Journal of Theoretical Probability, Vol. 18, No. 2, April 2005 (© 2005)
DOI: 10.1007/s10959-003-2604-9

A Functional LIL for Stochastic Integrals and the Lévy Area Process

James Kuelbs[1,*] **and Wenbo Li**[2]

Received September 2, 2003; revised December 15, 2003

A functional law of the iterated logarithm is obtained for processes given by certain stochastic integrals. This extends earlier results by Shi(12) and Rémillard(10) who established analogues of the classical limit results of Chung(4) for a variety of processes, including Lévy's stochastic area process. The functional aspects of our results are motivated by a paper of Wichura(13) on Brownian motion. Proofs depend on small ball probability estimates, and yield the small ball probabilities of the weighted sup-norm for the processes given by these stochastic integrals.

KEY WORDS: Functional LIL; Lévy's area process; small ball probabilities.

1. INTRODUCTION

If $\{X(t): t \geqslant 0\}$ is a symmetric stable process of index $\alpha \in (0, 2]$ with stationary independent increments such that with probability one the sample paths are in $D[0, \infty)$, and $X(0) = 0$, then for $t \geqslant 0, n \geqslant 1$, we define $M(t) = \sup_{0 \leqslant s \leqslant t} |X(s)|$, and

$$\eta_n(t) = M(nt)/(c_\alpha n/LLn)^{1/\alpha},$$

[1] Department of Mathematics, University of Wisconsin, Madison, WI 53706. E-mail: kuelbs@math.wisc.edu
[2] Department of Mathematical Sciences, University of Delaware, New York, DE 19711. E-mail: wli@math.udel.edu
[*] To whom correspondence should be addressed.

where the constant $0 < c_\alpha < \infty$ is given by

$$c_\alpha = -\lim_{\varepsilon \to 0^+} \varepsilon^\alpha \log P(\sup_{0 \leqslant s \leqslant 1} |X(s)| \leqslant \varepsilon)$$

and $LLn = \max(1, \log(\log n))$. Throughout $\mathcal{M}$ denotes the space of functions $f: [0, \infty) \to [0, \infty]$ such that $f(0) = 0$, f is right continuous on $(0, \infty)$, non-decreasing, and $\lim_{t \to +\infty} f(t) = \infty$, and we define

$$K_\alpha = \left\{ f \in \mathcal{M}: \int_0^\infty f^{-\alpha}(t) dt \leqslant 1 \right\}.$$

The topology on $\mathcal{M}$ is the topology of weak convergence, i.e. pointwise convergence at all continuity points of the limit function. This topology is metrizable and separable on $\mathcal{M}$ and if $\{f_n\}$ is a sequence of points in $\mathcal{M}$, then $C(\{f_n\})$ denotes the cluster set of $\{f_n\}$, i.e. all possible subsequential limits of $\{f_n\}$ in the weak topology. If $A \subseteq \mathcal{M}$, we write $\{f_n\} \twoheadrightarrow A$ if $\{f_n\}$ is relatively compact in $\mathcal{M}$ and $C(\{f_n\}) = A$. The functional law of the iterated logarithm obtained in Ref. 3 proves that

$$P(\{\eta_n\} \twoheadrightarrow K_\alpha) = 1.$$

If $\{X(s): s \geqslant 0\}$ is a sample continuous γ-fractional Brownian motion with $X(0) = 0$ and $0 < \gamma < 2$, then a similar result holds, and in[8] we proved

$$P(\{\eta_n\} \twoheadrightarrow K_\gamma) = 1,$$

where

$$\eta_n(t) = M(nt) / (c_\gamma n / LLn)^{\gamma/2},$$

$$K_\gamma = \left\{ f \in \mathcal{M}: \int_0^\infty f^{-2/\gamma}(s) ds \leqslant 1 \right\}$$

and

$$c_\gamma = -\lim_{\varepsilon \to 0^+} \varepsilon^{2/\gamma} \log P(\sup_{0 \leqslant s \leqslant 1} |X(s)| \leqslant \varepsilon).$$

Perhaps $\{M(t): t \geqslant 0\}$ and $\{X(t): t \geqslant 0\}$ should also be indexed by α or γ as appropriate, but we chose not to do so to simplify the notation.

Both of these functional laws of the iterated logarithm depend on suitable scaling and asymptotic independence properties for the process $\{X(t): t \geqslant 0\}$, and are refinements of Chung's LIL for Brownian motion. Furthermore, although quite similar in statement, they have substantially different proofs. There are also remarkably similar weighted occupation measure applications in both settings. These follow from the corresponding functional LIL and the relevant scaling property. Hence it seemed to be of interest to see if there were similar results for other classes of processes, and here we examine the situation when $\{X(t):t\geqslant 0\}$ is given by stochastic integrals of the form

$$X(t)=\int_0^t \langle AW(s),\ dW(s)\rangle,\quad t\geqslant 0, \tag{1.1}$$

where A is a real nonzero skew symmetric d by d matrix and $\{W(t):t\geqslant 0\}$ is standard sample continuous Brownian motion in R^d.

We were first motivated to study these processes by the interesting paper,[10] where Rémillard proved an analogue of Chung's LIL in this setting. That is, in Ref. 10 it was shown that there exists a constant $a(A)\in(0,\infty)$ such that with probability one

$$\varliminf_{t\to\infty}(LLt/t)\sup_{0\leqslant s\leqslant t}\left|\int_0^s\langle AW(u),\ dW(u)\rangle\right|=-a(A).$$

Here we refine this result obtaining a functional version which is an analogue of that for the stable processes and fractional Brownian motions. Our proofs involve methods quite different from those in Rémillard, who exploited the Markov nature of the problem, and first identified $a(A)$ analytically via a variational formula, and eventually in terms of the eigenvalues of A. In particular, we identify the limiting value in our theorems immediately in terms of the eigenvalues of A by using the representation of $\{X(t):t\geqslant 0\}$ determined in Section 3.

Another point of interest, which developed as we studied these stochastic integrals, is their relationship to Lévy's stochastic area process. The Lévy stochastic area process is important in its own right and arises immediately if $\{W(t):t\geqslant 0\}$ is standard Brownian motion in R^2 and A is the 2 by 2 skew symmetric matrix with $-1/2$ above the diagonal of zeros and 1/2 below the diagonal. However, this is only the beginning, and we will see that $\{X(t):t\geqslant 0\}$ as given in (1.1) can always be written as a linear combination of independent processes of this type, and it is this fact that we exploit in part of the proof. Linear combinations of independent Lévy stochastic area processes also arise as limits in a generalization of a functional central limit theorem for certain stochastic integrals, so they are important from that point of view as well. Theorem 2.1 of the paper by S. Janson and M.J. Wichura[7] contains such a result, and it is interesting to point out that the functional LIL in Refs. 3 and 8 for the special case of Brownian motion appeared in an earlier unpublished paper of Wichura.[13] Wichura's proof for the Brownian motion case was also built on diffusion process techniques and is entirely different than the methods in Refs. 3 and 8. It is important to mention that this is not merely a matter of choice, but of necessity at this time. The papers Refs. 3 and 8 also contain further background material and applications. In particular,[3] describes the topology of $\mathcal{M}$ in more detail.

The paper by Z. Shi[12] also obtains Chung's LIL for Lévy's stochastic area process, and here it is proved via methods that are from stochastic analysis. This approach motivates the representation for $\{X(t): t \geqslant 0\}$ obtained in Section 3 and it is this representation which is useful for many of the probability estimates we use throughout the paper. Nevertheless, the defining formulas for $\{X(t): t \geqslant 0\}$ in terms of stochastic integrals are also used in an important way in our proof of Proposition 4 in Section 5. Hence in our approach both representations are exploited.

2. STATEMENT OF RESULTS

Let $\{X(t): t \geqslant 0\}$ be given by stochastic integrals of the form

$$X(t) = \int_0^t \langle AW(s), dW(s)\rangle, \quad t \geqslant 0, \tag{2.1}$$

where A is a real nonzero skew symmetric d by d matrix and $\{W(t): t \geqslant 0\}$ is standard sample continuous Brownian motion in R^d. Let $M(t) = \sup_{0 \leqslant s \leqslant t} |X(s)|$, and assume $\mathcal{M}$ is as in the introduction, and for $t \geqslant 0, n \geqslant 1$, define

$$\eta_n(t) = M(nt)/(c_A n/LLn), \tag{2.2}$$

where the constant $0 < c_A < \infty$ is uniquely determined by A as specified below. Then the following holds.

Theorem 1. Let $\{X(t): t \geqslant 0\}$ be given by (2.1) with A a real nonzero skew symmetric matrix. Then there exist unique strictly positive constants $\alpha_1, \ldots, \alpha_r$, $1 \leqslant r \leqslant d/2$, depending only on A, such that for $c_A = (\pi/2)(\sum_{j=1}^r \alpha_j)$ we have

$$P(\{\eta_n\} \twoheadrightarrow K) = 1, \tag{2.3}$$

where

$$K = \left\{ f \in \mathcal{M}: \int_0^\infty f^{-1}(s)ds \leqslant 1 \right\}. \tag{2.4}$$

Remark. We will see from Section 3 that the non zero eigenvalues of A, repeated with necessary multiplicities, are $\pm\alpha_1 i, \ldots, \pm\alpha_r i$, $1 \leqslant r \leqslant d/2$, and hence $\alpha_1, \ldots, \alpha_r$, are uniquely determined by A and sometimes easily computible.

Two immediate corollaries are the following which apply when $\{X(t): t \geqslant 0\}$ is Levy's area process, or more generally as in (2.1). Corollary 1 is known from Ref. 12 and Corollary 2 from Ref. 10. They are included to motivate the form of the occupation measure results that follow from Theorem 1.

Corollary 1. Let $\{\eta_n\}$ be as in (2.2) with A skew symmetric 2 by 2 matrix with $-1/2$ above the diagonal of zeros and 1/2 below the diagonal. Then

$$X(t) = (1/2) \int_0^t (B_1(s) dB_2(s) - B_2(s) dB_1(s)) \tag{2.5}$$

is Levy's stochastic area process and

$$P(\underline{\lim}_{n\to\infty} \eta_n(1) = \underline{\lim}_{n\to\infty} \sup_{0\leqslant t\leqslant 1} |X(nt)| LLn/(c_A n) = 1) = 1, \tag{2.6}$$

where $c_A = \pi/4$.

Remark. $c_A = \pi/4$ is consistent with the result from Ref. 12 as there the stochastic area process is twice ours.

Corollary 2. Let $\{X(t): t \geqslant 0\}$ be given by (2.1) with A a real non-zero d by d skew symmetric matrix and $\{W(t): t \geqslant 0\}$ a standard sample continuous Brownian motion in R^d. Then there exist unique strictly positive constants $\alpha_1, \ldots, \alpha_r$, $1 \leqslant r \leqslant d/2$, depending only on A, such that for $c_A = (\pi/2)(\sum_{j=1}^r \alpha_j)$ we have

$$P(\underline{\lim}_{n\to\infty} \eta_n(1) = \underline{\lim}_{n\to\infty} \sup_{0\leqslant t\leqslant 1} |X(nt)| LLn/(c_A n) = 1)) = 1. \tag{2.7}$$

Occupation measure results similar to those in Refs. 3 and 8 can also be obtained for the maximal process $\{M(t): t \geqslant 0\}$ derived from the X process when X is given as in (2.1). These applications are very much in the spirit of those for Strassen's functional law of the iterated logarithm, but the details are quite different. Furthermore, it should be observed that these two functional limit theorems, even for Brownian motion, involve very different aspects of the process. What is so surprising is that the maximal process for the stable processes of Ref. 3, the fractional Brownian motions of Ref. 8, and the processes given by (2.1) have these very similar occupation measure results.

To motivate these results, note that Corollary 2 implies that with probability one $\underline{\lim}_{n\to\infty} \eta_n(1) = 1$, and hence it is natural to ask how fast does the function $\eta_n(\cdot)$ get away from the zero function, say on [0, 1], or how many samples $\eta_n(1), n \leqslant t$, fall in the interval $[0, c], c \geqslant 1$?

Several measures of these quantities are examined in Refs. 3 and 8, but here we only include results for the weighted occupation measures

$$\Psi_c(t) = t^{-1} \int_0^t I_{[0,c]}(\eta_s(1)\theta(s/t))ds, \tag{2.8}$$

where $c \geqslant 1, \theta$ maps $(0, 1]$ into $[1, \infty)$ with $\theta(1) = 1$, $\eta_s(u) = M(su)/(c_A s/LLs)$ for $s > 0$, $u \geqslant 0$, and $\eta_0(u) = 0$ for all $u \geqslant 0$. The interested reader can consult Refs. 3 and 8 for other relevant measures, and the necessary motivation for their analogues in this setting. Of course, the choice of the weight function θ with $\theta(1) = 1$ results from (2.7).

We also assume

$$\theta(s) \text{ is non-increasing on } (0, 1] \tag{2.9}$$

and define the function

$$h(s) = \theta(s) + \int_s^1 (\theta(u)/u)du, \quad 0 < s \leqslant 1. \tag{2.10}$$

Note that if θ is continuous on (0, 1] and (2.9) holds with $\theta(1) = 1$, then $h(s)$ is strictly decreasing and continuous on (0, 1]. Furthermore, under these conditions it is easy to see that the range of $h(s)$ is all of $[1, \infty)$. The functions $\theta(s) = 1, \theta(s) = k - (k-1)s, k \geqslant 1, \theta(s) = \log(e/s)$, and $\theta(s) = s^{-\beta+1}$, where $\beta > 1$, are interesting weights which satisfy the condition (2.9) and $\theta(1) = 1$. Now we can state our weighted occupation measure result for $\Psi_c(t)$.

Theorem 2. Let $\theta : (0, 1] \to [1, \infty)$ such that $\theta(1) = 1$. In addition, assume (2.9) and θ is continuous on (0, 1]. Then $h(s)$ defined as in (2.10) is strictly decreasing and continuous from (0, 1] onto $[1, \infty)$, and for each $c \geqslant 1$ we have with probability one that

$$\limsup_{t \to \infty} \Psi_c(t) = 1 - s_c, \tag{2.11}$$

where $s = s_c$ is the unique solution to $h(s) = c$ on the interval (0, 1].

Remark. If $h(\cdot)$ is defined as in (2.10) with $\theta(1) = 1$ and θ satisfying (2.9), then $h(\cdot)$ is strictly decreasing and continuous from (0, 1] to $[1, \infty)$ if and only if θ is continuous on (0, 1]. Hence the conditions on $h(\cdot)$ in Ref. 8 are the same as those in Theorem 2. The reader should also note that the conditions on θ in (2.9) are weaker than those in Ref. 8, and this is because in Section 7 we are able to improve Lemma 4.2 of Ref. 8 somewhat.

Examples. If $\theta(s)=k-(k-1)s$ for $0\leqslant s\leqslant 1$ and $k\geqslant 1$, then writing $\theta(s)=1+(k-1)(1-s)$ it is easy to see that $h(s)=1-k\log s$. Hence by solving $h(s)=c$ for $0<s\leqslant 1$ and $c\geqslant 1$, we get $s=s_c=\exp\{-(c-1)/k\}$, and thus with probability one,

$$\limsup_{t\to\infty}\Psi_c(t)=1-\exp\{-(c-1)/k\}$$

for $c\geqslant 1$.

If $\theta(s)=\log(e/s)$ on $(0,1]$, then for $0<s\leqslant 1, h(s)=1-2\log s+(\log s)^2/2$. Solving $h(s)=c$, for $0<s\leqslant 1$ and $c\geqslant 1$, we get $s=s_c=\exp\{2-2\sqrt{1+(c-1)/2}\}$, and hence with probability one

$$\limsup_{t\to\infty}\Psi_c(t)=1-\exp\{2-2\sqrt{1+(c-1)/2}\}$$

for $c\geqslant 1$.

Let $\theta(s)=s^{-\beta+1}$ where $\beta>1$. Then $s=s_c=([1+c(\beta-1)]/\beta)^{1/(1-\beta)}$ and with probability one

$$\limsup_{t\to\infty}\Psi_c(t)=1-([1+c(\beta-1)]/\beta)^{1/(1-\beta)}.$$

Theorem 2 is proved in Section 7, and requires that the parameter s in $\{\eta_s(\cdot)\}$ converge to infinity continuously rather than through the integers. In particular, $\{\eta_s(\cdot)\}$ must satisfy (5.1)–(5.3) as $s\to\infty$, but this follows from easy modifications of what is done in Section 5 and the related material in Ref. 3. Hence these details are not included here.

3. A REPRESENTATION FOR $\{X(t):t\geqslant 0\}$

First we prove some lemmas. The first shows how the $\{\alpha_j\}$ are related to A, and is the key step in showing how $\{X(t):t\geqslant 0\}$ can be written as a linear combination of independent Lévy stochastic area processes. Of course, when A is skew symmetric it is obvious that $\{X(t):t\geqslant 0\}$ is a linear combination of such processes, but the independence requires a proof. It is also worth mentioning that the eventual identification of the constant c_A in terms of the eigenvalues of A depends on some stochastic analysis facts that appeared in the context of this sort of problem in Ref. 12, and is quite different from what is done in Ref. 10 to identify $a(A)$. The representation obtained in this section allows us to make the probability estimates obtained in Section 4, which eventually allow us to identify c_A in terms of the eigenvalues of A.

Lemma 3.1. Let A be a real nonzero d by d skew symmetric matrix. Then there exists a d by d orthogonal matrix Q such that $Q^t AQ$ is a block diagonal matrix where each block is a 1 by 1 zero matrix or a 2 by 2 matrix of the form

$$\begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$$

and $\alpha > 0$. Futhermore, if for $1 \leqslant r \leqslant d/2$ the matrices

$$\begin{bmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{bmatrix}$$

are the 2 by 2 matrices in the block diagonal form for A, then $\alpha_1, \dots, \alpha_r > 0$ are unique.

Proof. Consider the matrix A as a linear operator for R^d to R^d. Then A is a normal nonzero linear operator, and hence there is an orthonormal basis $\{q_j : 1 \leqslant j \leqslant d\}$ of R^d with respect to which A has a block diagonal matrix where each block is a 1 by 1 matrix or a 2 by 2 matrix of the form

$$\begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}$$

with $\alpha > 0$, (see Ref. 1, p. 134). In particular, if C is the matrix of A with respect to this orthonormal basis, then $C = Q^t AQ$ where Q is the orthogonal matrix whose columns are the orthonormal basis $\{q_j : 1 \leqslant j \leqslant d\}$ in the canonical basis coordinates. Furthermore, $C^t = (Q^t AQ)^t = Q^t A^t (Q^t)^t = -C$ since $A^t = -A$ by the skew symmetry of A and that Q is orthogonal. Hence C is skew symmetic, and since A is nonzero, so also is C nonzero. Hence C has zeros on its diagonal and $C = Q^t AQ$ is as indicated.

To see that the $\{\alpha_j : 1 \leqslant j \leqslant r\}$ are unique observe that the nonzero eigenvalues of the block diagonal matrix C, repeated with necessary multiplicities, are $\{\pm\alpha_1 i, \dots, \pm\alpha_r i\}$. Since A is similar to C, these are the nonzero eigenvalues of A as well, and hence the uniqueness is proved. $\square$

Lemma 3.2. If $X(t)=\int_0^t \langle AW(s), dW(s)\rangle$, $t \geqslant 0$, where A is a real nonzero skew symmetric d by d matrix and $\{W(t): t \geqslant 0\}$ is standard sample continuous Brownian motion in R^d, then there exist strictly positive constants $\{\alpha_j : 1 \leqslant j \leqslant r\}$, with $1 \leqslant r \leqslant d/2$, uniquely determined by A, such that with probability one

$$X(t)=\sum_{j=1}^{r} 2\alpha_j A_j(t), \tag{3.1}$$

where $A_1, \ldots, A_r$ are independent Lévy stochastic area processes.

Proof. Let $C=Q^t AQ$ be as in Lemma 3.1 with the 2 by 2 block matrices of the form

$$\begin{bmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{bmatrix}$$

for $j=1, \ldots, r$. Then, for $t \geqslant 0$, we have $X(t)=\int_0^t \langle QC^t Q^t W(s), dW(s)\rangle = \int_0^t \langle C\hat{W}(s), d\hat{W}(s)\rangle$, where C has block diagonal form as given in Lemma 3.1 and $\{\hat{W}(t): t \geqslant 0\}=\{Q^t W(t): t \geqslant 0\}$ is a standard sample continuous Brownian motion due to the orthogonal invariance of Brownian motion. Thus Lemma 3.2 follows with $A_1, \ldots, A_r$ independent Lévy stochastic area processes as claimed.

Applying Lemma 3.2 we henceforth may assume for some integer r, $1 \leqslant r \leqslant d/2$, that we have

$$X(t)=\sum_{j=1}^{r} \alpha_i \int_0^t (B_{2j-1}(s)dB_{2j}(s) - B_{2j}(s)dB_{2j-1}(s))$$

and

$$\tilde{W}(t)=(B_1(t), B_2(t), \ldots, B_{2r}(t), B(t))$$

is a standard sample continuous Brownian motion in R^{2r+1}. The next step of the proof uses this fact to show the process $\{X(t): t \geqslant 0\}$ is equivalent to the the process

$$\tilde{X}(t)=B\left(\sum_{j=1}^{r} \alpha_j^2 \int_0^t (B_{2j-1}^2(s)+B_{2j}^2(s))ds\right), \quad t \geqslant 0,$$

where B is the $(2r+1)^{st}$ coordinate of $\tilde{W}$. In particular, this implies B is independent of the random clock

$$c(t)=\sum_{j=1}^{r}\alpha_j^2\int_0^2(B_{2j-1}^2(s)+B_{2j}^2(s))ds,\quad t\geqslant 0,$$

which is a fact we will exploit later. As mentioned previously, the proof of this representation uses an argument from the very nice paper.[12] □

Lemma 3.3. Let $\tilde{W}(t)=(B_1(t),B_2(t),\ldots,B_{2r}(t),B(t))$ be a standard sample continuous Brownian motion on R^{2r+1} with $\{X(t):t\geqslant 0\}$ as in (3.1) and $A_1,\ldots,A_r$ independent Lévy area processes. Then the law of the process

$$\tilde{X}(t)=B\left(\sum_{j=1}^{r}\alpha_j^2\int_0^t\left(B_{2j-1}^2(s)+B_{2j}^2(s)\right)ds\right),\quad t\geqslant 0,\tag{3.2}$$

is equivalent to that for $\{X(t):t\geqslant 0\}$.

Proof. Assume the process X is given as in (3.1) where $A_1,\ldots,A_r$ are independent Lévy area processes. Then the argument of Ref. 12, depending on stochastic analysis results from Refs. 6 and 11, implies there are Brownian motions $\tilde{B}_1,\ldots,\tilde{B}_r$ and random clocks $\tilde{C}_1,\ldots,\tilde{C}_r$ such that these processes are all independent, the processes $\{\tilde{B}_j(\tilde{C}_j(t)):t\geqslant 0\}$, $1\leqslant j\leqslant r$, have the same joint (product)law as $A_1,\ldots,A_r$, and the joint(product)law of $\{\tilde{C}_1,\ldots,\tilde{C}_r\}$ is the same as that of $\{C_1,\ldots,C_r\}$, where

$$C_j(t)=\frac{1}{4}\int_0^t(B_{2j-1}^2(s)+B_{2j}^2(s))ds,\quad t\geqslant 0$$

for $j=1,\ldots,r$.

Thus the law of $\{X(t):t\geqslant 0\}$ is the same as that of $\{Z(t)=\sum_{j=1}^{r}2\alpha_j\tilde{B}_j(\tilde{C}_j(t)):t\geqslant 0\}$, and the conditional law of the Z process given $\tilde{C}_1,\ldots,\tilde{C}_r$ is the same as that of a mean zero Gaussian process with independent increments and variance at time t equal to $\sum_{j=1}^{r}4\alpha_j^2\tilde{C}_j(t)$. The conditional probability law of the process $\tilde{X}$ given $C_1,\ldots,C_r$ is identical, as $\{B(t):t\geqslant 0\}$ is independent of $\{\sum_{j=1}^{r}\alpha_j^2\int_0^t(B_{2j-1}^2(s)+B_{2j}^2(s))ds:t\geqslant 0\}$. Since the clocks $\tilde{C}_1,\ldots,\tilde{C}_r$ have the same product law as that of $C_1,\ldots,C_r$ we thus have by integrating out the clocks that Lemma 3.3 is proved. □

4. SOME PROBABILITY ESTIMATES

Let $\{X(t): t \geqslant 0\}$ be given by the stochastic integrals in (2.1) where A is a real nonzero skew symmetric d by d matrix. Applying Lemmas 3.1–3.3 we have strictly positive constants $\alpha_1, \ldots, \alpha_r$, $1 \leqslant r \leqslant d/2$, depending only on A, such that the process

$$\tilde{X}(t) = B\left(\int_0^t \sum_{j=1}^r \alpha_j^2 \left(B_{2j-1}^2(s) + B_{2j}^2(s)\right) ds\right), \quad t \geqslant 0 \tag{4.1}$$

has the same probability law as $\{X(t): t \geqslant 0\}$ and $B_1, \ldots, B_{2r}, B$ are independent Brownian motions.

Here we provide the necessary probability estimates for the results of this paper. These estimates are similar to those obtained for the stable processes in Ref. 3, and for fractional Brownian motion in Ref. 8. A main new feature here is to use the representation in (4.1), where time is now given by a random clock and, although the final estimates are similar, modifications need to be made in their proofs. It is in the proof of the lower bound results where the modifications are most pronounced. It is also interesting to note that the proof of the upper bound results here, and in Refs. 3 and 8, all depend on a iterative scheme. Key to this scheme here is inequality (4.4) below, which extends Lemma 1 of Ref. 12 in an obvious way. Once one has (4.4), the upper bounds of Proposition 1 follow via Lemma 4.1, which depends on the results in Ref. 9 and the exponential Tauberian theorem.

The upper bounds for our probability estimates are given in the following proposition.

Proposition 1. Let $\{X(t): t \geqslant 0\}$ be as in (2.1), and assume $M(t) = \sup_{0 \leqslant s \leqslant t} |X(s)|$, for $t \geqslant 0$. Fix sequences $\{t_i\}_{i=0}^m, \{a_i\}_{i=1}^m$ such that $0 = t_0 < t_1 < \cdots < t_m$, $0 \leqslant a_i < b_i$ for $i = 1, \ldots, m$, and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_m$. Then for all $m \geqslant 1$

$$\begin{aligned} &\overline{\lim_{\epsilon \to 0^+}} \epsilon \log P(a_i\epsilon \leqslant M(t_i) \leqslant b_i\epsilon, i = 1, \ldots, m) \\ &\quad \leqslant -\frac{\pi}{2}\left(\sum_{j=1}^r \alpha_j\right) \sum_{i=1}^m \left(t_i - t_{i-1}\right)/b_i, \end{aligned} \tag{4.2}$$

where $\alpha_1, \ldots, \alpha_r$, $1 \leqslant r \leqslant d/2$, are strictly positive numbers uniquely determined by A as in (4.1).

Proof. Since the process $\{X(t):t\geqslant 0\}$ has the same law as $\{\tilde{X}(t):t\geqslant 0\}$ where $\tilde{X}(t)$ is as in (4.1), we will replace $\{X(t):t\geqslant 0\}$ by $\{\tilde{X}(t):t\geqslant 0\}$ in the proof. Hence it suffices to prove (4.2) with $M(t)$ replaced by $\tilde{M}(t)$, where $\tilde{M}(t)=\sup_{0\leqslant s\leqslant t}|\tilde{X}(s)|$, for $t\geqslant 0$.

For $j=1,\dots,2r$, let $\{B_j(t):t\geqslant 0\}$ be the independent Brownian motions of (4.1), and for $t\geqslant 0$ set

$$c(t)=\sum_{j=1}^{r}\alpha_j^2\int_0^t\left(B_{2j-1}^2(s)+B_{2j}^2(s)\right)ds. \tag{4.3}$$

Then, given $\{t_i\}_{i=0}^m, \{a_i\}_{i=1}^m, \{b_i\}_{i=1}^m$ as indicated, the first step of the proof is to show that

$$\begin{aligned} &p(a_i\epsilon\leqslant\tilde{M}(t_i)\leqslant b_i\epsilon, i=1,\dots,m)\\ &\quad\leqslant(4/\pi)^m E\left(\exp\left\{-\frac{\pi^2}{8}\epsilon^{-2}\sum_{i=1}^m b_i^{-2}\Delta_i c\right\}\right)\end{aligned} \tag{4.4}$$

where $\Delta_i c=c(t_i)-c(t_{i-1})$ for $i=1,\dots,m$.

To verify (4.4) let P_c denote the conditional probability $P(\cdot|c)$ and define $A_i=\{\sup_{t_{i-1}\leqslant s<t_i}|\tilde{X}(s)|\leqslant b_i\epsilon\}$ for $i=1,\dots,m$. Then it follows easily that

$$P(a_i\epsilon\leqslant\tilde{M}(t_i)\leqslant b_i\epsilon, i=1,\dots,m)\leqslant P(\cap_{i=1}^m A_i)$$

and letting μ_c, t_{m-1} denote $P(\tilde{X}(t_{m-1})\epsilon\cdot|c)$ we have that

$$\begin{aligned}&P_c(\cap_{i=1}^m A_i)\\ &\quad=\int_{\mathbb{R}}P_c(\cap_{i=1}^{m-1}A_i,\sup_{t_{m-1}\leqslant s<t_m}|\tilde{X}(s)-\tilde{X}(t_{m-1})+x|\leqslant b_m\epsilon|\tilde{X}(t_{m-1})=x)d\mu_{c,t_{m-1}}(x)\\ &\quad=\int_{\mathbb{R}}P_c(\sup_{t_{m-1}\leqslant s<t_m}|\tilde{X}(s)-\tilde{X}(t_{m-1})+x|\leqslant b_m\epsilon)P_c(\cap_{i=1}^m A_i|\tilde{X}(t_{m-1}=x)d\mu_{c,t_{m-1}}(x),\end{aligned}$$

since $\sup_{t_{m-1}\leqslant s<t_m}|\tilde{X}(s)-\tilde{X}(t_{m-1})+x|$ is P_c independent of $\tilde{X}(t_{m-1})$ and $\cap_{i-1}^m A_i$ by the P_c independent increments of $\{\tilde{X}(t):t\geqslant 0\}$. Since $\{\tilde{X}(s):s\geqslant 0\}$ is a centered Gaussian process with respect to P_c, we have by Anderson's inequality that

$$\begin{aligned}&P_c\left(\sup_{t_{m-1}\leqslant s<t_m}|\tilde{X}(s)-\tilde{X}(t_{m-1})+x|\leqslant b_m\epsilon\right)\\ &\quad\leqslant P_c(\sup_{t_{m-1}\leqslant s<t_m}|\tilde{X}(s)-\tilde{X}(t_{m-1})|\leqslant b_m\epsilon)\\ &\quad=P_c(\sup_{0\leqslant s\leqslant 1}|B(s)|\leqslant b_m\epsilon/(\triangle_m c)^{1/2}),\end{aligned}$$

where the equality follows by the scaling property of Brownian motion and the homogeneity of its increments. Thus

$$P_c(\cap_{i=1}^{m} A_i) \leqslant P_c(\cap_{i=1}^{m-1} A_i) P_c(\sup_{0\leqslant s\leqslant 1} |B(s)| \leqslant b_m\epsilon/(\Delta_m c)^{1/2})$$

and iterating this estimate we see that

$$P_c(\cap_{i=1}^{m} A_i) \leqslant \prod_{i=1}^{m} P_c(\sup_{0\leqslant s\leqslant 1} |B(s)| \leqslant b_i\epsilon/(\Delta_i c)^{1/2}).$$

Hence by (1.5.2) of Ref. 5, p. 43 we see

$$P_c(\cap_{i=1}^{m} A_i) \leqslant (4/\pi)^m \exp\left\{-\frac{\pi^2}{8}\epsilon^{-2}\sum_{i=1}^{m} b_i^{-2}\Delta_i c\right\},$$

and taking expectations we thus have (4.4).

Our next step is to understand the righthand term in (4.4) by use of the exponential Tauberian theorem. That is, we prove the following lemma. □

Lemma 4.1. Let $0 < d_1 \leqslant d_2 \leqslant \cdots \leqslant d_m$ and suppose $c(t)$ is as in (4.3). Let $0 = t_0 < t_1 < t_2 < \cdots < t_m$ and $\Delta_i c = c(t_i) - c(t_{i-1})$ for $i = 1, \ldots, m$. Then

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda^{-1/2} \log E\left(\exp\left\{-\lambda\sum_{i=1}^{m}\Delta_i c/d_i^2\right\}\right)\\ = -2^{1/2}\left(\sum_{j=1}^{r}\alpha_j\right)\sum_{i=1}^{m}(t_i - t_{i-1})/d_i\end{aligned} \tag{4.5}$$

and

$$\lim_{\epsilon\to 0^+} \epsilon\log P\left(\sum_{i=1}^{m}\Delta_i c/d_i^2 \leqslant \epsilon\right) = -\left(\sum_{j=1}^{r}\alpha_j\right)^2\left(\sum_{i=1}^{m}(t_i - t_{i-1})/d_i\right)^2\Big/2. \tag{4.6}$$

Proof. Applying Theorem 6.4 of Ref. 9 with $\rho(t) = \sum_{i=1}^{m} d_i^{-1} I_{[t_{i-1}-t_i]}(t)$ for $t \geqslant 0$, we have for each $\alpha > 0$ and $j = 1, \ldots, 2r$ that

$$\lim_{\epsilon\to 0^+} \epsilon\log P\left(\alpha^2\sum_{i=1}^{m} d_i^{-2}\int_{t_{i-1}}^{t_i} B_j^2(s)ds \leqslant \epsilon\right) = -\frac{\alpha^2}{8}\left(\sum_{i=1}^{m}(t_i - t_{i-1})/d_i\right)^2. \tag{4.7}$$

Hence by the exponential Tauberian theorem as in Ref. 2, p. 254 we have that as $\lambda \to \infty$

$$\log E\left(\exp\left\{-\lambda\alpha^2\sum_{i=1}^{m} d_i^{-2}\int_{t_{i-1}}^{t_i} B_j^2(s)ds\right\}\sim -2^{-1/2}\alpha\left(\sum_{i=1}^{m}(t_i - t_{i-1})/d_i\right)\lambda^{1/2}\right. \tag{4.8}$$

for $j = 1, \ldots, 2r$. Hence by the independence of the Brownian motions (4.8) easily implies (4.5). Applying the exponential Tauberian theorem again we see that (4.5) gives (4.6). Hence Lemma 4.1 holds.

To finish the proof of Proposition 1 we apply (4.5) to (4.4) with $\lambda = \frac{\pi^2}{8}\epsilon^{-2}$ and $\{b_i\} = \{d_i\}$. This yields (4.2) and Proposition 1 holds.

Now we turn to the lower bounds which are companions for the upper bounds of Proposition 1. □

Proposition 2. Fix sequences $\{t_i\}_{i=0}^m$, $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=0}^m$ such that $0 = t_0 < t_1 < \cdots < t_m$ and $0 = b_0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_m < b_m$ and assume $\{c(t) : t \geqslant 0\}$ is as in (4.3). Then for $\gamma > 0$ we take $0 < \delta < \gamma$ such that $a_i(1+\delta) < b_i(1-\delta)$ for $i = 1, \ldots, m$ and we set $d_i(\delta) = a_i(1+\delta)$ or $d_i(\delta) = b_i(1-\delta)$ for $i = 1, \ldots, m$. Let G denote all possible sequences $\{d_i(\delta)\}_{i=1}^m$ with at least one $d_i(\delta) = a_i(1+\delta)$ for some $i = 1, \ldots, m$. Then

$$\begin{aligned}
&P(a_i\epsilon \leqslant M(t_i) \leqslant b_i\epsilon, 1 \leqslant i \leqslant m, |X(t_m)| \leqslant b_m\gamma\epsilon)\\
&\quad\geqslant (2/\pi)^m E\left(\exp\left\{\frac{-\pi^2}{8}\epsilon^{-2}\sum_{i=1}^{m}(b_i(1-\delta))^{-2}\Delta_i c\right\}\prod_{i=1}^{m} P_c\left(|B(1)| \leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right)\right)\\
&\qquad -(4/\pi)^m \sum_{\{d_i(\delta)\}\in G} E\left(\exp\left\{\frac{-\pi^2}{8}\epsilon^{-2}\sum_{i=1}^{m}(d_i(\delta))^{-2}\Delta_i c\right\}\right),
\end{aligned} \tag{4.9}$$

where $\Delta_i c = c(t_i) - c(t_{i-1})$ and $\Delta_i b = b_i - b_{i-1}$ for $i = 1, \ldots, m$.

Proof. Replacing the X and M processes by $\{\tilde{X}(t) : t \geqslant 0\}$ and $\{\tilde{M}(t) : t \geqslant 0\}$ as in Proposition 1, we define

$$B_i = \{a_i\epsilon \leqslant \sup_{t_{i-1}\leqslant s\leqslant t_i} |\tilde{X}(s)| \leqslant b_i\epsilon, \quad |\tilde{X}(t_i)| \leqslant b_i\delta\epsilon\}$$

for $i = 1, \ldots, m$. Then

$$\{a_i\epsilon \leqslant \tilde{M}(t_i) \leqslant b_i\epsilon, 1 \leqslant i \leqslant m, |\tilde{X}(t_m)| \leqslant b_m\gamma_\epsilon\} \supseteq \cap_{i=1}^m B_i \tag{4.10}$$

and if for $i=1,\ldots,m$ we define

$$A_i=\Big\{a_i(1+\delta)\epsilon \leqslant \sup_{t_{i-1}\leqslant s\leqslant t_i} |\tilde{X}(s)-\tilde{X}(t_{i-1})| \leqslant b_i(1-\delta)\epsilon, |\tilde{X}(t_i)-\tilde{X}(t_{i-1})| \leqslant \Delta_i b\delta\epsilon\Big\},$$

it follows that with $P_c(\cdot)=P(\cdot|c)$ we have

$$\begin{aligned}
P_c(\cap_{i=1}^m B_i) &\geqslant P_c(\cap_{i=1}^{m-1} B_i\cap A_m)\\
&= P_c(\cap_{i=1}^{m-1} B_i) P_c(A_m)\\
&\geqslant \prod_{i=1}^m P_c(A_i),
\end{aligned}\tag{4.11}$$

where the last inequality follows by iteration. If $\Delta_i c>0$, then

$$P_c(A_i)=P_c\left(\frac{a_i(1+\delta)\epsilon}{(\Delta_i c)^{1/2}} \leqslant \sup_{0\leqslant s\leqslant 1} |B(s)| \leqslant \frac{b_i(1-\delta)\epsilon}{(\Delta_i c)^{1/2}}, |B(1)| \leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right)$$

and by Sidák's lemma we therefore have

$$P_c(A_i)\geqslant (f_i-g_i)P_c\left(|B(1)|\leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right),\tag{4.12}$$

where $f_i=P_c\left(\sup_{0\leqslant s\leqslant 1}|B(s)|\leqslant \frac{b_i(1-\delta)\epsilon}{(\Delta_i c)^{1/2}}\right)$ and $g_i=P_c\left(\sup_{0\leqslant s\leqslant 1}|B(s)|\leqslant \frac{a_i(1+\delta)\epsilon}{(\Delta_i c)^{1/2}}\right)$ for $i=1,\ldots,m$. Of course, f_i and g_i depend on a_i, b_i, δ and $\Delta_i c$, but we suppress that. Combining (4.11) and (4.12) we therefore have

$$\begin{aligned}
P(\cap_{i=1}^m B_i) &\geqslant E\left(\prod_{i=1}^m P_c(A_i)\right)\\
&\geqslant E\left(\prod_{i=1}^m (f_i-g_i)P_c\left(|B(1)|\leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right)\right)\\
&\geqslant E\left(\prod_{i=1}^m f_i P_c\left(|B(1)|\leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right)\right)\\
&\quad -\sum_{\{h_i\}_{i=1}^m\in H} E\left(\prod_{i=1}^m h_i P_c\left(|B(1)|\leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right)\right)\\
&\geqslant E\left(\prod_{i=1}^m f_i P_c\left(|B(1)|\leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right)\right)-\sum_{\{h_i\}\in H} E\left(\prod_{i=1}^m h_i\right),
\end{aligned}\tag{4.13}$$

where $H=\{\{h_i\}_{i=1}^m : h_i=f_i$ or g_i and at least one $h_i=g_i\}$. Furthermore, if $\Delta_i c=0$ for some i, then $P_c(A_i)=0$, $f_i=g_i$, and (4.13) is still valid.

Since

$$(2/\pi)\exp\left\{\frac{-\pi^2}{8}x^{-2}\right\} \leqslant P(\sup_{0\leqslant s\leqslant 1}|B(s)|\leqslant x) \leqslant (4/\pi)\exp\left\{\frac{-\pi^2}{8}x^{-2}\right\}$$

for all $x>0$, it follows for $\Delta_i c\geqslant 0$ that

$$\prod_{i=1}^m f_i \geqslant (2/\pi)^m \exp\left\{\frac{-\pi^2}{8}\epsilon^{-2}\sum_{i=1}^m (b_i(1-\delta))^{-2}\Delta_i c\right\}. \tag{4.14}$$

Now for each $\{h_i\}\in H$ there corresponds one and only one $\{d_i(\delta)\}\in G$ and hence for this $\{d_i(\delta)\}$ we have

$$E\left(\prod_{i=1}^m h_i\right) \leqslant (4/\pi)^m E\left(\exp\left\{\frac{-\pi^2}{8}\epsilon^{-2}\sum_{i=1}^m d_i^{-2}(\delta)\Delta_i c\right\}\right). \tag{4.15}$$

Combining (4.10), (4.13), (4.14), and (4.15) we have (4.9) holding. Hence Proposition 2 is proved. □

In order to obtain lower bound analogue for (4.2) we need several additional lemmas.

Lemma 4.2. For each $\{h_i\}\in H$ and uniquely corresponding $\{d_i(\delta)\}\in G$ we have for $\delta>0$ that

$$\overline{\lim_{\epsilon\to 0^+}}\, \epsilon \log E\left(\prod_{i=1}^m h_i\right) \leqslant -\frac{\pi}{2}\left(\sum_{j=1}^r \alpha_j\right)\left(\sum_{i=1}^m (t_i-t_{i-1})/d_i(\delta)\right). \tag{4.16}$$

Proof. Using (4.15) we see

$$\log E\left(\prod_{i=1}^m h_i\right) \leqslant \log E\left(\exp\left\{\frac{-\pi^2}{8}\epsilon^{-2}\sum_{i=1}^m d_i^{-2}(\delta)\Delta_i c\right\}\right) + \log\left(\frac{4}{\pi}\right)^m$$

and hence by (4.5) of Lemma 4.1, with $\lambda=(\pi^2/8)\epsilon^{-2}$, we have (4.6).

Our next lemma is a variation on the lower bound in the proof of the exponential Tauberian theorem which we need. □

Lemma 4.3. Let $\{\beta_i\}_{i=1}^m$ and $\{\gamma_i\}_{i=1}^m$ be sequences of strictly positive real numbers and assume $\{\eta_i\}_{i=1}^m$ are non-negative random variables such that for some $\alpha > 0$ and $V(\beta_1, \dots, \beta_m) > 0$ we have

$$\lim_{\epsilon \to 0^+} \epsilon^\alpha \log P\left(\sum_{i=1}^m \eta_i/\beta_i \leqslant \epsilon\right) = -V(\beta_1, \dots, \beta_m). \tag{4.17}$$

If G is a normal random variable with mean zero and variance one, then

$$\begin{aligned} &\underline{\lim}_{\lambda \to \infty} \lambda^{-\alpha/(\alpha+1)} \log E\left(\exp\left\{-\lambda \sum_{i=1}^m \eta_i/\beta_i\right\} \prod_{i=1}^m P(|G| \leqslant \gamma_i(\eta_i \lambda)^{-1/2})\right) \\ &\quad \geqslant -(\alpha+1)\alpha^{-\alpha/(\alpha+1)} V(\beta_1, \dots, \beta_m)^{1/(\alpha+1)}. \end{aligned} \tag{4.18}$$

Proof. First observe that for every $L > 0$, there exists $c_L > 0$ such that

$$\begin{aligned} &E\left(\exp\left\{-\lambda \sum_{i=1}^m \eta_i/\beta_i\right\} \prod_{i=1}^m P(|G| \leqslant \gamma_i(\eta_i \lambda)^{-1/2})\right) \\ &\quad \geqslant E\left(\exp\left\{-\lambda \sum_{i=1}^m \eta_i/\beta_i\right\} I\left(\sum_{i=1}^m \eta_i/\beta_i \leqslant L\right) \prod_{i=1}^n P(|G| \leqslant \gamma_i(\eta_i \lambda)^{-1/2})\right) \\ &\quad \geqslant (c_L/\lambda^{1/2})^m E\left(\exp\left\{-\lambda \sum_{i=1}^m \eta_i/\beta_i\right\} I\left(\sum_{i=1}^m \eta_i/\beta_i \leqslant L\right)\right) \\ &\quad = (c_L/\lambda^{1/2})^m \int_0^L e^{-\lambda x} d\mu(x), \end{aligned} \tag{4.19}$$

where $\mu = \mathcal{L}(\sum_{i=1}^m \eta_i/\beta_i)$. The existence of c_L in (4.19) follows easily since $\sum_{i=1}^m \eta_i/\beta_i \leqslant L$ implies $\inf_{1 \leqslant i \leqslant m} \gamma_i/\eta_i^{1/2} \geqslant \inf_{1 \leqslant i \leqslant m} \frac{\gamma_i}{(\beta_i L)^{1/2}} > 0$ under the positivity assumptions on $\{\beta_i\}_{i=1}^m, \{\gamma_i\}_{i=1}^m$, and $\{\eta_i\}_{i=1}^m$. Thus for $\xi > 0$ and $\theta = \lambda^{-1/(\alpha+1)} \xi^{-1/\alpha}$ we have

$$\begin{aligned} &\underline{\lim}_{\lambda \to \infty} \lambda^{-\alpha/(\alpha+1)} \log \int_0^L e^{-\lambda x} d\mu(x) \\ &\quad \geqslant \underline{\lim}_{\lambda \to \infty} \lambda^{-\alpha/(\alpha+1)} \log \int_0^\theta \exp\{-\lambda x) d\mu(x) \\ &\quad \geqslant \underline{\lim}_{\lambda \to \infty} \lambda^{-\alpha/(\alpha+1)} [-\theta\lambda + \log \mu((0, \theta])] \\ &\quad \geqslant -\xi^{-1/\alpha} - V(\beta_1, \dots, \beta_m)\xi \\ &\quad \geqslant -(1+\alpha) V(\beta_1, \dots, \beta_m)^{1/(\alpha+1)} \alpha^{-\alpha/(\alpha+1)}, \end{aligned} \tag{4.20}$$

where the last inequality holds by setting $\xi = (V(\beta_1, \ldots, \beta_m)\alpha)^{-\frac{\alpha}{(\alpha+1)}}$. Combining (4.19) and (4.20), Lemma 4.3 now follows. □

Lemma 4.4. Under the previous conditions

$$\varliminf_{\epsilon \to 0^+} \epsilon \log E\left(\prod_{i=1}^m f_i P_c\left(|B(1)| \leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right)\right)$$
$$\geqslant -\frac{\pi}{2}\left(\sum_{j=1}^r \alpha_j\right)\left(\sum_{i=1}^m (t_i - t_{i-1})/(b_i(1-\delta))\right). \tag{4.21}$$

Proof. Since $\prod_{i=1}^m f_i \geqslant (2/\pi)^m \exp\{-(\pi^2/8)\epsilon^{-2}\sum_{i=1}^m (b_i(1-\delta))^{-2}\Delta_i c\}$, by (4.14), (4.21) follows from (4.6) and Lemma 4.3, (4.18), with $\lambda = \epsilon^{-2}\pi^2/8, \alpha = 1, \eta_i = \Delta_i c, \beta_i = (b_i(1-\delta))^2$ for $i = 1, \ldots, m$, and $V(x_1^2, \ldots, x_m^2) = (\sum_{j=1}^r \alpha_j)^2(\sum_{i=1}^m (t_i - t_{i-1})/(x_i))^2/2$ for $\{x_i\}_{i=0}^m$ a positive sequence. □

Proposition 3. Assume the condition in Proposition 2. Then for all $m \geqslant 1$

$$\varliminf_{\epsilon \to 0^+} \epsilon \log P(a_i\epsilon \leqslant M(t_i) \leqslant b_i\epsilon, i = 1, \ldots, m)$$
$$\geqslant -\frac{\pi}{2}\left(\sum_{j=1}^r \alpha_j\right)\sum_{i=1}^m (t_i - t_{i-1})/b_i. \tag{4.22}$$

Proof. Combining (4.16) and (4.21) and that

$$\sum_{i=1}^m (t_i - t_{i-1})/(b_i(1-\delta)) < \sum_{i=1}^m (t_i - t_{i-1})/d_i(\delta)$$

for all $\delta > 0$ we have the limit as $\epsilon \to 0^+$ of

$$E\left(\exp\left\{-\frac{\pi^2}{8}\epsilon^{-2}\sum_{i=1}^m (d_i(\delta))^{-2}\Delta_i c\right\}\right)$$

divided by

$$E\left(\exp\left\{-\frac{\pi^2}{8}\epsilon^{-2}\sum_{i=1}^m (b_i(1-\delta))^{-2}\Delta_i c\right\}\prod_{i=1}^m P_c\left(|B(1)| \leqslant \frac{\Delta_i b\delta\epsilon}{(\Delta_i c)^{1/2}}\right)\right)$$

is zero for each $\{d_i(\delta)\} \in G$.

Thus (4.9), Lemma 4.4, and that $\log(A-B)=\log A+\log(1-B/A)$ easily yields (4.22) with $\sum_{i=1}^{m}(t_i-t_{i-1})/b_i$ replaced by $\sum_{i=1}^{m}(t_i-t_{i-1})/(b_i(1-\delta))$. Since $\delta>0$ is arbitrary and does not appear in the left-hand term of (4.22), the limit as $\delta\to 0^+$ yields (4.22). □

5. PROOF OF THEOREM 1

Given the probability estimates of Proposition 1 and 3, the proof of Theorem 1 follows once one establishes the following three facts. That is, we need to prove

$$P(C(\{\eta_n\})\subseteq K)=1, \tag{5.1}$$

$$P(\{\eta_n\} \text{ is relatively compact in } \mathcal{M})=1 \tag{5.2}$$

and

$$P(K\subseteq C(\{\eta_n\}))=1. \tag{5.3}$$

The topology on $\mathcal{M}$ is given by weak convergence as described above, and is discussed in detail in Ref. 3. As mentioned previously, the weak topology $\mathcal{M}$ is separable and metric, and a subset F of $\mathcal{M}$ is relatively compact if for every $\Gamma>0$ there exists $t_0=t_0(\Gamma)$ such that $t\geqslant t_0$ implies $\inf_{f\in F} f(t)\geqslant\Gamma$. The proofs of (5.1) and (5.2) follow as for their analogues in Ref. 3. Hence we do not include the details, but turn to (5.3), whose proof is different in this setting.

Proposition 4. $P(K\subset C(\{\eta_n\}))=1.$

Proof. Following the argument for the analogue in Ref. 3 it suffices to show that for every $f\in K$ which is strictly increasing where it is finite, and satisfying

$$\Lambda(f)=\int_0^\infty (f(t))^{-1}\,dt<1,$$

we have for every weak neighborhood N_f of f that

$$P(\eta_n\in N_f \ \ i.o.)=1. \tag{5.4}$$

The fact that $f\in K$ can be taken strictly increasing where it is finite can be handled by replacing f by $\tilde{f}(t)=f(t)+\beta t$ where β is sufficiently small in the argument in Ref. 3.

Let $t_f^* = \sup\{t : f(t) < \infty\}$. Then $t_f^* = \infty$ or $0 \leqslant t_f^* < \infty$. If $t_f^* = \infty$, then a typical weak neighborhood of f is of the form $N = \cap_{j=1}^m \Gamma_j$ where $0 < t_1 < \cdots < t_m$

$$\Gamma_j = \{g : f(t_j) - \theta < g(t_j) < f(t_j) + \theta\} \tag{5.5}$$

and $\theta > 0$. If $0 < t_f^* < \infty$, then a typical weak neighborhood of f is of the form

$$N_f = \Big(\bigcap_{j=1}^m \Gamma_j\Big) \cap \Big(\bigcap_{k=1}^s R_{m+k}\Big), \tag{5.6}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_f^* \leqslant t_{m+1} < \cdots < t_{m+s}, \Gamma_j$ is as in (5.5), and $R_{m+k} = \{g : g(t_{m+k}) > m_k\}$. When $t_f^* = 0$, then a typical neighborhood of f is of the form

$$N_f = \bigcap_{k=1}^s R_k,$$

where $R_k = \{g(t_k) > m_k\}$.

Assuming $\Lambda(f) < 1$ and f is strictly increasing, we consider only the case $t_f^* = \infty$ (the other cases being much the same). Then $N_f = \bigcap_{j=1}^m \Gamma_j$ where Γ_j is as in (5.5), and we turn to verifying (5.4).

Let $n_k = k^k$ and define for $k \geqslant 1$ the processes

$$X_k(s) = \int_{n_{2k-1}}^{s+n_{2k-1}} \langle A(W(u) - W(n_{2k-1})), dW(u)\rangle \quad s \geqslant 0. \tag{5.7}$$

Of course, if $s \leqslant 0$, then we define $X_k(s) = 0$. Then the law of $\{X_k(s) : s \geqslant 0\}$ is the same as that of $\{X(s) : s \geqslant 0\}$, and

$$\begin{aligned} X_k(s) &= X(s + n_{2k-1}) - X(n_{2k-1}) \\ &\quad - \langle AW(n_{2k-1}), W(s + n_{2k-1}) - W(n_{2k-1})\rangle, \quad s \geqslant 0. \end{aligned} \tag{5.8}$$

Hence for $n_{2k}s \geqslant n_{2k-1}$ we have

$$\begin{aligned} X(n_{2k}s) - X_k(n_{2k}s - n_{2k-1}) &= X(n_{2k-1}) \\ &\quad + \langle AW(n_{2k-1}), W(n_{2k}s) - W(n_{2k-1})\rangle, \end{aligned}$$

and for $0 \leqslant n_{2k}s \leqslant n_{2k-1}$ we have

$$X(n_{2k}s) - X_k(n_{2k}s - n_{2k-1}) = X(n_{2k}s),$$

which implies

$$\sup_{0 \leqslant s < t_m} |X(n_{2k}s) - X_k(n_{2k}s - n_{2k-1})| \leqslant I_{1,k} + I_{2,k}, \tag{5.9}$$

where

$$I_{1,k} = \sup_{0 \leqslant s \leqslant n_{2k-1}/n_{2k}} |X(n_{2k}s)| = M(n_{2k-1}) \tag{5.10}$$

and

$$I_{2,k} = \sup_{n_{2k-1}/n_{2k} \leqslant s \leqslant t_m} |X(n_{2k-1}) + \langle AW(n_{2k-1}), W(n_{2k}s) - W(n_{2k-1})\rangle|. \tag{5.11}$$

Next we define the maximal process $\{M_k(t) : t \geqslant 0\}$ in terms of the X_k process, shifted somewhat. That is, we define

$$M_k(t) = \sup_{0 \leqslant s \leqslant t} |X_k(n_{2k}s - n_{2k-1})|, \quad t \geqslant 0$$

and since $X_k(s) = 0$ for $s \leqslant 0$ we see that

$$M_k(t) = \sup_{n_{2k-1}/n_{2k} \leqslant s \leqslant t} |X_k(n_{2k}s - n_{2k-1})|, \quad t \geqslant 0. \tag{5.12}$$

In particular, since the law of $\{X_k(t) : t \geqslant 0\}$ is the same as that of $\{X(t) : t \geqslant 0\}$, we have the law of $\{M_k(t) : t \geqslant 0\}$ equal to the law of $\{M(n_{2k}(t - n_{2k-1}/n_{2k})) : t \geqslant 0\}$, where we understand $M(s) = 0$ for $s \leqslant 0$. In addition, we then have from (5.9) to (5.11) that

$$\sup_{1 \leqslant i \leqslant m} |M_k(t_i) - M(n_{2k}t_i)| \leqslant I_{1,k} + I_{3,k}, \tag{5.13}$$

where

$$\begin{aligned} I_{3,k} &= M(n_{2k-1}) \\ &\quad + \sup_{n_{2k-1}/n_{2k} \leqslant s \leqslant t_m} |\langle AW(n_{2k-1}), W(n_{2ks}) - W(n_{2k-1})\rangle|. \end{aligned} \tag{5.14}$$

Next define $a_n = c_A n/LLn$ where $c_A = \pi/2 \sum_{j=1}^{r} \alpha_j$ as in Theorem 1, and let

$$A_k = \{M(n_{2k}t_i)/a_{n_{2k}} \in \Gamma_i, i = 1, \ldots, m\},$$

$$B_k = \{M_k(t_i)/a_{n_{2k}} \in (f(t_i) - \theta/2, f(t_i) + \theta/2), i = 1, \ldots, m\}, \tag{5.15}$$

and

$$C_k = \{\sup_{1 \leqslant i \leqslant m} |M_k(t_i) - M(n_{2k}t_i)|/a_{n_{2k}} \geqslant \theta/2\}$$

for $k = 1, 2, \ldots,$ and $\theta > 0$. From (5.7) we easily see the $\{B_k : k \geqslant k_0\}$ are independent events for k_0 sufficiently large because the increments of the

Brownian motion are independent and $n_{2k-1} \leqslant n_{2k}t_m \leqslant n_{2k+1}$ for all k sufficiently large. In addition we have

$$B_k \subseteq A_k \cup C_k$$

and hence $P(A_k \ i.o.) = 1$ if we show $P(B_k \ i.o.) = 1$ and $P(C_k \ i.o.) = 0$.

Hence the proof of Proposition 4 will follow from the next two lemmas. □

Lemma 5.1. $P(C_k \ i.o.) = 0$.

Proof. It suffices to show $\sum_{k=1}^{\infty} P(C_k) < \infty$, and using (5.12)–(5.14) we need only show for every $\delta > 0$ that

$$\sum_{k=1}^{\infty} P(M(n_{2k-1}) > \delta a_{n_{2k}}) < \infty \tag{5.16}$$

and

$$\sum_{k=1}^{\infty} P\left(\sup_{\epsilon_k \leqslant s \leqslant t_m} |\langle AW(n_{2k-1}), W(n_{2k}s) - W(n_{2k-1})\rangle| > \delta a_{n_{2k}} \right) < \infty, \tag{5.17}$$

where $\epsilon_k = n_{2k-1}/n_{2k}$.

Now define $b_k = n_{2k}/(n_{2k-1} LLn_{2k})$. Then

$$\begin{aligned} P(M(n_{2k-1}) > \delta a_{n_{2k}}) &= P(M(1) > \delta c_A b_k) \\ &= P(\sup_{0 \leqslant s \leqslant 1} |B(c(s))| > \delta c_A b_k) \end{aligned}$$

and since

$$\begin{aligned} P(\sup_{o \leqslant s \leqslant 1} |B(c(s))| > x) &\leqslant 4P(B(c(1)) > x) \\ &\leqslant 4E(B^2(c(1)))/x^2 \\ &= 4E(c(1)E(B^2(1)|c))/x^2 \\ &= 4E(c(1))/x^2, \end{aligned}$$

we have

$$\begin{aligned} P(M(n_{2k-1}) > \delta a_{n_{2k}}) &\leqslant 4E(c(1))/(\delta c_A b_k)^2 \\ &= O(LK/k^2). \end{aligned}$$

Thus (5.16) holds.

To verify (5.17) set

$$p_k = P\left(\sup_{\epsilon_k \leqslant s \leqslant t_m} |\langle AW(n_{2k-1}), W(n_{2k}s) - W(n_{2k-1})\rangle| > \delta a_{n_{2k}}\right)$$

and then observe that by the independent increments of $\{W(s) : s \geqslant o\}$ we have

$$\begin{aligned} p_k &= \int_{R^d} P\left(\sup_{\epsilon_k \leqslant s \leqslant t_m} |\langle y, W(n_{2k}s) - W(n_{2k-1})\rangle| > \delta a_{n_{2k}}\right) dP_{AW(n_{2k-1})}(y) \\ &\leqslant 4\int_{R^d} P\left(|\langle y, W(n_{2k}t_m) - W(n_{2k-1})\rangle| > \delta a_{n_{2k}}\right) dP_{AW(n_{2k-1})}(y) \\ &\leqslant 4\exp\left\{-\lambda\delta a_{n_{2k}}\right\} \int_{R^d} E\left(e^{\langle y, Z_k\rangle}\right) dP_{AW(n_{2k-1})}(y), \end{aligned}$$

where $Z_k = W(n_{2k}t_m) - W(n_{2k-1})$. Therefore

$$p_k \leqslant 4\exp\left\{-\lambda\delta a_{n_{2k}}\right\} E\left(\exp\left\{\lambda^2(n_{2k}t_m - n_{2k-1})n_{2k-1}|AW(1)|^2/2\right\}\right)$$

and setting $\xi_k = \lambda^2(n_{2k}t_m - n_{2k-1})n_{2k-1}|A|^2$ we have

$$p_k \leqslant 4\exp\{-\lambda\delta a_{n_{2k}}\} \int_{R^d} \exp\{\xi_k|y|^2/2 - |y|^2/2\}dy.$$

Hence letting $\lambda = (2(n_{2k}t_m - n_{2k-1})|A|^2)^{-1/2}$ we see

$$\begin{aligned} p_k &\leqslant 4\exp\{-\delta(2(n_{2k}t_m - n_{2k-1})n_{2k-1}|A|^2/2)^{-1/2}c_A n_{2k}/LLn_{2k}\} \\ &\quad \times \int_{R^d} \exp\{-|y|^2/4\}dy \\ &\leqslant 4\exp\{-Ck^{1/2}/Lk\} \end{aligned}$$

for some strictly positive constant C. Thus (5.17) holds and Lemma 5.1 is proven. □

Lemma 5.2. $P(B_k \ i.o.) = 1.$

Proof. since the $\{B_k : k \geqslant k_0\}$ are independent for some k_0 sufficiently large, it suffices to show $\sum_{k=1}^{\infty} P(B_k) = \infty$.

Recalling $\epsilon_k = n_{2k-1}/n_{2k}$ and that $\{X(t) : t \geqslant 0\}$ and $\{X_k(t) : t \geqslant 0\}$ are equal in law, we see

$$P(B_k) = P(M(t_i - \epsilon_k) \in c_A (LL_{n_{2k}})^{-1} E_i(\theta/2), 1 \leqslant i \leqslant m),$$

where $E_i(\theta/2) = (f(t_i) - \theta/2, f(t_i) + \theta/2), 1 \leqslant i \leqslant m$. Therefore $P(B_k)$ is greater than or equal to $P(M(t_i) \in c_A (LLn_{2k})^{-1} E_i(\theta/4), 1 \leqslant i \leqslant m)$ minus the quantity $P(\sup_{1 \leqslant i \leqslant m} |M(t_i) - M(t_i - \epsilon_k)| \geqslant c_A \theta/(4LLn_{2k}))$, so it suffices to prove

$$\sum_{k \geqslant 1} P(M(t_i) \in c_A (LLn_{2k})^{-1} E_i(\theta/4), i = 1, \ldots, m) = \infty \tag{5.18}$$

and

$$\sum_{k \geqslant 1} P(\sup_{1 \leqslant i \leqslant m} |M(t_i) - M(t_i - \epsilon_k)| \geqslant c_A \theta/(4LLn_{2k}) < \infty \tag{5.19}$$

for all $\theta > 0$.

Since $P(\sup_{1 \leqslant i \leqslant m} |M(t_i) - M(t_i - \epsilon_k)| \geqslant x) \leqslant \sum_{i=1}^{m} P(|M(t_i) - M(t_i - \epsilon_k)| \geqslant x)$ for all $x > 0$, to prove (5.19) it suffices to show that for all $t > 0, \delta > 0$.

$$\sum_{k \geqslant 1} P(|M(t) - M(t - \epsilon_k)| > \delta/(LLn_{2k})) < \infty. \tag{5.20}$$

Now $M(t) = \max\{M(t - \epsilon_k), \sup_{t - \epsilon_k \leqslant s \leqslant t} |X(t - \epsilon_k) + (X(s) - X(t - \epsilon_k))|\}$ and since $M(t) \geqslant M(t - \epsilon_k)$ because $\epsilon_k \geqslant 0$ we easily see that

$$M(t) \leqslant M(t - \epsilon_k) + \sup_{t - \epsilon_k \leqslant s \leqslant t} |X(s) - X(t - \epsilon_k)|.$$

Therefore (5.19) follows if we show for all $t > 0, \delta > 0$ that

$$\sum_{k \geqslant 1} P(\sup_{t - \epsilon_k \leqslant s \leqslant t} |X(s) - X(t - \epsilon_k)| > \delta/(LLn_{2k})) < \infty. \tag{5.21}$$

To verify (5.21) recall by Lemma 3.3 (see (3.4)) that $\{X(t) : t \geqslant 0\}$ is equal in law to $\{B(c(t)) : t \geqslant 0\}$, where the clock process $\{c(t) : t \geqslant 0\}$, is given by (4.3) and $\{B(t) : t \geqslant 0\}$, is a standard one-dimensional sample continuous

Brownian motion independent of the clock process. Hence by Markov's inequality for all $x > 0$ we have

$$
\begin{aligned}
&P\left(\sup_{t-\epsilon_k \leqslant s \leqslant t} |X(s) - X(t-\epsilon_k)| > x\right) \leqslant E\left(\sup_{t-\epsilon_k \leqslant s \leqslant t} |X(s) - X(t-\epsilon_k)|^4\right) \Big/ x^4 \\
&\quad = E\left(E\left(\sup_{t-\epsilon_k \leqslant s \leqslant t} |B(c(s)) - B(c(t-\epsilon_k))|^4 | c\right)\right) \Big/ x^4 \\
&\quad = E\left(|c(t) - c(t-\epsilon_k)|^2\right) E\left(\sup_{0 \leqslant s \leqslant 1} B^4(s)\right) \Big/ x^4 \\
&\quad \leqslant 12E\left(\left|\int_{t-\epsilon_k}^{t} \sum_{j=1}^{r} \alpha_j^2 (B_{2j-1}^2(s) + B_{2j}^2(s)) ds\right|^2\right) \Big/ x^4.
\end{aligned}
$$

Setting $m_r = \Sigma_{k=1}^{r} \alpha_k^4$ we easily see by applying Jensen's inequality to obtain the second inequality below that

$$
\begin{aligned}
&P\left(\sup_{t-\epsilon_k \leqslant s \leqslant t} |X(s) - X(t-\epsilon_k)| > x\right) \\
&\quad \leqslant 12 m_r^2 \epsilon_k^2 E\left(\left|\int_{t-\epsilon_k}^{t} \sum_{j=1}^{r} \frac{\alpha_j^2}{m_r} \frac{\left(B_{2j-1}^2(s) - B_{2j}^2(s)\right)}{2} \frac{ds}{\epsilon_k}\right|^2\right) \Big/ x^4 \\
&\quad \leqslant 12 x^{-4} m_r^2 \epsilon_k^2 \int_{t-\epsilon_k}^{t} \frac{\sum_{j=1}^{r} \alpha_j^4}{m_r} \frac{(3s^2 + 3s^2)}{2} \frac{ds}{\epsilon_k} \\
&\quad = 12 x^{-4} m_r^2 \epsilon_k (t^3 - (t-\epsilon_k)^3) \\
&\quad = 12 x^{-4} m_r^2 \epsilon_k^2 (3t^2 - 3\epsilon_k t + \epsilon_k^2).
\end{aligned}
$$

Since $t > 0$ is fixed and $\delta > 0$ is arbitrary with $\epsilon_k = n_{2k-1}/n_{2k}$, we see (5.21) holds, and hence as indicated above (5.19) follows.

Thus it remains to verify (5.18). To check this recall Proposition 3 to obtain

$$
\begin{aligned}
&\underline{\lim}_{k\to\infty} \frac{c_A}{LLn_{2k}} \log P(M(t_i) \in c_A (LLn_{2k})^{-1} E_i(\theta/4), 1 \leqslant i \leqslant m) \\
&\quad \geqslant -c_A \sum_{i=1}^{m} (t_i - t_{i-1})/(f(t_i) + \theta/4) \\
&\quad \geqslant -c_A \int_0^{\infty} (f(s) + \theta/4)^{-1} ds,
\end{aligned}
$$

where in the second inequality we use that $f(\cdot)$ is increasing on $[0,\infty)$. Since $\int_0^\infty f(s)^{-1}ds<1$, if necessary we can choose $\theta>0$ sufficiently small so that $\int_0^\infty (f(s)+\theta/4)^{-1}ds<1-\delta$ for some $\delta\in(0,1)$. Thus for k sufficiently large we see

$$P(M(t_i)\in c_A(LLn_{2k})^{-1}E_i(\theta/4),\, 1\leqslant i\leqslant m)\geqslant \exp\{-LLn_{2k}(1-\delta/2)\}.$$

Since $n_k=k^k$, we thus have (5.18) holding and Lemma 5.2 is proven.

This completes the proof of Proposition 4, and hence Theorem 1 is proven. □

6. PROOF OF COROLLARIES

If $\underline{\lim}_{n\to\infty} M(n)(\omega)LLn/(c_A n)=d(\omega)<1$ on a set Ω_0 with $P(\Omega_0)>0$, then $\underline{\lim}_{n\to\infty} M(nt)(\omega)LLn/(c_A n)\leqslant d(\omega)<1$ for all $t\in[0,1]$ and $\omega\in\Omega_0$. If $\Omega_1=\{\omega : C(\{\eta_n(\omega)\})$ is relatively compact in $\mathcal{M}\}$, then $P(\Omega_0\cap\Omega_1)=P(\Omega_0)>0$ and for $\omega\in\Omega_0\cap\Omega_1$ there is a subsequence $\{n_k\}=\{n_k(\omega)\}$ such that for some $f(\cdot)=f(\omega)(\cdot)\in K$ the sequence $\{\eta_{n_k(\omega)}\}$ converges weakly to f and

$$\lim_{n_k(\omega)\to\infty} M(n_k(\omega))(\omega)LLn_k(\omega)/(c_A n_k(\omega))=d(\omega)<1.$$

Since f is non-decreasing we then have for all but possibly countably many t that f is continuous at t, so at such t we have

$$\lim_{n_k(\omega)\to\infty} \eta_{n_k(\omega)}(t)(\omega)=f(t)\leqslant \lim_{n_k(\omega)\to\infty} \eta_{n_k(\omega)}(1)(\omega)=d(\omega)<1.$$

Thus $\int_0^\infty f(s)^{-1}ds\geqslant \int_0^1 f(s)^{-1}ds>1$, which contradicts $f\in K$. Hence $d(\omega)\geqslant 1$.

If d is any number strictly bigger than one, define $f(t)=d$ for $0\leqslant t<1+\delta$ and $f(t)=+\infty$ for $1+\delta\leqslant t<\infty$. Then for δ sufficiently small we have $f\in K$. Hence with probability one there is a subsequence $\{n_k\}=\{n_k(\omega)\}$ such that $\{\eta_{n_k}(\omega)\}$ converges weakly to f, and the fact that f is also continuous at $t=1$ implies

$$\lim_{n_k(\omega)\to\infty} \eta_{n_k(\omega)\to\infty}(1)(\omega)=f(1)=d.$$

Hence with probability one, $\underline{\lim}_{n\to\infty}\eta_n(1)\leqslant d$, and since $d>1$ was arbitrary this implies with probability one that $\underline{\lim}_{n\to\infty}\eta_n(1)\leqslant 1$. Combining this with the fact that this liminf must be at least one proves both corollaries.

7. PROOF OF THEOREM 2

The proof requires two lemmas. The first is a slight modification of Lemma 4.1 in Ref. 3, and its proof easily follows the ideas in Ref. 3. Hence we omit these details. The second is an extension of Lemma 4.2 in Ref. 8, which allows us to cover a broader class of weight fuctions θ in Theorem 2. In particular, assumption (2.9) is now weaker that its companion assumption (1.11) in Ref. 8.

Lemma 7.1. Let $F_c(f)=\int_0^1 I_{[0,c]}(f(u)r(u))du$, and

$$G_c(t)=\int_0^1 I_{[0,c]}\left(\eta_t(u)r(u)\left(\frac{LLtu}{LLt}\right)\right)du,$$

where $r:(0,1]\to[0,\infty)$ is measurable. Then for each $c>0$, with probability one

$$\overline{\lim_{t\to\infty}}\, G_c(t)\leqslant \sup_{f\in K} F_c(f). \tag{7.1}$$

Furthermore, we have equality in (7.1) whenever $\sup_{f\in k} F_c(f)$ is left continuous at c.

Lemma 7.2. Let g be real-valued, non-negative, and continuous on (0,1] with $0<g(1)<1$. If $tg(t)$ is non-negative on (0,1] and $\lim\limits_{t\downarrow 0} tg(t)>0$, then

$$\sup_{f\in K_0}\int_0^1 I_{\{t:f(t)\geqslant g(t)\}}(x)dx=1-u_0, \tag{7.2}$$

where K_0 is the set of non-negative, non-increasing, right-continuous functions f on (0,1] with $\int_0^1 f(t)dt\leqslant 1$, and u_0 satisfies

$$u_0g(u_0)+\int_{u_0}^1 g(u)du=1. \tag{7.3}$$

Proof. If $\lim_{t\downarrow 0} tg(t)>1$, then Lemma 7.2 follows from Lemma 4.2 in Ref. 8. If $\lim_{t\downarrow 0} tg(t)=\delta$, where $0<\delta\leqslant 1$, then we take $0<\epsilon_0<1$ such that $\epsilon_0 g(\epsilon_0)>2\delta/3$. Let $j(t)=tg(t)$ for $0<t\leqslant 1$ and set $j(0)=\delta$. Then j is continuous and non-increasing on [0,1], and we define for $0<\epsilon\leqslant\epsilon_0$ the functions $j_\epsilon(t)$ where $j_\epsilon(t)=j(t)$ for $\epsilon<t\leqslant 1$ and $j_\epsilon(t)=\max\{j(\epsilon)k_\epsilon(t), j(t)\}$, where $k_\epsilon(t)=1+2(1-t/\epsilon)/\delta$ on $[0,\epsilon]$.

Then $j_\epsilon(t)\geqslant j(t)$ on [0,1], and it is also non-increasing and continuous there. Furthermore, $j_\epsilon(0)=\max\{j(\epsilon)k_\epsilon(0), j(0)\}\geqslant\max\{j(\epsilon_0)(1+$

$2/\delta), \delta\} \geqslant 2\delta(1+2/\delta)/3 > 4/3$. Since $j_\epsilon(t) \geqslant j(t)$ on [0,1], we thus have $g_\epsilon(t) = j_\epsilon(t)/t \geqslant g(t)$ on (0,1] and also that $\lim_{t\downarrow 0} t g_\epsilon(t) \geqslant 4/3$. In particular, we also have $g_\epsilon(1) = g(1) < 1$, so Lemma 4.2 of Ref. 8 applies to g_ϵ.

To finish the proof let

$$\Lambda(g) = \sup_{f\in K_0} \int_0^1 I_{\{t:f(t)\geqslant g(t)\}}(x)dx$$

and for $s \in (0, 1]$ define

$$\lambda(s) = sg(s) + \int_s^1 g(u)du$$

and

$$\lambda_\epsilon(s) = sg_\epsilon(s) + \int_s^1 g_\epsilon(u)du.$$

Then by Lemma 4.2 of Ref. 8 we see that $\Lambda(g_\epsilon) = 1 - u_\epsilon$, where u_ϵ is the solution to $\lambda_\epsilon(s) = 1$ in (0,1].

Since $g_\epsilon \geqslant g$ on [0,1], it follows immediately that $\Lambda(g_\epsilon) \leqslant \Lambda(g)$. However, since $g_\epsilon = g$ on $[\epsilon, 1]$, we see that

$$\Lambda(g) \leqslant \epsilon + \sup_{f\in K_0} \int_\epsilon^1 I_{\{t:f(t)\geqslant g_\epsilon(t)\}}(x)dx. \tag{7.4}$$

Thus $\Lambda(g) \leqslant \epsilon + \Lambda(g_\epsilon)$ and also for $s \in [\epsilon, 1]$ we have $\lambda(s) = \lambda_\epsilon(s)$. In particular, if $\epsilon > 0$ is sufficiently small, then the unique number u_0 in [0,1] which solves $\lambda(s) = 1$ is such that $0 < \epsilon < u_o \leqslant 1$, and $u_\epsilon = u_0$ since for $s \geqslant \epsilon, \lambda(s) = \lambda_\epsilon(s)$.

Thus for all $\epsilon \in (0, u_0)$ we have $1 - u_0 = 1 - u_\epsilon = \Lambda(g_\epsilon) \leqslant \Lambda(g)$, and since $\Lambda(g) \leqslant \epsilon + \Lambda(g_\epsilon)$ follows from (7.4) with $\Lambda(g_\epsilon) = 1 - u_\epsilon$, we see that

$$1 - u_0 \leqslant \Lambda(g) \leqslant \epsilon + (1 - u_0).$$

Since $\epsilon > 0$ is arbitrary, this implies $\Lambda(g) = 1 - u_0$, and Lemma 7.2 is proven. $\square$

Proof of Theorem 2. Since $\eta_s(1) = \eta_t(s/t)(tLLs/sLLt)$ for $s, t > 0$, letting $u = s/t$ implies $\Psi_c(t)$ as given in (2.8), satisfies

$$\Psi_c(t) = \int_0^1 I_{[0,c]}\left(\eta_t(u)u^{-1}\theta(u)\left(\frac{LLtu}{LLt}\right)\right)du.$$

Applying Lemma 7.1 with $r(u) = u^{-1}\theta(u)$ implies $\limsup_{t\to\infty} \Psi_c(t) = \sup_{f\in K} F_c(f)$ with probability one, provided $\sup_{f\in K} F_x(f)$ is left continuous in x at c.

When $c > 1$, Lemma 7.2 implies

$$\sup_{f \in K} F_c(f) = \sup_{f \in K} \int_0^1 I_{[0,c]}(f(u)u^{-1}\theta(u))du = 1 - s_c. \tag{7.5}$$

That is, if $f(u) > cu/\theta(u)$ on [0,1] then $f^{-1}(u) < \theta(u)/(cu)$ on (0,1]. Letting $g(u) = \theta(u)/(cu)$ on (0,1], we see from (2.9) that g satisfies the conditions in Lemma 7.2. Hence from (7.2) we have

$$\sup_{f \in K} \int_0^1 I_{\{t: f^{-1}(t) \geqslant g(t)\}}(x)dx = 1 - s_c, \tag{7.6}$$

where s_c is as defined in Theorem 2. However, since

$$\{t : f^{-1}(t) \geqslant g(t)\} = \{t : f(t)t^{-1}\theta(t) \leqslant c\},$$

we have (7.5) holding.

Therefore for $c > 1, \sup_{f \in K} F_c(f) = 1 - s_c$, and since h as given in (2.10) is one-to-one and continous from (0,1] onto $[1, \infty)$ with $h(1) = 1$ we have s_c continuous in c for each $c > 1$. Thus Lemma 7.1 and (7.5) imply (2.11) for $c > 1$. If $c = 1$, then for $\delta > 0$

$$0 \leqslant \sup_{f \in K} F_1(f) \leqslant \sup_{f \in K} F_{1+\delta}(f) = 1 - s_{1+\delta},$$

and since $\lim_{\delta \downarrow 0} s_{1+\delta} = s_1 = 1$, we have $\sup_{f \in K} F_c(f) = 0$ when $c = 1$. Thus the upper bound in (7.1) implies with probability one that $\limsup_{t \to \infty} \Psi_c(t) \leqslant 0$ when $c = 1$. However, this lim sup is clearly non-negative, so (2.11) holds even when $c = 1$. Thus Theorem 2 is proven.

ACKNOWLEDGMENTS

Supported in part by NSF.

REFERENCES

1. Axler, S. (1996). *Linear Algebra Done Right*, Springer, New York.
2. Bingham, N. H., Goldie, C. M., and Teugles, J. L. (1987). *Regular Variation*, Cambridge University Press, Cambridge.
3. Chen, X., Kuelbs, J., and Li, W. (2000). A functional LIL for symmetric stable processes. *Ann. of Probab.* **28**, 258–276.
4. Chung, K. L. (1948). On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* **64**, 205–233.
5. Csörgő, M., and Révész, P. (1981). *Strong Approximations in Probability and Statistics*, Academic Press, New York.
6. Ikeda, N., and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, North-Holland Publishing Company, Amsterdam.

7. Janson, S., and Wichura, M. (1983). Invariance principles for stochastic area and related stochastic integrals. *Stoch. Proc. Applic.* **16**, 71–84.
8. Kuelbs, J., and Li, W. V. (2002). A functional LIL and some weighted occupation measure results for fractional Brownian motion. *J. Theoret. Probab.* **15**, 1007–1030.
9. Li, W. V., and Shao, Q. M. (2001). *Gaussian processes : Inequalities, small ball probabilities, and applications, Stochastic processes: Theory and Methods. Handbook of Statistics* Vol. 19, North-Holland Publishing Company, Amsterdam.
10. Rémillard, B. (1994). On Chung's law of the iterated logarithm for some stochastic integrals. *Ann. Probab.* **22**, 1794–1802.
11. Rogers, L. C. G., and Williams, D. (1987). Diffusions, Markov Processes and Martingales, Vol.II, Ito Calculus, Wiley, New York.
12. Shi, Z. (1994). Liminf behaviors of the windings and Lévy's stochastic areas of planar Brownian motion. In: Azéma, J., Meyer, P. A., and Yor, M. (eds.), *Séminaire de Probabilitiés XXVIII*, Lectures Notes in Mathematics, Vol. 1583, Springer, Berlin.
13. Wichura, M. (1973), A functional form of Chung's law of the iterated logarithm for maximum absolute partial sums, unpublished.