

For every $\varepsilon > 0$ and a subset A of S, let $N(A,\varepsilon)$ denote the minimal number of open balls of radius ε for the metric d that are necessary to cover A. For $t \in S$ and $h > 0$, let $B(t,h) = \{s \in S\colon d(t,s) \le h\}$, and define a locally and uniformly Dudley type entropy (LUDE) integral

$$Q = \sup_{h>0}\sup_{t\in S}\int_0^\infty (\log N(B(t,h),\varepsilon h))^{1/2}\,d\varepsilon.$$

Assume $Q < \infty$ and let $D = \sup_{s,t\in S} d(t,s)$ be the diameter of the set S. For $\theta = 1000(1+Q)$, define

$$\mathcal{A}_{-1} = \{t \in S\colon d(t,t_0) \le \theta^{-1}x\},$$
$$\mathcal{A}_k = \{t \in S\colon \theta^{k-1}x < d(t,t_0) \le \theta^k x\}, \quad k = 0,1,2,\ldots,L,$$

where $L = 1+[\log_\theta(D/x)]$. Let $N_k(x) := N(\mathcal{A}_k,\, \theta^{k-2}x)$ denote the minimal number of open balls of radius $\theta^{k-2}x$ for the metric d that are necessary to cover $\mathcal{A}_k$, $k = 0,1,\ldots,L$, and let

$$N(x) = 1 + \sum_{0\le k\le L} N_k(x).$$

The results in this section are attributed to Li and Shao (2001b). We have the following general lower bound.

Theorem 2.1. *Assume that* $Q < \infty$ *and*

$$\mathbb{E}\,((X_s - X_{t_0})(X_t - X_{t_0})) \ge 0 \quad \textit{for } s,t \in S \tag{2.1}$$

Then we have

$$\mathbb{P}\left(\sup_{t\in S}(X_t - X_{t_0}) \le x\right) \ge e^{-N(x)}. \tag{2.2}$$

The upper bound can be obtained under a different set of conditions.

Theorem 2.2. *For $x > 0$, let $s_i \in S$, $i = 1, \ldots, M$ be a sequence such that for every i*

$$\sum_{j=1}^{M} |\mathrm{Corr}(X_{s_i} - X_{t_0}, X_{s_j} - X_{t_0})| \le 5/4 \tag{2.3}$$

and

$$d(s_i, t_0) = (\mathbb{E}\,|X_{s_i} - X_{t_0}|^2)^{1/2} \ge x/2. \tag{2.4}$$

Then

$$\mathbb{P}\left(\sup_{t\in S}(X_t - X_{t_0}) \le x\right) \le e^{-M/10}. \tag{2.5}$$

To match the lower bound given in Theorem 2.1, one can select the sequence $\{s_i\}$ in Theorem 2.2 as follows. Let $q > 1$. For $k = 1, 2, \ldots, L-1$, choose $s_{k,j}, j = 1, \ldots, M_k$ such that

$$(1/2)q^k x \le d(s_{k,j}, t_0) \le q^k x.$$

Hopefully, when q is large, $\{s_{k,j}, 1 \le j \le M_k, 1 \le k < L\}$ satisfies (2.3).

The bounds provided by Theorems 2.1 and 2.2 are sharp under certain regular conditions.

Theorem 2.3. *Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and stationary increments, that is*

$$\forall\, t, s \in [0,1]^d, \quad \sigma^2(|t-s|) = \mathbb{E}\,(X_t - X_s)^2$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. If there are $0 < \alpha \le \beta < 1$ such that $\sigma(h)/h^\alpha$ is non-decreasing and $\sigma(h)/h^\beta$ non-increasing. Then there exist $0 < c_1 \le c_2 < \infty$ depending only on α, β and d such that for $0 < x < 1/2$

$$\begin{aligned}\exp(-c_2 \log(1/x)) &\le \mathbb{P}\Big(\sup_{t\in[0,1]^d} X(t) \le \sigma(x)\Big) \\ &\le \exp(-c_1 \log(1/x)).\end{aligned} \tag{2.6}$$

Theorem 2.4. *Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and*

$$\mathbb{E}\,(X_t X_s) = \prod_{i=1}^{d} \frac{1}{2}(\sigma^2(t_i) + \sigma^2(s_i) - \sigma^2(|t_i - s_i|)) \tag{2.7}$$

for $t = (t_1, \ldots, t_d)$ and $s = (s_1, \ldots, s_d)$. If there are $0 < \alpha \le \beta < 1$ such that $\sigma(h)/h^\alpha$ is non-decreasing and $\sigma(h)/h^\beta$ non-increasing. Then there exists

$0 < c_3 \le c_4 < \infty$ *depending only on* α, β *and* d *such that for* $0 < x < 1/2$

$$\exp(-c_4 \log^d(1/x)) \le \mathbb{P}\Big(\sup_{t\in[0,1]^d} X(t) \le \sigma^d(x)\Big) \le \exp(-c_3 \log^d(1/x)). \tag{2.8}$$

In particular, for the fractional Levy Brownian motion and the fraction Brownian sheet, one has the following

- Let $\{L_\alpha(t),\ t \in [0,1]^d\}$ be the fractional Levy's Brownian motion of order α, $0 < \alpha < 2$, i.e. $L_\alpha(0) = 0$, $\mathbb{E}\, L_\alpha(t) = 0$ and $\mathbb{E}\,(L_\alpha(t) - L_\alpha(s))^2 = |t-s|^\alpha$, $0 < \alpha < 2$,

$$\exp(-c_2 \log(1/x)) \le \mathbb{P}\Big(\sup_{t\in[0,1]^d} L_\gamma(t) \le x\Big) \le \exp(-c_1 \log(1/x)).$$

- Let $\{B_\alpha(t),\ t \in [0,1]^d\}$ be the fractional Brownian sheet of order α, $0 < \alpha < 2$, i.e. $B_\alpha(0) = 0$, $\mathbb{E}\, B_\alpha(t) = 0$ and

$$\mathbb{E}\,(B_\alpha(t)B_\alpha(s)) = \prod_{i=1}^d \frac{1}{2}(t_i^\alpha + s_i^\alpha - |t_i - s_i|^\alpha)$$

for $t = (t_1, \dots, t_d)$ and $s = (s_1, \dots, s_d)$. Then there exists $0 < c_3 \le c_4 < \infty$ depending only on α and d such that for $0 < x < 1/2$

$$\exp(-c_4 \log^d(1/x)) \le \mathbb{P}\Big(\sup_{t\in[0,1]^d} B_\alpha(t) \le x\Big) \le \exp(-c_3 \log^d(1/x)).$$

For the lower tial probability for the two-dimensional Brownian sheet $\{W(t), t \in [0,1]^2\}$ or B_1 above, Csáki, Khoshnevvisan and Shi (2000) proved that

$$\exp(-c_5 \log^2(1/x)) \le \mathbb{P}\Big(\sup_{t\in[0,1]^2} W(t) \le x\Big) \le \exp\left(-\frac{c_6 \log^2(1/x))}{\log\log(1/x)}\right).$$

For the one-dimensional fractional Brownian motion of order α, $0 < \alpha < 2$, Molchan (1999) obtained an exact rate of the lower tail probability

$$\mathbb{P}\left(\sup_{0\le t\le 1} B_\alpha(t) \le x\right) = x^{(2-\alpha)/\alpha + o(1)} \tag{2.9}$$

as $x \to 0$.

It should be pointed out that the lower probability is very different from the small ball probability under the sup-norm, which considers the absolute value of the supremum of a Gaussian process. In particular, the small ball problem for

Brownian sheet $W(t)$ on $\mathbb{R}^d$ under the sup-norm is still open for $d \geq 3$. The best known results are

$$\ln \mathbb{P}\Big(\sup_{\mathbf{t}\in[0,1]^2} |W(\mathbf{t})| \leq x \Big) \approx -x^{-2} \log^3(1/x)$$

and

$$-c_2 x^{-2} \log^{2d-1}(1/x) \leq \ln \mathbb{P}\Big(\sup_{\mathbf{t}\in[0,1]^d} |W(\mathbf{t})| \leq x \Big) \leq -c_1 x^{-2} \log^{2d-2}(1/x).$$

for $d \geq 3$, as $x \to 0$. We refer to a recent survey paper of Li and Shao (2001a) for more information on the small ball probability and its applications.

3. Lower Tail Probabilities for Stationary Gaussian Processes

Let $\{W(t), t \geq 0\}$ be the Brownian motion and $\{U(t), t \geq 0\}$ be the Ornstein–Uhlenbeck process. It is known that $\{U(t), t \geq 0\}$ and $\{W(e^t)/e^{t/2}, t \geq 0\}$ have the same distribution. Moreover

$$\mathbb{P}\left(\sup_{0\leq t\leq 1} W(t) \leq x \right) = \mathbb{P}\left(|W(1)| \leq x\right) \sim (2/\pi)^{1/2} x$$

as $x \to 0$ and

$$\mathbb{P}\left(\sup_{0\leq t\leq T} U(t) \leq 0 \right) = \exp(-T/2 + o(T))$$

as $T \to \infty$. It would be interesting to find the connection between these two types of lower tail probabilities. To this end, we first prove a general result for stationary Gaussian processes.

Theorem 3.1. *Let $\{Y_t, t \geq 0\}$ be an almost surely continuous stationary Gaussian process with $\mathbb{E}\, Y_t = 0$ and $\mathbb{E}\, Y_t^2 = 1$ for $t \geq 0$. Put $\rho(t) = \mathbb{E}\, Y_0 Y_t$. Assume that $\rho(t) \geq 0$. We have*

(i) *The limit*

$$p(x) := \lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\left(\sup_{0\leq t\leq T} Y_t \leq x \right) \tag{3.1}$$

exists and

$$p(x) = \sup_{T>0} T^{-1} \ln \mathbb{P}\left(\sup_{0\leq t\leq T} Y_t \leq x \right)$$

for every $x \in \mathbb{R}^1$.

(ii) *If* $\rho(t)$ *is decreasing and*

$$a_{h,\theta}^2 := \inf_{0<t\le h} \frac{\rho(\theta t)-\rho(t)}{1-\rho(t)} > 0, \tag{3.2}$$

then

$$\mathbb{P}\Big(\sup_{0\le t\le nh} Y_t \le x+y\Big) \le \Phi^{-n}\left(\frac{-y+x((1+a_{h,\theta}^2)^{1/2}-1)}{a_{h,\theta}}\right)\mathbb{P}\Big(\sup_{0\le t\le \theta nh} Y_t \le x\Big) \tag{3.3}$$

for $x \in \mathbb{R}^1$, $y>0$, $0<\theta<1$ *and* $n \ge 1$, *where* Φ *is the standard normal distribution function.*

(iii) *If* $\rho(t)$ *is decreasing and* $a_{h,\theta} > 0$ *for every* $0<h<\infty$ *and* $0<\theta<1$, *then* $p(x)$ *is continuous.*

Proof. The existence of the limit $p(x)$ is ensured by sub-additivity as proven in Li and Shao (2001b). Parts (ii) and (iii) are new. We only need to prove (ii) since (iii) is a direct consequence of (ii). To prove (ii), let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal random variables independent of $\{Y_t, t\ge 0\}$ and write $a = a_{h,\theta}$. Observe that

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1\le i\le n}\sup_{(i-1)h\le t\le ih}\frac{Y_t+aZ_i}{\sqrt{1+a^2}} \le x\Big)\\
&\ge \mathbb{P}\Big(\max_{1\le i\le n}\sup_{(i-1)h\le t\le ih} Y_t \le x+y,\ \max_{1\le i\le n} aZ_i \le -y+x(1+a^2)^{1/2}-x\Big)\\
&= \mathbb{P}\Big(\max_{1\le i\le n}\sup_{(i-1)h\le t\le ih} Y_t \le x+y\Big)\mathbb{P}\Big(\max_{1\le i\le n} aZ_i \le -y+x(1+a^2)^{1/2}-x\Big)\\
&= \mathbb{P}\Big(\max_{1\le i\le n}\sup_{(i-1)h\le t\le ih} Y_t \le x+y\Big)\Phi^n\left(\frac{-y+x(1+a^2)^{1/2}-x}{a}\right).
\end{aligned}$$

Also note that for $(i-1)h \le s,t \le ih$

$$\begin{aligned}
&\mathbb{E}\left\{\frac{(Y_s+aZ_i)(Y_t+aZ_i)}{1+a^2}\right\} - \mathbb{E}\,Y_{\theta s}Y_{\theta t}\\
&= \frac{\rho(|t-s|)+a^2}{1+a^2} - \rho(\theta|t-s|)\\
&= \frac{1}{1+a^2}\big(a^2(1-\rho(\theta|t-s|)) - (\rho(\theta|t-s|)-\rho(|t-s|))\big)\\
&\le \frac{1}{1+a^2}\big(a^2(1-\rho(|t-s|)) - (\rho(\theta|t-s|)-\rho(|t-s|))\big)\\
&= \frac{1-\rho(|t-s|)}{1+a^2}\left(a^2 - \frac{\rho(\theta|t-s|)-\rho(|t-s|)}{1-\rho(|t-s|)}\right)\\
&\le 0
\end{aligned}$$

by the definition of a. Similarly, for $(i-1)h \le s \le ih$, $(j-1)h < t < jh$ and $i \ne j$

$$\mathbb{E}\left\{\frac{(Y_s + aZ_i)(Y_t + aZ_i)}{1+a^2}\right\} = \frac{\rho(|t-s|)}{1+a^2} \le \rho(\theta|t-s|) = \mathbb{E}\, Y_{\theta s} Y_{\theta t}.$$

Therefore, by the Slepian lemma

$$\mathbb{P}\left(\max_{1\le i\le n} \sup_{(i-1)h\le t\le ih} \frac{Y_t + aZ_i}{\sqrt{1+a^2}} \le x\right) \le \mathbb{P}\left(\max_{1\le i\le n} \sup_{(i-1)h\le t\le ih} Y_{\theta t} \le x\right)$$
$$= \mathbb{P}\left(\sup_{0\le t\le \theta nh} Y_t \le x\right).$$

This proves (3.3) by the above inequalities. □

Remark 3.1. It is easy to see that condition (3.2) is satisfied if

$$\rho(t) = 1 - ct^\alpha + o(t^\alpha)$$

as $t \to 0$ for some $c > 0$ and $\alpha > 0$.

To state the connection between lower tail probabilities of a non-stationary Gaussian process and its dual stationary Gaussian process, let $\{X_t, t \ge 0\}$ be a Gaussian process with $X_0 = 0$, $\mathbb{E}\, X_t = 0$. Assume that:

(A1) $\mathbb{E}\, X_s X_t \ge 0$ and $\mathbb{E}\, X_t^2 = t^\alpha$ for $\alpha > 0$;
(A2) $\{Y_t = X(e^t)/e^{\alpha/2}, t \ge 0\}$ is a stationary Gaussian process;
(A3) $\{X_{at}, 0 \le t \le 1\}$ and $\{a^{\alpha/2} X_t, 0 \le t \le 1\}$ have the same distribution for each fixed $a > 0$.
(A4) $\rho(t) := \mathbb{E}\, Y_t Y_0$ is decreasing and condition (3.2) holds.

By Theorem 3.1

$$c_\alpha := -\lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\left(\sup_{0\le t\le T} Y_t \le 0\right)$$
$$= -\sup_{T>0} \frac{1}{T} \ln \mathbb{P}\left(\sup_{0\le t\le T} Y_t \le 0\right)$$

exists. The next result shows that the constant c_α is closely related to the rate of the lower tail probability $\mathbb{P}\left(\sup_{0\le t\le 1} X_t \le x\right)$.

Theorem 3.2. *Under conditions* (A1)–(A4), *we have*

$$\mathbb{P}\left(\sup_{0\le t\le 1} X_t \le x\right) = x^{2c_\alpha/\alpha + o(1)}$$

as $x \to 0$.

The above result was proved in Li and Shao (2001b) when X is the fractional Brownian motion of order α by using the Slepian lemma, the scaling property of X, and the continuity of $p(x)$ proved in Theorem 3.1. And the proof is quite complicated when $0 < \alpha < 1$.

It is easy to see that conditions (A1)–(A4) are satisfied for the fractional Brownian motion B_α of order α. Therefore, by (2.9)

$$\mathbb{P}\left(\sup_{0\le t\le T} B_\alpha(e^t)/e^{t\alpha/2} \le 0\right) = \exp(-T(1-\alpha/2)+o(T))$$

as $T\to\infty$.

Another class of Gaussian processes satisfying (A1)–(A4) are $\{X_\alpha(t), t\ge 0\}$ with the covariance function

$$\mathbb{E}\, X_\alpha(s)X_\alpha(t) = \frac{2^\alpha (st)^{(1+\alpha)/2}}{(s+t)^\alpha},$$

where $\alpha>0$. This family of Gaussian processes seems new and has Brownian like properties

(1) $\mathbb{E}\, X_\alpha^2(t)=t$;
(2) $\{X_\alpha(at), t\ge 0\}$ and $\{\sqrt{a}X_\alpha(t), t\ge 0\}$ have the same distribution for fixed $a>0$.

Their detailed properties are studied in Li and shao (2001e).

When $\alpha=1$, the process is also related to the probability that a random polynomial has no real zero. To be more precise, let $\{Z_i, i\ge 0\}$ be independent standard normal random variables. In their study on the probability that the random polynomial $\sum_{i=0}^n Z_i x^i$ does not have real root in $\mathbb{R}$, Dembo, Poonen, Shao and Zeitouni (2001) obtain

$$\mathbb{P}\left(\sum_{i=0}^n Z_i x^i \le 0,\ \forall\, x\in\mathbb{R}^1\right) = n^{-b+o(1)}$$

as $n\to\infty$ through even integers, where

$$b = -4\lim_{T\to\infty}\frac{1}{T}\ln\mathbb{P}\left(\sup_{0\le t\le T} Y_1(t)\le 0\right)$$

and $Y_1(t)=X_1(e^t)/e^{t/2}$ is a centered stationary Gaussian process with

$$\mathbb{E}\, Y_1(s)Y_1(t) = \frac{2e^{-|t-s|/2}}{1+e^{-|t-s|}} \quad \text{for } s,t\ge 0.$$

It is proved in Li and Shao (2001c,e) that $0.5<b<1$, but the exact value of b is still unknown. A different limiting representation for the decay exponent b is given in Li and Shao (2001e).

At the end of this section, it is worthy to mention a new Gaussian comparison inequality proven in Li and Shao (2001c).

Let $\{\xi_i, 1\le i\le n\}$ and $\{\eta_i, 1\le i\le n\}$ be two normal random vectors with mean zero and variance one. Assume that

$$\mathbb{E}\,\xi_i\xi_j \ge E\eta_i\eta_j \ge 0$$

for $1 \le i, j \le n$. Then for $x \ge 0$

$$\mathbb{P}\Big(\max_{1\le i\le n} \xi_i \le x\Big) \le \mathbb{P}\Big(\max_{1\le i\le n} \eta_i \le x\Big) \prod_{1\le i<j\le n} \left(\frac{\pi - 2\arcsin(E\eta_i\eta_j)}{\pi - 2\arcsin(E\xi_i\xi_j)}\right)^{e^{-x^2/(1+E\xi_i\xi_j)}}.$$

4. An Application to the Capture Time of the Fractional Brownian Motion Pursuit

Let $\{B_{k,\alpha}(t);\ t \ge 0\}(k = 0, 1, 2, \ldots, n)$ be independent fractional Brownian motions of order $\alpha \in (0, 2)$ and set

$$\tau_{n,\alpha} = \inf\Big\{t > 0\colon \max_{1\le k\le n} B_{k,\alpha}(t) = B_{0,\alpha}(t) + 1\Big\}.$$

The stopping time $\tau_{n,\alpha}$ can be viewed as the capture time in the random pursuit problem for the fractional Brownian particles; see Bramson and Griffeath (1991), Kesten (1992) and Li and Shao (2001d) for more details. A natural question is: when is $\mathbb{E}(\tau_{n,\alpha})$ finite? The question is the same as estimating the lower tail probability of $\max_{1\le k\le n}\sup_{0\le t\le 1}(B_{k,\alpha}(t) - B_{0,\alpha}(t))$. In fact, for any $s > 0$, by the fractional Brownian scaling,

$$\begin{aligned}\mathbb{P}(\tau_{n,\alpha} > s) &= \mathbb{P}\left(\max_{1\le k\le n} \sup_{0\le t\le s} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < 1\right)\\ &= \mathbb{P}\left(\max_{1\le k\le n} \sup_{0\le t\le 1} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < s^{-\alpha/2}\right).\end{aligned}$$

Li and Shao (2001b) show that

$$\mathbb{P}\left(\max_{1\le k\le n} \sup_{0\le t\le 1} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < x\right) = x^{2\gamma_{n,\alpha}/\alpha + o(1)}$$

as $x \to 0$, where

$$\gamma_{n,\alpha} := -\lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\le t\le T} \max_{1\le k\le n} (Y_{k,\alpha}(t) - Y_{0,\alpha}(t)) \le 0\Big) \tag{4.4}$$

and $Y_{k,\alpha}(t) = e^{-t\alpha/2}B_{k,\alpha}(e^t)$, $k = 0, 1, \ldots, n$, are the fractional Ornstein–Uhlenbeck process of order α. In other words,

$$\mathbb{P}\big(\tau_{n,\alpha} > t\big) = t^{-\gamma_{n,\alpha} + o(1)}$$

as $t \to \infty$ for fixed n.

In the Brownian motion case, $\alpha = 1$, with $\tau_n = \tau_{n,1}$, $\mathbb{E}\,\tau_5 < \infty$ and $\mathbb{E}\,\tau_3 = \infty$ are proved in Li and Shao (2001d) by using some distribution identities and the Faber–Krahn isoperimetric inequality. It is still a conjecture due to Bramson and Griffeath (1991) that $\mathbb{E}\,\tau_4 < \infty$. Their simulation suggested that $\gamma_4 \approx 1.032$. Also for the Brownian motion case, Kesten (1992) proved that $\gamma_n = \gamma_{n,1}$ is of order $\ln n$ when n is large. More precisely, Kesten showed that

$$0 < \liminf_{n\to\infty} \gamma_n/\ln n \le \limsup_{n\to\infty} \gamma_n/\ln n \le 1/4$$

and conjectured the existence of $\lim_{n\to\infty} \gamma_n / \ln n$. Li and Shao (2001c) give an affirmative answer to Kesten's conjecture and obtain

$$\frac{1}{d_\alpha} \le \liminf_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} \le \limsup_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} < \infty$$

where $d_\alpha = 2\int_0^\infty (e^{x\alpha} + e^{-x\alpha} - (e^x - e^{-x})^\alpha)\,dx$. Furthermore, for $\gamma_n = \gamma_{n,1}$, we proved that

$$\lim_{n\to\infty} \frac{\gamma_n}{\ln n} = \frac{1}{4}.$$

We also conjecture that indeed

$$\lim_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} = \frac{1}{d_\alpha}.$$

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