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Large and moderate deviations for intersection local times

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Abstract. We study the large and moderate deviations for intersection local times generated by, respectively, independent Brownian local times and independent local times of symmetric random walks. Our result in the Brownian case generalizes the large deviation principle achieved in Mansmann (1991) for the L_2-norm of Brownian local times, and coincides with the large deviation obtained by Csörgö, Shi and Yor (1999) for self intersection local times of Brownian bridges. Our approach relies on a Feynman-Kac type large deviation for Brownian occupation time, certain localization techniques from Donsker-Varadhan (1975) and Mansmann (1991), and some general methods developed along the line of probability in Banach space. Our treatment in the case of random walks also involves rescaling, spectral representation and invariance principle. The law of the iterated logarithm for intersection local times is given as an application of our deviation results.

1. Introduction

The mathematical notion of various intersection local times was motivated by the models of polymer physics and quantum field theory. For an expository paper on mathematical polymer models, see den Hollander (1996). For a survey on results for one-dimensional polymers, see van der Hofstad and Klenke (2001). For an introduction to polymers from a physicist's point of view, see Vanderzande (1998). For the latest work on attractive random polymer, see van der Hofstad and Klenke (2001) and van der Hofstad, Klenke, and König (2002). For large deviation results on the one-dimensional Edwards model, see van der Hofstad, den Hollander and König (2003).

One of the basic quantity in the study is the associated Hamiltonian (energy function) H which is a nonnegative function of the paths. The asymptotic behavior

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of the partition function (normalizing constant) $\mathbb{E}\, e^{-\lambda H}$ for $\lambda > 0$ is of great interests and it is directly connected with the lower tail behavior $\mathbb{P}(H \le \epsilon)$ for $\epsilon > 0$ under appropriate scaling. The upper tail behavior $\mathbb{P}(H \ge x)$ is also important and appears in certain self-attracting models with weight such as $\mathbb{E}\, e^{\lambda H}$. There are several other motivations given later for the study of the upper tails of H_t and h_n, defined in (1.4) and (1.10) respectively, which are the main subject of this paper. Our approach is to combine abstract tools from probability in Banach space with those existing methods developed in the large deviation theory of Donsker and Varadhan. We mainly deal with the one-dimensional case in this paper.

Before we present our motivations and main results, we need some standard notations. Unless mentioned otherwise, $W(t)$; $W_1(t), \cdots, W_m(t)$ are independent 1-dimensional Brownian motions with the local times $L(t,x)$; $L_1(t,x), \cdots, L_m(t,x)$ $(t \ge 0, x \in \mathbb{R})$, respectively. We also use δ_x to denote the Dirac measure at x. There are two kinds of basic intersections: Formally, the quantity

$$\int_{-\infty}^{\infty} L^p(t,x)dx = \int_0^t \cdots \int_0^t \prod_{j=1}^{p-1} \delta_0\big(W(s_{j+1}) - W(s_j)\big)ds_1 \cdots ds_p$$

measures the 'amount' of time spent by the path in p-multiple self-intersections up to the time t for an integer $p > 1$; and the quantity

$$\int_{-\infty}^{\infty} \prod_{j=1}^{m} L_j(t,x)dx = \int_0^t \cdots \int_0^t \prod_{j=1}^{m-1} \delta_0\big(W_{j+1}(s_{j+1}) - W_j(s_j)\big)ds_1 \cdots ds_m \tag{1.1}$$

measures the 'amount' of time that m independent Brownian trajectories intersect together up to t. These random quantities are called intersection local times in literature. The basic idea to define them rigorously is to replace the Dirac measure by a suitable approximation.

There are several motivations for this work. In a study of uniform empirical process, Csáki, König and Shi (1999) established the following result, for $p = 2$ or $p \ge 3$, on the large deviation for the self-intersection local time of Brownian bridge:

$$\lim_{\lambda\to\infty} \lambda^{-2/(p-1)} \log \mathbb{P}\left\{ \int_{-\infty}^{\infty} \xi^p(1,x)dx \ge \lambda \right\} = -C_2(1,p) \tag{1.2}$$

where $\xi(t,x)$ stands for the local time of a Brownian bridge and the explicit constant $C_2(1,p)$ is given in (1.9). They raise the question on what to expect in the case of Brownian motion. Through a subadditivity technique utilized in Khoshnevisan-Lewis (1998), the large deviation for L_p-norm of Brownian local time with any $p > 1$ can been obtained with right rate but without explicit constant.

For $p = 2$, an interesting development is made by Mansmann (1991) who proved

$$\lim_{\lambda\to\infty} \lambda^{-1} \log \mathbb{E} \exp\left\{ \sqrt{\lambda} \int_{-\infty}^{\infty} L^2(1,x)dx \right\} = \frac{1}{6} \tag{1.3}$$

on the study of free energy of the Dirac polaron. See also Csáki-König-Shi (1999) for a numerical correction of Mansmann's result and Borodin (1982) for some related results. By a standard argument via Gärtner-Ellis theorem, (1.3) is equivalent to

$$\lim_{\lambda\to\infty} \lambda^{-2} \log \mathbb{P}\left\{ \int_{-\infty}^{\infty} L^2(1,x)dx \geq \lambda \right\} = -\frac{3}{2}$$

which takes exactly same form as (1.2) in the context of Brownian bridge for $p = 2$. Csáki, König and Shi (1999) intuitively explain why this is so by representing Brownian bridge as normalized Brownian motion over one excursion. According to their explanation, it becomes natural to expect that the large deviation described in (1.2) holds also in the case of Brownian motion for all real number $p > 1$. As a corollary of what we shall establish in this paper, this is confirmed to be the case.

The notion of intersection local times is also connected to other problems. In Khoshnevisan-Lewis (1998) and Csáki-König-Shi (1999), a simple connection on upper tail behaviors has been established between 2-multiple self-intersection local times and the so-called Brownian motion in Brownian sceneries $\int_{-\infty}^{\infty} L(t,x)B(dx)$ through the equality

$$\mathbb{E} \exp\left\{\lambda \int_{-\infty}^{\infty} L(t,x)B(dx)\right\} = \mathbb{E} \exp\left\{\frac{\lambda^2}{2} \int_{-\infty}^{\infty} L^2(t,x)dx\right\}, \qquad \forall \lambda \in \mathbb{R}$$

where $\{B(x);\ -\infty < x < \infty\}$ is a two-sided Brownian motion (serving as scenery) independent of $W(t)$.

During our study, we also learned the connection to the local times of additive Lévy (Brownian) process

$$\eta_m(I,x) = \int_I \delta_x\big(W_1(s_1) + \cdots + W_m(s_m)\big)ds_1 \cdots ds_m \qquad x \in \mathbb{R} \quad I \subset [0,\infty)^m.$$

See Khoshnevisan-Xiao-Zhong (2003a, 2003b) for some recent progress related to this subject. Indeed, one can easily see that

$$\left\{\eta_2\big([0,t]^2, 0\big);\quad t \geq 0\right\} \stackrel{d}{=} \left\{ \int_{-\infty}^{\infty} L_1(t,x)L_2(t,x)dx;\quad t \geq 0\right\}.$$

A direct consequence of our work is an understanding of the upper tail behaviors of local times of additive Brownian process in the case $m = 2$. The situation is different for $m \geq 3$. However, in view of the representation

$$\eta_m\big([0,t]^m, x\big) = \int \cdots \int_{x_1+\cdots+x_m=x} \prod_{j=1}^m L_j(t,x_j)dx_1 \cdots dx_{m-1}$$

it is our hope that the techniques developed in this work may be of use in the general cases.

It is well known that the intersection behaviors have strong dimension dependence. According to the work of Dvoretzky-Erdös-Kakutani (1950, 1954), given

m independent d-dimensional Brownian motions $W_1(t), \cdots, W_m(t)$, the set of intersection

$$\bigcap_{j=1}^{m} \{x \in \mathbb{R}; \quad x = W_j(t) \text{ for some } t \geq 0\}$$

contains points different from 0 if and only if $(d-2)m < d$. The interested reader is referred to a recent survey paper by Khoshnevisan (2003) for an elementary proof of the above result and for an overview of various results and techniques. A natural problem is to investigate, in the case when $(d-2)m < d$, the long term behaviors of the intersection local time given in (1.1). On the other hand, self-intersection of a multi-dimensional Brownian path is a complicated issue in which case the self-intersection local times can not be directly defined in any reasonable way. As $d = 2$, the renormalized self-intersection local times are constructed essentially as centered self-intersection local times through an approximation procedure (see Le Gall (1992) for details). As pointed out in Westwater (1980), even the renormalized Brownian self-intersection local times can not be properly defined as $d \geq 3$. Le Gall (1994) proves existence of non-trivial critical value for exponential integrability of renormalized 2-multiple self intersection local time of a 2-dimensional Brownian motion. We also refer the interested reader to Le Gall (1992) and the references therein for the study of some other aspects of intersection local times in the multi-dimensional case. Recently, König and Mörters (2002) obtained the upper tail asymptotics for the (projected) Brownian intersection local times on $\mathbb{R}^d$, $d \geq 2$, with application to thick points. Their main tools are moment methods and analysis of variational formulas. After this paper was submitted, some new results were obtained on the large deviations and related results for the intersection local times of multi-dimensional Brownian motions and random walks in Chen (2003), and for the renormalized self-intersection local time of a 2-dimensional Brownian motion in Bass and Chen (2003).

In this paper, we only consider the case $d = 1$ and self-attractions rather than the case $d = 1$ and self-repellence, the Edwards model. More specifically, our first goal is the large deviation principle for the mixed intersection local time

$$H_t = \int_{-\infty}^{\infty} \prod_{j=1}^{m} L_j^p(t,x)dx, \qquad t \geq 0 \tag{1.4}$$

where $m \geq 1$ is an integer and real number $p > 0$ satisfying $mp > 1$. When p is an integer, the above quantity measures the duration that m independent trajectories intersect together, while each of them intersects itself p times:

$$\begin{aligned}\int_{-\infty}^{\infty} \prod_{j=1}^{m} L_j^p(t,x)dx = \int_0^t \cdots \int_0^t & ds_1 \cdots ds_{mp} \delta_0\big(W_1(s_2) - W_1(s_1)\big) \\ & \cdots \delta_0\big(W_1(s_p) - W_1(s_{p-1})\big)\delta_0\big(W_2(s_{p+1}) - W_1(s_p)\big) \\ & \cdots \delta_0\big(W_m(s_{mp}) - W_m(s_{mp-1})\big).\end{aligned}$$

By the scaling property of Brownian motions, for each $t \geq 0$ and $a \geq 0$,

$$\int_{-\infty}^{\infty} \prod_{j=1}^{m} L_j^p(at, x)dx \stackrel{\mathrm{d}}{=} a^{(mp+1)/2} \int_{-\infty}^{\infty} \prod_{j=1}^{m} L_j^p(t, x)dx. \tag{1.5}$$

Without loss of generality, we let time $t = 1$ in the following statement. We also use $B(\cdot, \cdot)$ to denote the beta function.

Theorem 1.1. *For each integer $m \geq 1$ and real number $p > 0$ with $mp > 1$,*

$$\lim_{\lambda \to \infty} \lambda^{-2mp/(mp+1)} \log \mathbb{E} \exp\left\{\lambda\left(\int_{-\infty}^{\infty} \prod_{j=1}^{m} L_j^p(1, x)dx\right)^{1/mp}\right\} = C_1(m, p) \tag{1.6}$$

and thus equivalently,

$$\lim_{\lambda \to \infty} \lambda^{-2/(mp-1)} \log \mathbb{P}\left\{\int_{-\infty}^{\infty} \prod_{j=1}^{m} L_j^p(1, x)dx \geq \lambda\right\} = -C_2(m, p) \tag{1.7}$$

where

$$\begin{aligned} C_1(m, p) = &\left(m^{3mp-1} p^{2mp}\right)^{-1/(mp+1)} \\ &\times \left(\frac{\sqrt{2}}{(mp-1)(mp+1)} B\left(\frac{1}{mp-1}, \frac{1}{2}\right)\right)^{-2(mp-1)/(mp+1)} \end{aligned} \tag{1.8}$$

and

$$C_2(m, p) = -\frac{m}{4(mp-1)} \left(\frac{mp+1}{2}\right)^{(3-mp)/(mp-1)} B\left(\frac{1}{mp-1}, \frac{1}{2}\right)^2. \tag{1.9}$$

When $m = 1$, from (1.7) we see that the self-intersection local times obey the same large deviation described in (1.2), at least in the case $p = 2$ or $p \geq 3$. It is natural to ask whether or not that (1.2) holds for all $p > 1$, and more generally, Theorem 1.1 holds if Brownian motions are replaced by Brownian bridges.

Our second goal is to establish a moderate deviation principle for mixed intersection local times of 1-dimensional random walks with integer values. To avoid some technical difficulties, we only deal with symmetric random walks. Except in section 4 (where we also deal with multi-dimensional random walks), $S(n) = \sum_{k=1}^{n} X_k$, $n = 1, 2, \cdots$ is always a random walk generated by an integer-valued symmetric i.i.d. sequence $\{X_n\}_{n\geq 1}$ with $\sigma^2 \equiv \mathbb{E}\, X_1^2 < \infty$. Without compromising generality we always assume that the smallest group that supports $\{S(n)\}_{n\geq 1}$ is $\mathbb{Z}$. Define the local time

$$l(n, x) = \sum_{k=1}^{n} I_{\{S(k)=x\}} \qquad x \in \mathbb{Z}, n = 1, 2, \cdots .$$

Let $\{S_1(n)\}_{n\geq 1}, \cdots, \{S_m(n)\}_{n\geq 1}$ be m independent copies of $\{S_n\}_{n\geq 1}$ with their local times being denoted by $l_1(n,x), \cdots, l_m(n,x)$. When $p > 0$ is an integer, the random quantity

$$h_n = \sum_{x\in\mathbb{Z}} \prod_{j=1}^m l_j^p(n,x) = \sum_{k_1,\cdots,k_{mp}=1}^n I_{\{S_1(k_1)=\cdots=S_1(k_p)=\cdots=S_m(k_{(m-1)p+1})=\cdots=S_m(k_{mp})\}} \tag{1.10}$$

counts the number of times that up to the time n, the trajectories of m independent random walks meet together, while each of them intersects itself p times. Then the following weak law holds.

Theorem 1.2. *Let $m \geq 1$ be an integer and let $p > 0$ be real such that $mp > 1$. Then*

$$n^{-(mp+1)/2} \sum_{x\in\mathbb{Z}} \prod_{j=1}^m l_j^p(n,x) \xrightarrow{d} \sigma^{-(mp-1)} \int_{-\infty}^{\infty} \prod_{j=1}^m L_j^p(1,x)dx.$$

Next we turn to moderate deviation for the mixed intersection local times of random walks. Throughout, $\{b_n\}$ represents a positive sequence satisfying

$$b_n \to \infty \quad \text{and} \quad b_n/n \to 0. \tag{1.11}$$

In the light of Theorem 1.2, the following result becomes natural.

Theorem 1.3. *Let $m \geq 1$ be an integer and let $p > 0$ be real such that $mp > 1$. Then for any positive sequence $\{b_n\}$ satisfying (1.11),*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{ \left(\frac{b_n}{n}\right)^{(mp+1)/2mp} \left(\sum_{x\in\mathbb{Z}} \prod_{j=1}^m l_j^p(n,x) \right)^{1/mp} \right\} = \sigma^{-2(mp-1)/(mp+1)} C_1(m,p) \tag{1.12}$$

and thus equivalently,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left\{ \sum_{x\in\mathbb{Z}} \prod_{j=1}^m l_j^p(n,x) \geq n^{(mp+1)/2} b_n^{(mp-1)/2} \right\} = -\sigma^2 C_2(m,p) \tag{1.13}$$

where $C_1(m,p)$ and $C_2(m,p)$ are given in (1.8) and (1.9) respectively.

An important application of the large and moderate deviations we establish is to obtain the law of the iterated logarithm. Indeed, we have

Theorem 1.4. *For each integer $m \geq 1$ and real number $p > 0$ with $mp > 1$,*

$$\limsup_{t\to\infty} t^{-(mp+1)/2} \big(\log\log t\big)^{-(mp-1)/2} \int_{-\infty}^{\infty} \prod_{j=1}^m L_j^p(t,x)dx = C_3(m,p) \quad a.s. \tag{1.14}$$

and

$$\limsup_{n\to\infty} n^{-(mp+1)/2}\big(\log\log n\big)^{-(mp-1)/2}\sum_{x\in\mathbb{Z}}\prod_{j=1}^{m} l_j^p(n,x)$$
$$= \sigma^{-(mp-1)}C_3(m,p) \qquad a.s. \tag{1.15}$$

where

$$C_3(m,p) = \Big(\frac{4(mp-1)}{m}\Big)^{(mp-1)/2}\Big(\frac{mp+1}{2}\Big)^{(mp-3)/2}B\Big(\frac{1}{mp-1},\frac{1}{2}\Big)^{-(mp-1)} \tag{1.16}$$

Let us make some comments on the results for random walks. The weak convergence of intersection local times of random walks has been studied by Le Gall (1986) and Rosen (1990) in the general dimension. In addition, it is always of interest to ask for the long term behaviors of the intersection local times whenever they have unbounded growth. It is known, for example, for m independent, identically distributed lattice valued d-dimensional random walks with mean zero and finite variance, their trajectories meet together infinitely often if and only if $(d-2)m \le d$. The law of the iterated logarithm has been obtained for two critical cases "$d = 4$, $m = 2$" in Marcus-Rosen (1997) and "$d = m = 3$" in Rosen (1997). For the non-critical cases, i.e. $(d-2)m < d$, a natural procedure is first to work on the Brownian motions and then to extend achieved results to the random walks in the spirit of invariance principle. The difficulty one has to overcome is discontinuity of the functionals involved. In the case of 1-dimensional symmetric simple random walks, Révész (1990) obtains the strong approximation

$$\sup_{x\in\mathbb{Z}}|L(n,x)-l(n,x)| = o\big(n^{\frac{1}{4}+\epsilon}\big) \qquad a.s. \quad (n\to\infty).$$

in an enlarged probability space. Révész also points out the rate of approximation is nearly best possible. His result is strong enough to extend the law of the iterated logarithm for intersection local times from Brownian motions to symmetric simple random walks, but not enough to extend the large deviation in Theorem 1.1 to the moderate deviation given in Theorem 1.3, even in the case of symmetric simple random walks. It is worth to point out that the techniques used in Révész (1990) depend significantly on the unique structure of 1-dimensional symmetric simple random walks and do not apply to general random walks.

Next we briefly outline some key technical points in each section. In Section 2 we derive the large deviation (1.6) with right hand sides being variations of a supremum which is solved in Section 7. Our method is different from the one used in Mansmann (1991), where the Donsker-Varadhan (1974) large deviation is essential. In fact, the well known Donsker-Varadhan large deviation is no longer applicable in our investigation partially due to discontinuity of Brownian local time as a functional defined on the space of probability measures on $\mathbb{R}^m$ endowed with the topology of weak convergence. In addition, for $m > 1$, even the lower semi-continuity is not available, which is crucially needed in Mansmann's approach for the lower bounds. Furthermore, the proof of Theorem 1.3 demands a new approach which can later be extended (at least partially) to the case of random walks. A key idea in our treatment is to embed the Brownian local times into the Banach space $\mathcal{L}^p(\mathbb{R})$ after reducing the problem to the special case when $m = 1$ and $p > 1$. The computation is based on a Feynman-Kac type large deviation result given in Remillard (1998) for Brownian occupation times, which immediately yields the desired lower bound after a deterministic comparison. On the other hand, the upper bound is harder to establish due to the fact that the Brownian local time fails to be exponentially tight when embedded into $\mathcal{L}^p(\mathbb{R})$, as we shall point out in detail in Section 2. In the proof of the upper bound, we map the Brownian motion into a circle, an idea developed in Donsker-Varadhan (1975) and in Mansmann (1991), and then prove that the local time of "Brownian motion on the circle" is exponentially tight, by a very general result due to de Acosta (1985).

From Section 3 to 5, we work on Theorem 1.2 and Theorem 1.3 on random walks. One of the key ideas is to rescale the space variable of the local times of random walks, which makes it possible to apply invariance principle. In Section 3, we prove that a properly rescaled and normalized local time of a random walk converges weakly to Brownian local time when it is viewed as a process with values in $\mathcal{L}^p(\mathbb{R})$ for $p > 1$. This leads directly to Theorem 1.2. In Section 4, we obtain a Feynman-Kac type exponential estimate for the lower bound of moderate deviation given in Theorem 1.3 with the aid of the spectral theory on Hilbert space. We anticipate that Theorem 4.1 established in Section 4 may have some interesting applications to other related problems. In Section 5, we prove Theorem 1.3. The following observation is helpful throughout: For any integer $n, k \geq 1$, the increment

$$\sum_{x\in\mathbb{Z}} \big(l(n+k,x) - l(n,x)\big)^p$$

is independent of $\{S_1, \cdots, S_n\}$. In the proof of Theorem 1.1 and Theorem 1.3 in Section 2 and Section 5, respectively, we only prove (1.6) and (1.12), since (1.7) and (1.13) are their direct consequences through Gärtner-Ellis Theorem (see, e.g., Theorem 2.3.6 in Dembo-Zeitouni (1993)).

In Section 6, we prove Theorem 1.4, the strong limit laws of the iterated logarithm, which are based on the large deviation given in Theorem 1.1 and the moderate deviation in Theorem 1.3 through the Borel-Cantelli lemma. All arguments here are more or less standard once deviation results are obtained.

In Section 7, we prove two analytic lemmas. One is a technical fact needed for the upper bound of the large deviation given in Section 2. Another is a solution of the variation problem presented in Section 2 and Section 5. Our solution is of independent interest and a few ideas from Strassen (1964) are exploited.

It is interesting to point out that despite of some essential difference between one and multi-dimensional cases, some of the ideas developed in this work has been utilized in Bass and Chen (2003) and Chen (2003) in their study of the intersection local times in the multi-dimensional case.

Finally, we mention a more general problem on the deviation behaviors of

$$\int_{-\infty}^{\infty}\prod_{j=1}^{m}L_j^{p_j}(t,x)dx,\qquad t\ge 0\quad\text{and}\quad\sum_{x\in\mathbb{Z}}\prod_{j=1}^{m}l_j^{p_j}(n,x).$$

When p_j are integers, the above quantities measure the duration (number) that m independent trajectories intersect together, while the jth one intersect itself p_j times. Based on proofs of Theorem 1.1, we have for $P=\sum_{j=1}^{m}p_j>1$

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\lambda\Big(\int_{-\infty}^{\infty}\prod_{j=1}^{m}L_j^{p_j}(t,x)dx\Big)^{1/P}\right\}$$
$$\le\lambda^{2P/(P+1)}\cdot\sum_{j=1}^{m}\Big(P^{-2}p_j\Big)^{2P/(P+1)}\cdot\Big(\frac{\sqrt{2}}{P^2-1}B\Big(\frac{1}{P-1},\frac{1}{2}\Big)\Big)^{-2(P-1)/(P+1)}.$$

It is plausible that the above upper bound is tight. New ideas are needed for this general problem.

2. Large deviations for Brownian intersection local times

In view of Lemma 7.2 in section 7, we will have (1.6) (and therefore Theorem 1.1) after we prove

$$\lim_{\lambda\to\infty}\lambda^{-2mp/(mp+1)}\log\mathbb{E}\exp\left\{\lambda\Big(\int_{-\infty}^{\infty}\prod_{j=1}^{m}L_j^{p}(1,x)dx\Big)^{1/mp}\right\}$$
$$=m^{-(mp-1)/(mp+1)}\sup_{g\in\mathcal{F}}\left\{\Big(\int_{-\infty}^{\infty}|g(x)|^{2mp}dx\Big)^{1/mp}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}\tag{2.1}$$

where $\mathcal{F}$ is the set of absolutely continuous functions g on $(-\infty,\infty)$ with

$$\int_{-\infty}^{\infty}|g(x)|^2dx=1\quad\text{and}\quad\int_{-\infty}^{\infty}|g'(x)|^2dx<\infty.\tag{2.2}$$

From (1.5), (2.1) is equivalent to

$$\lim_{t\to\infty}t^{-1}\log\mathbb{E}\exp\left\{\Big(\int_{-\infty}^{\infty}\prod_{j=1}^{m}L_j^{p}(t,x)dx\Big)^{1/mp}\right\}$$
$$=m^{-(mp-1)/(mp+1)}\sup_{g\in\mathcal{F}}\left\{\Big(\int_{-\infty}^{\infty}|g(x)|^{2mp}dx\Big)^{1/mp}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.\tag{2.3}$$

Our starting point is the following result based on the Feynman-Kac formula (see, e.g., Remillard (1998)):

$$\lim_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\int_0^t f\big(W(s)\big)ds\right\}$$
$$= \sup_{g\in\mathcal{F}}\left\{\int_{-\infty}^{\infty} f(x)g^2(x)dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}. \tag{2.4}$$

where f can be any measurable, bounded function on $(-\infty,\infty)$.

To establish the lower bound for (2.3), we first consider the special case $m = 1$ and real $p > 1$. We claim that for any $a > 0$ and $\gamma > 0$,

$$\liminf_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\gamma\Big(\int_{-a}^{a} L^p(t,x)dx\Big)^{1/p}\right\}$$
$$\geq \sup_{g\in\mathcal{F}}\left\{\gamma\Big(\int_{-a}^{a}|g(x)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}. \tag{2.5}$$

Indeed, if we let $q > 1$ be the conjugate of p defined by $p^{-1}+q^{-1} = 1$ and let f in (2.4) satisfy $f \equiv 0$ outside $[-a,a]$ and

$$\int_{-a}^{a}|f(x)|^q dx = 1, \tag{2.6}$$

Then

$$\Big(\int_{-a}^{a} L^p(t,x)dx\Big)^{1/p} \geq \int_{-\infty}^{\infty} L(t,x)f(x)dx = \int_0^t f\big(W(s)\big)ds.$$

Consequently,

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\gamma\Big(\int_{-a}^{a} L^p(t,x)dx\Big)^{1/p}\right\}$$
$$\geq \sup_{g\in\mathcal{F}}\left\{\gamma\int_{-a}^{a} f(x)g^2(x)dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.$$

Note that the set of f satisfying the above inequality is dense in the unit sphere of $\mathcal{L}^q[-a,a]$ and taking supremum on the right hand side over such f we obtain (2.5). One can easily see how the lower bound of (2.3) follows from (2.5) in the case $m = 1$ and $p > 1$. To extend it to the general case, we prove that, when viewed as stochastic process taking values in the Banach space $\mathcal{L}^p[-a,a]$ ($p > 1$), the truncated, normalized local time $\{t^{-1}L(t,x);\ -a \leq x \leq a\}$ is exponentially tight: For any $M > 0$, there is a compact set $K_M \subset \mathcal{L}^p[-a,a]$ such that

$$\limsup_{t\to\infty} t^{-1}\log\mathbb{P}\Big\{t^{-1}L(t,\cdot)\notin K_M\Big\} \leq -M. \tag{2.7}$$

Indeed, by Lemma 3.4 of Donsker-Varadhan (1977), for any $\epsilon > 0$

$$\lim_{\delta\to 0^+}\limsup_{t\to\infty} t^{-1}\log\mathbb{P}\Big\{\sup_{|x-y|\leq\delta}|L(t,x)-L(t,y)| > \epsilon t\Big\} = -\infty. \tag{2.8}$$

Due to joint continuity of $L(t,x)$ as function of t and x, this can be slightly strengthened into: for any $\epsilon > 0$ and $N > 0$, there is a $\delta > 0$ such that

$$\sup_{t\geq 1} t^{-1}\log \mathbb{P}\Big\{ \sup_{|x-y|\leq\delta} |L(t,x)-L(t,y)| > \epsilon t\Big\} \leq -N.$$

Hence, for any integer $k \geq 1$, there is a $\delta_k > 0$ such that

$$\sup_{t\geq 1} t^{-1}\log \mathbb{P}\Big\{ \sup_{|x-y|\leq\delta_k} |L(t,x)-L(t,y)| > k^{-1}t\Big\} \leq -kM.$$

We take K_M as the closure (in $\mathcal{L}^p[-a,a]$) of the set

$$A = \bigcap_{k=1}^{\infty}\Big\{f;\;\; f\geq 0,\;\; \int_{-a}^{a} f(x)dx \leq 1 \text{ and } \sup_{|x-y|\leq\delta_k} |f(x)-f(y)| \leq k^{-1}\Big\}.$$

Note that A is an equi-continuous family. According to the Ascoli theorem, A is relatively compact when viewed as a subset of $C[-a,a]$. We claim that A is also relatively compact in $\mathcal{L}^p[-a,a]$ – therefore K_M is compact in $\mathcal{L}^p(\mathbb{R})$. This follows from the fact that for any sequence $\{f_n\} \subset A$, the uniform convergence $\sup_{x\in[-a,a]} |f_n(x) - f(x)| \to 0$ leads to $\mathcal{L}^p$-convergence $\int_{-a}^{a} |f_n(x) - f(x)|^p dx \to 0$ as $n \to \infty$. Furthermore, for any $t \geq 1$,

$$\begin{aligned}\mathbb{P}\Big\{t^{-1}L(t,\cdot)\notin K_M\Big\} &\leq \sum_{k=1}^{\infty}\mathbb{P}\Big\{ \sup_{|x-y|\leq\delta_k} |L(t,x)-L(t,y)| > k^{-1}t\Big\}\\ &\leq \sum_{k=1}^{\infty} e^{-kMt} = \big(1-e^{-Mt}\big)^{-1}e^{-Mt}\end{aligned}$$

which leads to (2.7).

Next we present the proof of the lower bound for (2.3) in the general case. Note that the functional Ψ defined by

$$\Psi(f_1,\cdots,f_m) = \frac{1}{m}\sum_{j=1}^{m}\bigg(\int_{-a}^{a}|f_j(x)|^{mp}dx\bigg)^{1/mp} - \bigg(\int_{-a}^{a}\prod_{j=1}^{m}|f_j(x)|^{p}dx\bigg)^{1/mp}$$

is non-negative for $mp \geq 1$ and continuous on the Banach space $\otimes_{j=1}^{m}\mathcal{L}_j^{mp}[-a,a]$, and that $\Psi \equiv 0$ on the diagonal

$$\{(f_1,\cdots,f_m);\quad f_1 = \cdots = f_m\}.$$

Hence, for given $\epsilon > 0$ and any $f \in \mathcal{L}^{mp}[-a,a]$ there exists a $\delta = \delta(f,\epsilon) > 0$ such that

$$\Psi(f_1,\cdots,f_m) \leq \epsilon \qquad \text{for } f_j \in B(f,\delta),\;\; \forall 1\leq j\leq m$$

where $B(f, \delta)$ stands for the open ball in $\mathcal{L}^{mp}[-a, a]$ with the center f and the radius δ. Therefore, if we view $L_j(t, \cdot)$ $(1 \le j \le m)$ as stochastic processes taking values in $\mathcal{L}^{mp}[-a, a]$ by limiting the space variable x to $[-a, a]$, we have

$$\begin{aligned}
&\mathbb{E} \exp\left\{\left(\int_{-a}^{a} \prod_{j=1}^{m} L_j^p(t, x)dx\right)^{1/mp}\right\} \\
&\quad \ge e^{-t\epsilon}\mathbb{E}\left(\exp\left\{\frac{1}{m}\sum_{j=1}^{m}\left(\int_{-a}^{a} L_j^{mp}(t, x)dx\right)^{1/mp}\right\};\right. \\
&\qquad \left. t^{-1}L_j(t, \cdot) \in B(f, \delta)\ \forall 1 \le j \le m\right) \\
&\quad = e^{-t\epsilon}\left(\mathbb{E}\left(\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\};\ t^{-1}L(t, \cdot) \in B(f, \delta)\right)\right)^m.
\end{aligned}$$

Let $K_M \subset \mathcal{L}^{mp}(\mathbb{R})$ be the compact set given by (2.7) (with p being replaced by mp). Then by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\};\ t^{-1}L(t, \cdot) \notin K_M\right) \\
&\quad \le \left(\mathbb{E} \exp\left\{\frac{2}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\}\right)^{1/2}\left(\mathbb{P}\left\{t^{-1}L(t, \cdot) \notin K_M\right\}\right)^{1/2}.
\end{aligned}$$

Note that

$$\limsup_{t\to\infty} t^{-1} \log \mathbb{E} \exp\left\{\frac{2}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\} < \infty.$$

Hence, for sufficiently large M,

$$\limsup_{t\to\infty} t^{-1} \log \mathbb{E}\left(\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\};\ t^{-1}L(t, \cdot) \notin K_M\right) < 0$$

which leads to

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\};\ t^{-1}L(t, \cdot) \in K_M\right) \\
&\quad \sim \mathbb{E} \exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\} \qquad (t \to \infty).
\end{aligned}$$

Let $\{B(g_1, \delta_1), \cdots, B(g_N, \delta_N)\}$ be a collection of finite number of open balls in $\mathcal{L}^{mp}[-a, a]$ that covers K_M. Then

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\};\ t^{-1}L(t, \cdot) \in K_M\right) \\
&\quad \le \sum_{i=1}^{N} \mathbb{E}\left(\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t, x)dx\right)^{1/mp}\right\};\ t^{-1}L(t, \cdot) \in B(g_i, \delta_i)\right).
\end{aligned}$$

Therefore,

$$\liminf_{t\to\infty} t^{-1}\log \max_{1\le i\le N}\left\{\mathbb{E}\left(\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t,x)dx\right)^{1/mp}\right\};\right.\right.$$
$$\left.\left. t^{-1}L(t,\cdot)\in B(g_i,\delta_i)\right)\right\}$$
$$\ge \liminf_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t,x)dx\right)^{1/mp}\right\}.$$

Summarizing the above discussion, we have

$$\liminf_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\left(\int_{-a}^{a}\prod_{j=1}^{m} L_j^{p}(t,x)dx\right)^{1/mp}\right\}$$
$$\ge -\epsilon + m\liminf_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\frac{1}{m}\left(\int_{-a}^{a} L^{mp}(t,x)dx\right)^{1/mp}\right\}$$
$$\ge -\epsilon + m\sup_{g\in\mathcal{F}}\left\{\frac{1}{m}\left(\int_{-a}^{a}|g(x)|^{2mp}dx\right)^{1/mp} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}$$

where the last step follows from (2.5) with $\gamma = m^{-1}$ and p being replaced by mp. Letting $\epsilon\to 0$ and then taking supremum over $a>0$ leads to the lower bound

$$\liminf_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\left(\int_{-\infty}^{\infty}\prod_{j=1}^{m} L_j^{p}(t,x)dx\right)^{1/mp}\right\}$$
$$\ge m\sup_{g\in\mathcal{F}}\left\{\frac{1}{m}\left(\int_{-\infty}^{\infty}|g(x)|^{2mp}dx\right)^{1/mp} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}$$

which is the lower bound in (2.3) by using the substitution $g(x)\mapsto m^{-mp/2(2mp+1)} g\big(m^{-mp/(2mp+1)}x\big)$. We thus finished the proof of the lower bound for (2.3).

To establish the upper bound for (2.3), we first deal with the case $m=1$ and $p>1$. That is, we shall prove

$$\limsup_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\left(\int_{-\infty}^{\infty} L^{p}(t,x)dx\right)^{1/p}\right\}$$
$$\le \sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}. \tag{2.9}$$

Before the proof, let us mention that the situation we face is quite different from the one in the proof of the lower bound. Indeed, the approach of truncation is no longer working as the tails of

$$\int_{a}^{\infty} L^{p}(t,x)dx \quad\text{and}\quad \int_{-\infty}^{-a} L^{p}(t,x)dx$$

are too heavy to be cut off. In fact, based on (2.4) and by an argument similar to the one for (2.5), we have

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-1}\log\mathbb{E}\,\exp\left\{\left(\int_a^\infty L^p(t,x)dx\right)^{1/p}\right\}\\
&\ge \sup_{g\in\mathcal{F}}\left\{\left(\int_a^\infty |g(x)|^{2p}dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^\infty |g'(x)|^2dx\right\}\\
&= \sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^\infty |g(x)|^{2p}dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^\infty |g'(x)|^2dx\right\}
\end{aligned}$$

where the second step follows from shifting the variable x. This shows that no matter how large a is, the tails have the same weight as the whole integral $\int_{-\infty}^\infty L^p(t,x)dx$ in the sense of large deviation. It also indicates that $\{t^{-1}L(t,\cdot)\}$ is no longer exponentially tight when embedded into $\mathcal{L}^p(\mathbb{R})$ without truncation. That is why we do truncation at a fixed level $a>0$ in the proof of the lower bound.

We shall localize the value space by mapping the Brownian path into a compact space, an approach developed by Donsker-Varadhan (1975) in study of the Wiener sausage (see, also Mansmann (1991) for an application close to our situation). Let $M>0$ be fixed and let T_M be the set of equivalent classes under the equivalence relation on $\mathbb{R}$ defined by $x\sim y$ if $x-y=M$. Let $\lambda(dx)$ be the Lebesgue (Haar) measure on the compact group T_M. Let $W_*(t)$ be the image of $W(t)$ under the quotient map $x\in\mathbb{R}\mapsto \bar{x}\in T_M$. In the literature, the process $W_*(t)$ is called the Brownian motion on T_M. It can be seen that $W_*(t)$ is a Markov process with independent increments. It can also be verified that the process

$$L_*(t,\bar{x})=\sum_{k\in\mathbb{Z}}L(t,x+kM)\qquad t\ge 0\quad x\in\mathbb{R}$$

is the local time of $W_*(t)$. That is, for each $t\ge 0$, $L_*(t,\cdot)$ is the density of the occupation measure $\int_0^t I_{\{W_*(s)\in A\}}ds$, $A\subset T_M$, with respect to $\lambda(dx)$. Note that

$$\begin{aligned}
\int_{-\infty}^\infty L^p(t,x)dx &= \sum_{k\in\mathbb{Z}}\int_0^M L^p(t,x+kM)dx\\
&\le \int_0^M\left(\sum_{k\in\mathbb{Z}}L(t,x+kM)\right)^p dx=\int_{T_M}\big(L_*(t,\bar{x})\big)^p\lambda(d\bar{x}).
\end{aligned}$$

For any measurable function f on T_M, define $f_*(x)=f(\bar{x})$. Then f_* is a periodic function with the period M and

$$\begin{aligned}
\int_{T_M} f(\bar{x})L_*(t,\bar{x})\lambda(d\bar{x}) &= \int_0^M f_*(x)\sum_{k\in\mathbb{Z}}L(t,x+kM)dx\\
&= \int_{-\infty}^\infty f_*(x)L(t,x)dx=\int_0^t f_*\big(W(s)\big)ds.
\end{aligned}$$

From (2.4),

$$\lim_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\int_{T_M} f(\bar{x})L_*(t,\bar{x})\lambda(d\bar{x})\right\}$$
$$= \sup_{g\in\mathcal{F}}\left\{\int_{-\infty}^{\infty} f_*(x)g^2(x)dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}$$
$$= \sup_{g\in\mathcal{F}}\left\{\int_0^M f_*(x)\Big(\sum_{k\in\mathbb{Z}} g^2(x+kM)\Big)dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}. \quad (2.10)$$

In the following lemma, we show that the process $\{L_*(t,\cdot)\}$ can be embedded into the separable Banach space $\mathcal{L}^p(T_M)$ and that $\{t^{-1}L_*(t,\cdot)\}$ is exponentially tight in $\mathcal{L}^p(T_M)$.

To this end, let us recall some concepts related to Minkowski functionals. Given a Banach space $\mathbb{B}$, a set $K\subset\mathbb{B}$ is called positively balanced, if $\lambda x\in K$ whenever $\lambda\in[0,1]$ and $x\in K$. The Minkowski functional $q_K(\cdot)$ of a convex and positively balanced set K is defined by

$$q_K(x) = \inf\{\lambda>0;\quad x\in\lambda K\}$$

with the customary convention that $\inf\phi = \infty$. Then $q_K(\cdot)$ is subadditive and positively homogeneous:

$$q_K(x+y)\le q_K(x)+q_K(y) \quad\text{and}\quad q_K(\lambda x)=\lambda q_K(x) \qquad x,y\in\mathbb{B},\quad \lambda\ge 0.$$

Lemma 2.1. *For each $t\ge 0$,*

$$\mathbb{P}\big\{L_*(t,\cdot)\in\mathcal{L}^p(T_M)\big\} = 1. \quad (2.11)$$

Moreover, there is a compact, convex, positively balanced subset $K\in\mathcal{L}^p(T_M)$ such that

$$\limsup_{t\to\infty} t^{-1}\log\mathbb{E}\exp\Big\{q_K\big(L_*(t,\cdot)\big)\Big\} < \infty \quad (2.12)$$

where $q_K(\cdot)$ is the Minkowski functional of K.

Proof. For simplicity we only prove (2.11) in the case $t=1$ and (2.12) in the case $t=n$ runs along positive integers. Note that

$$\left(\int_{T_M} L_*^p(1,\bar{x})\lambda(d\bar{x})\right)^{1/p}$$
$$= \left(\int_0^M\left(\sum_{k\in\mathbb{Z}} L(1,x+kM)\right)^p dx\right)^{1/p}$$
$$\le \sum_{k\in\mathbb{Z}}\left(\int_0^M L^p(1,x+kM)dx\right)^{1/p}$$
$$= \sum_{k\in\mathbb{Z}}\left(\int_0^M L^p(1,x+kM)dx\right)^{1/p} I_{\{kM\in[\min_{0\le s\le 1}W(s),\ \max_{0\le s\le 1}W(s)+M]\}}$$

$$\leq \left(\sum_{k\in\mathbb{Z}} I_{\{kM \in [\min_{0\leq s\leq 1} W(s),\ \max_{0\leq s\leq 1} W(s)+M]\}} \right)^{1/q}$$
$$\times \left(\sum_{k\in\mathbb{Z}} \int_0^M L^p(1, x+kM)dx \right)^{1/p}$$
$$\leq 2M^{-1}\Big(\max_{0\leq s\leq 1} |W(s)| + 1 \Big)^{1/q} \Big(\int_{-\infty}^{\infty} L^p(1,x)dx \Big)^{1/p}$$
$$\leq 2M^{-1}\Big(\max_{0\leq s\leq 1} |W(s)| + 1 \Big)^{(p-1)/p} \sup_{x\in\mathbb{R}} L^{1/q}(1,x)$$
$$\leq M^{-1}\Big(\Big(\max_{0\leq s\leq 1} |W(s)| + 1 \Big)^{2/q} + \sup_{x\in\mathbb{R}} L^{2/q}(1,x) \Big).$$

Note that there is $\gamma_0 > 0$ such that

$$\mathbb{E}\exp\Big\{\gamma_0 \sup_{x\in\mathbb{R}} L^2(1,x)\Big\} < \infty \quad \text{and} \quad \mathbb{E}\exp\Big\{\gamma_0 \max_{0\leq s\leq 1} |W(s)|^2\Big\} < \infty.$$

In fact, the first is given in Theorem 1.7 of Borodin (1986), and the second is well known for any norm of Gaussian elements. Hence,

$$\mathbb{E}\exp\left\{\gamma\Big(\int_{T_M} L_*^p(1,\bar{x})\lambda(d\bar{x}) \Big)^{1/p}\right\} < \infty, \quad \text{for any } \gamma > 0. \tag{2.13}$$

In particular, we have (2.11).

Similarly, for any $\bar{y}, \bar{z} \in T_M$,

$$\Big(\int_{T_M} \big(L_*(1,\bar{x}+\bar{y}) - L_*(1,\bar{x}+\bar{z})\big)^p \lambda(d\bar{x}) \Big)^{1/p}$$
$$= \Big(\int_{T_M} \big(L_*(1,\bar{x}) - L_*(1,\bar{x}+\bar{z}-\bar{y})\big)^p \lambda(d\bar{x}) \Big)^{1/p}$$
$$\leq 2\cdot 2^{1/p} M^{-1} \sup_{x\in\mathbb{R}} \big(L(1,x) - L(1,x+z-y)\big)^{1/q} \Big(\max_{0\leq s\leq 1} |W(s)| + 1 \Big)^{1/q}.$$

By continuity of Brownian local time we have established the continuity of the random function $L_*(1,\bar{y}+\cdot)$ on $\mathcal{L}^p(T_M)$. It is well known, as a general result, that each random variable taking values in a separable Banach space is tight. In particular, $L_*(1,\bar{y}+\cdot)$ is tight for each $\bar{y} \in T_M$: Given $\epsilon > 0$ there is a compact set $C_y \in \mathcal{L}^p(T_M)$ such that $\mathbb{P}\{L_*(1,\bar{y}+\cdot) \in C_y\} \geq 1-\epsilon$. By continuity of $L_*(1,\bar{y}+\cdot)$ as a $\mathcal{L}^p(T_M)$-valued function of $\bar{y}$ and by the compactness of T_M, the family $\big\{L_*(1,\bar{y}+\cdot)\big\}_{\bar{y}\in T_M}$ is uniformly tight: Given $\epsilon > 0$ there is a compact set $C \in \mathcal{L}^p(T_M)$ such that $\inf_{y\in T_M} \mathbb{P}\{L_*(1,\bar{y}+\cdot) \in C\} \geq 1-\epsilon$. Note that for any $\bar{y} \in T_M$, $\int_{T_M} L_*^p(t,\bar{x}+\bar{y})\lambda(d\bar{x}) = \int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})$. Hence from (2.13)

$$\sup_{\bar{y}\in T_M} \mathbb{E}_{\bar{y}} \exp\left\{\gamma\Big(\int_{T_M} L_*^p(1,\bar{x})\lambda(d\bar{x}) \Big)^{1/p}\right\} < \infty \qquad \forall\gamma > 0.$$

By Theorem 3.1 in de Acosta (1985), there is a compact, convex, positively balanced subset $K \in \mathcal{L}^p(T_M)$, such that

$$\sup_{\bar{y}\in T_M} \mathbb{E}_{\bar{y}} \exp\left\{q_K\big(L_*(1,\cdot)\big)\right\} < \infty. \tag{2.14}$$

By the triangular inequality and the Markov property,

$$\begin{aligned}
&\mathbb{E}\exp\left\{q_K\big(L_*(n,\cdot)\big)\right\}\\
&\quad\le \mathbb{E}\exp\left\{q_K\big(L_*(n-1,\cdot)\big) + q_K\big(L_*(n,\cdot) - L_*(n-1,\cdot)\big)\right\}\\
&\quad\le \mathbb{E}\exp\left\{q_K\big(L_*(n-1,\cdot)\big)\right\} \sup_{\bar{y}\in T_M} \mathbb{E}_{\bar{y}} \exp\left\{q_K\big(L_*(1,\cdot)\big)\right\}.
\end{aligned}$$

Repeating this procedure gives

$$\mathbb{E}\exp\left\{q_K\big(L_*(n,\cdot)\big)\right\} \le \left(\sup_{\bar{y}\in T_M} \mathbb{E}_{\bar{y}} \exp\left\{q_K\big(L_*(1,\cdot)\big)\right\}\right)^n.$$

Hence (2.12) follows from (2.14). □

Let $\epsilon > 0$ and $\gamma > 0$ be fixed and let $K \subset \mathcal{L}^p(T_M)$ be the compact set given in Lemma 2.1. By the fact that the set of measurable, bounded functions on T_M is dense in the unit ball of $\mathcal{L}^q(T_M)$, and by the Hahn-Banach Theorem, for each $h \in \gamma K$, there is a bounded function f such that $\int_{T_M} |f(\bar{x})|^q \lambda(d\bar{x}) = 1$, and

$$\int_{T_M} f(\bar{x})h(\bar{x})\lambda(d\bar{x}) > \left(\int_{T_M} |h(\bar{x})|^p \lambda(d\bar{x})\right)^{1/p} - \epsilon.$$

Consequently, there are finitely many bounded functions $f_1, \cdots, f_N$ in the unit sphere of $\mathcal{L}^q(T_M)$ such that

$$\left(\int_{T_M} |h(\bar{x})|^p \lambda(d\bar{x})\right)^{1/p} < \max_{1\le i\le N} \int_{T_M} f_i(\bar{x})h(\bar{x})\lambda(d\bar{x}) + \epsilon \qquad \forall h \in \gamma K.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\left(\int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})\right)^{1/p}\right\};\ t^{-1}L_*(t,\cdot) \in \gamma K\right)\\
&\qquad \le e^{\epsilon t} \sum_{i=1}^N \mathbb{E}\exp\left\{\int_{T_M} f_i(x)L(t,x)dx\right\}.
\end{aligned}$$

In view of (2.10),

$$\begin{aligned}
&\limsup_{t\to\infty} t^{-1}\log\mathbb{E}\left(\exp\left\{\left(\int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})\right)^{1/p}\right\};\ t^{-1}L_*(t,\cdot) \in \gamma K\right)\\
&\quad \le \epsilon + \max_{1\le i\le N} \sup_{g\in\mathcal{F}} \left\{\int_0^M f_i(\bar{x})\Big(\sum_{k\in\mathbb{Z}} g^2(x+kM)\Big)dx - \frac{1}{2}\int_{-\infty}^{\infty} |g'(x)|^2 dx\right\}\\
&\quad \le \epsilon + \sup_{g\in\mathcal{F}} \left\{\left(\int_0^M \Big(\sum_{k\in\mathbb{Z}} g^2(x+kM)\Big)^p dx\right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |g'(x)|^2 dx\right\}
\end{aligned}$$

where the second step follows from Hölder's inequality and the fact $\int_0^M |f_i(\bar{x})|^q dx = 1$ for $1 \le i \le N$. Letting $\epsilon \longrightarrow 0$ gives

$$\limsup_{t\to\infty} t^{-1}\log \mathbb{E}\left(\exp\left\{\left(\int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})\right)^{1/p}\right\};\ t^{-1}L_*(t,\cdot)\in\gamma K\right)$$
$$\le \sup_{g\in\mathcal{F}}\left\{\left(\int_0^M \left(\sum_{k\in\mathbb{Z}} g^2(x+kM)\right)^p dx\right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2 dx\right\}. \quad (2.15)$$

By the Cauchy-Schwarz inequality, on the other hand,

$$\mathbb{E}\left(\exp\left\{\left(\int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})\right)^{1/p}\right\};\ t^{-1}L_*(t,\cdot)\notin\gamma K\right)$$
$$\le \left(\mathbb{E}\exp\left\{2\left(\int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})\right)^{1/p}\right\}\right)^{1/2}\left(\mathbb{P}\left\{t^{-1}L_*(t,\cdot)\notin\gamma K\right\}\right)^{1/2}.$$

Note that (2.13) implies that

$$\limsup_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{2\left(\int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})\right)^{1/p}\right\}\equiv C_1<\infty,$$

since for any integer n,

$$\left(\int_{T_M} L_*^p(n,\bar{x})\lambda(d\bar{x})\right)^{1/p}\le\sum_{k=1}^{n}\left(\int_{T_M}|L_*(k,\bar{x})-L_*(k-1,\bar{x})|^p\lambda(d\bar{x})\right)^{1/p}$$

with i.i.d. terms. Furthermore, by the Chebyshev inequality,

$$\mathbb{P}\left\{t^{-1}L_*(t,\cdot)\notin\gamma K\right\}=\mathbb{P}\left\{q_K\big(L_*(t,\cdot)\big)\ge\gamma t\right\}\le e^{-\gamma t}\mathbb{E}\exp\left\{q_K\big(L_*(t,\cdot)\big)\right\}.$$

Hence, by Lemma 2.1 there is a constant $C_2>0$ independent of γ, such that

$$\limsup_{t\to\infty} t^{-1}\log\mathbb{P}\left\{t^{-1}L_*(t,\cdot)\notin\gamma K\right\}\le-\gamma+C_2.$$

Combining above observations we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left(\exp\left\{\left(\int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})\right)^{1/p}\right\};\ t^{-1}L_*(t,\cdot)\notin\gamma K\right)$$
$$\le (C_1+C_2-\gamma)/2. \quad (2.16)$$

Note that $\gamma>0$ can be arbitrarily large. Hence from (2.15) and (2.16) we have

$$\limsup_{t\to\infty} t^{-1}\log\mathbb{E}\exp\left\{\left(\int_{T_M} L_*^p(t,\bar{x})\lambda(d\bar{x})\right)^{1/p}\right\}$$
$$\le \sup_{g\in\mathcal{F}}\left\{\left(\int_0^M\left(\sum_{k\in\mathbb{Z}} g^2(x+kM)\right)^p dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.$$

Thus, from (2.10) we obtain

$$\limsup_{t\to\infty} t^{-1}\log \mathbb{E}\,\exp\left\{\left(\int_{-\infty}^{\infty} L^p(t,x)dx\right)^{1/p}\right\}$$
$$\le \sup_{g\in\mathcal{F}}\left\{\left(\int_0^M \Big(\sum_{k\in\mathbb{Z}} g^2(x+kM)\Big)^p dx\right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |g'(x)|^2 dx\right\}$$

for all $M > 0$. Letting $M \longrightarrow \infty$ we have (2.9) by Lemma 7.1 in Section 7.

Finally, we come to the proof of the upper bound necessary for (2.3) in the case $m > 1$. Note that for $mp \ge 1$,

$$\left(\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(t,x)dx\right)^{1/mp} \le \frac{1}{m}\sum_{j=1}^m\left(\int_{-\infty}^{\infty} L_j^{mp}(t,x)dx\right)^{1/mp}$$

which gives that

$$\mathbb{E}\,\exp\left\{\left(\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(t,x)dx\right)^{1/mp}\right\}$$
$$\le \left(\mathbb{E}\,\exp\left\{\frac{1}{m}\left(\int_{-\infty}^{\infty} L^{mp}(t,x)dx\right)^{1/mp}\right\}\right)^m$$
$$= \left(\mathbb{E}\,\exp\left\{\left(\int_{-\infty}^{\infty} L^{mp}\Big(m^{-2mp/(mp+1)}t, x\Big)dx\right)^{1/mp}\right\}\right)^m$$

where the second step follows from (1.5). Replacing p by mp in (2.9) we obtain

$$\limsup_{t\to\infty} t^{-1}\log \mathbb{E}\,\exp\left\{\left(\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(t,x)dx\right)^{1/mp}\right\}$$
$$\le m^{-(mp-1)/(mp+1)}\sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty} g^{2mp}(x)dx\right)^{1/mp} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.$$

Therefore, (2.3) follows.

3. Weak law of convergence

In the rest of the paper, we use $[x]$ to denote the integer part of $x \in \mathbb{R}$. Note that

$$\sum_{x\in\mathbb{Z}}\prod_{j=1}^m l_j^p(n,x) = \int_{-\infty}^{\infty}\prod_{j=1}^m l_j^p(n,[x])dx = n^{1/2}\int_{-\infty}^{\infty}\prod_{j=1}^m l_j^p(n,[n^{1/2}x])dx.$$

So, Theorem 1.2 is equivalent to

$$n^{-mp/2}\int_{-\infty}^{\infty}\prod_{j=1}^m l_j^p(n,[n^{1/2}x])dx \xrightarrow{d} \sigma^{-(mp-1)}\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(1,x)dx.$$

Note that the function

$$\eta(f_1, \cdots, f_m) = \int_{-\infty}^{\infty} \prod_{j=1}^{m} f_j^p(x)dx$$

is continuous on the Banach space $\otimes_{j=1}^m \mathcal{L}_j^{mp}(\mathbb{R})$, we need only to show that as a $\mathcal{L}^{mp}(\mathbb{R})$-valued process, $l(n, [n^{1/2}\cdot])/\sqrt{n}$ weakly converges to $\sigma^{-1}L(1, \sigma^{-1}\cdot)$ as $n \to \infty$. Replacing mp by p, we need only to establish the following result.

Proposition 3.1. *For any real number $p > 1$, we have the weak convergence*

$$l(n, [n^{1/2}\cdot])/\sqrt{n} \xrightarrow{d} \sigma^{-1}L(1, \sigma^{-1}\cdot) \qquad (n \to \infty)$$

in the space $\mathcal{L}^p(\mathbb{R})$.

Proof. We first prove Proposition 3.1 under the extra assumption that $\{S(n)\}_{n\geq 1}$ is aperiodic, that is, the greatest common factor of $\{n \geq 1; \mathbb{P}\{S(n) = 0\} > 0\}$ is 1.

According to Theorem 2.4 of de Acosta (1970), we need to check two things: First, there is a w^*-dense set $D \in \mathcal{L}^q(\mathbb{R})$ such that for each $f \in D$,

$$n^{-1/2} \int_{-\infty}^{\infty} f(x)l(n, [n^{1/2}x])dx \xrightarrow{\mathrm{d}} \sigma^{-1} \int_{-\infty}^{\infty} f(x)L(1, x)dx,$$

and second, the sequence $\{l(n, [n^{1/2}\cdot])\}_{n\geq 1}$ is *flatly concentrated*, that is, for every $\epsilon > 0$, there is a finite-dimensional subspace F o f $\mathcal{L}^p(\mathbb{R})$ such that

$$\liminf_{n\to\infty} \mathbb{P}\Big\{n^{-1/2}l(n, [n^{1/2}\cdot]) \in F^\epsilon\Big\} \geq 1 - \epsilon$$

where F^ϵ is the ϵ-neighborhood of F.

To verify the first assertion, we take D as the set of uniformly continuous functions in $\mathcal{L}^q(\mathbb{R})$. For each $f \in D$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)l(n, [n^{1/2}x])dx &= \frac{1}{n^{1/2}} \int_{-\infty}^{\infty} f(x/n^{1/2})l(n, [x])dx \\ &= \frac{1}{n^{1/2}}\Big(o(n) + \sum_{x\in\mathbb{Z}} f(n^{-1/2}x)l(n, x)\Big) = \frac{1}{n^{1/2}}\Big(o(n) + \sum_{k=1}^{n} f(n^{-1/2}S_k)\Big)\end{aligned}$$

as $n \to \infty$. By invariance principle,

$$\begin{aligned}n^{-1/2} \int_{-\infty}^{\infty} f(x)l(n, [n^{1/2}x])dx &\xrightarrow{\mathrm{d}} \int_0^1 f\big(\sigma W(s)\big)ds \\ &= \sigma^{-1} \int_{-\infty}^{\infty} f(x)L(1, \sigma^{-1}x)dx.\end{aligned}$$

To check the second condition, we consider the partition $x_j = j\delta$, $j \in \mathbb{Z}$, for a given $\delta > 0$. Define

$$F = \Big\{f;\quad f(x) = 0 \text{ outside } [-M, M] \text{ and } f(x) \text{ equals constant between any two neighboring partition points}\Big\}$$

where the constant $M > 0$ is to be specified later. Clearly, F is a finite-dimensional subspace of $\mathcal{L}^p(\mathbb{R})$. Define

$$l_n(x) = \begin{cases} l(n, [n^{1/2}x_j]) & \text{for} \quad x_j \le x < x_{j+1} \\ l(n, [n^{1/2}x_{-j}]) & \text{for} \quad x_{-(j+1)} \le x < x_{-j}. \end{cases}$$

On the event $\{\max_{k\le n} |S_k| \le Mn^{1/2}\}$, $l_n(\cdot) \in F$. Therefore,

$$\mathbb{P}\Big\{n^{-1/2} l(n, [n^{1/2}\cdot]) \notin F^\epsilon\Big\} \le \mathbb{P}\big\{\max_{k\le n} |S_k| > Mn^{1/2}\big\} + \mathbb{P}\Big\{\Big(\int_{-\infty}^{\infty} |l(n, [n^{1/2}x]) - l_n(x)|^p dx\Big)^{1/p} \ge \epsilon n^{1/2}\Big\}.$$

By the invariance principle

$$\lim_{M\to\infty} \lim_{n\to\infty} \mathbb{P}\big\{\max_{k\le n} |S_k| > Mn^{1/2}\big\} = \lim_{M\to\infty} \mathbb{P}\big\{\max_{0\le s\le 1} |W(s)| \ge M\big\} = 0.$$

Hence, we need only to show, using symmetry,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n^{p/2}} \int_0^\infty \mathbb{E}\,|l(n, [n^{1/2}x]) - l_n(x)|^p dx = 0.$$

Define, for any $x \ge 0$,

$$\tau_n(x) = \inf\big\{k \ge 1; [n^{1/2}x_j] \le S_k \le [n^{1/2}x]\big\} \quad \text{for} \quad x \in [x_j, x_{j+1}).$$

For any $j \ge 0$ and $x \in [x_j, x_{j+1})$,

$$\begin{aligned}
&\mathbb{E}\,|l(n, [n^{1/2}x]) - l_n(x)|^p \\
&\quad = \mathbb{E}\,\Big|\big(l(n, [n^{1/2}x]) - l(\tau_n(x), [n^{1/2}x])\big) - \big(l(n, [n^{1/2}x_j]) - l(\tau_n(x), [n^{1/2}x_j])\big) \\
&\qquad + \big(I_{\{S_{\tau_n(x)} = [n^{1/2}x]\}} - I_{\{S_{\tau_n(x)} = [n^{1/2}x_j]\}}\big)\Big|^p I_{\{\tau_n(x) \le n\}} \\
&\quad = \sum_{k=1}^n \mathbb{P}\big\{\tau_n(x) = k\big\}\mathbb{E}\,\big|l(n-k, 0) - l(n-k, [n^{1/2}x] - [n^{1/2}x_j]) + 1\big|^p \\
&\quad \le \sum_{k=1}^n \mathbb{P}\big\{\tau_n(x) = k\big\} \sup_{|y|\le 2\delta n^{1/2}} \mathbb{E}\,\big|l(n-k, 0) - l(n-k, y) + 1\big|^p \\
&\quad \le \max_{k\le n} \sup_{|y|\le 2\delta n^{1/2}} \mathbb{E}\,\big|l(k, 0) - l(k, y) + 1\big|^p \cdot \sum_{k=1}^n \mathbb{P}\big\{\tau_n(x) = k\big\}
\end{aligned}$$

where the second equality follows from the Markov property and symmetry of the random walk. Observe that

$$\begin{aligned}\int_0^\infty \sum_{k=1}^n \mathbb{P}\{\tau_n(x)=k\}dx &= \int_0^\infty \mathbb{P}\{\tau_n(x)\le n\}dx\\ &= \sum_{j=0}^\infty \int_{x_j}^{x_{j+1}} \mathbb{P}\{\tau_n(x)\le n\}dx\\ &\le \delta \sum_{j=0}^\infty \mathbb{P}\{\max_{0\le k\le n} S_k \ge x_j n^{1/2}\}\\ &\le C\frac{1}{n^{1/2}}\mathbb{E}\max_{0\le k\le n} S_k \longrightarrow C\mathbb{E}\max_{0\le s\le 1} W(s) < \infty \quad \text{as} \quad n\to\infty\end{aligned}$$

for some constant $C > 0$ independent of δ and n. Therefore, it remains to show

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n^{p/2}}\max_{k\le n}\sup_{|y|\le 2\delta n^{1/2}} \mathbb{E}\left|l(k,0)-l(k,y)\right|^p = 0. \tag{3.1}$$

Indeed,

$$\mathbb{E}\left|l(k,0)-l(k,y)\right|^p \le \left(\mathbb{E}\left|l(k,0)-l(k,y)\right|^{2(p-1)}\right)^{1/2}\left(\mathbb{E}\left|l(k,0)-l(k,y)\right|^2\right)^{1/2}$$

and

$$\mathbb{E}\left|l(k,0)-l(k,y)\right|^{2(p-1)} \le \mathbb{E}\, l^{2(p-1)}(n,0)+\mathbb{E}\, l^{2(p-1)}(n,y) \le 1+2\mathbb{E}\, l^{2(p-1)}(n,0)$$

where the last step follows from Lemma 1 in Chen (2000). By the weak law for local times $l(n,0)/n^{1/2} \xrightarrow{d} \sigma^{-1}|\xi|$ where $\xi \sim N(0,1)$. Through a standard procedure we have

$$\mathbb{E}\, l^{2(p-1)}(n,0) \sim n^{p-1}\sigma^{-2(p-1)}\mathbb{E}\,|\xi|^{2(p-1)}, \quad \text{as} \quad n\to\infty.$$

Let $\psi_i = I_{\{S_i=0\}} - I_{\{S_i=y\}}$ $(i=0,1,2,\cdots)$. Then for any integers k and y with $1\le k\le n$ and $|y|\le 2\delta n^{1/2}$,

$$\begin{aligned}\mathbb{E}\left|l(k,0)-l(k,y)\right|^2 &\le 2\sum_{i=1}^k\sum_{j=i}^k \mathbb{E}\,(\psi_i\psi_j) = 2\sum_{i=1}^k \mathbb{E}\left(\psi_i \sum_{j=0}^{k-i}\mathbb{E}_{S_i}\psi_j\right)\\ &\le 2\sum_{i=1}^n \mathbb{E}\left(|\psi_i|\sum_{j=1}^\infty |\mathbb{E}_{S_i}\psi_j|\right)\\ &\le 2C|y|\sum_{i=1}^n \mathbb{E}\,|\psi_i|\\ &\le 4Cn^{1/2}\delta\{\mathbb{E}\, l(n,0)+\mathbb{E}\, l(n,y)\}\\ &\le 4Cn^{1/2}\delta\{2\mathbb{E}\, l(n,0)+1\} \sim 8C\delta\sigma^{-1}\mathbb{E}\,|\xi| n \quad \text{as} \quad n\to\infty\end{aligned}$$

where $C > 0$ is a universal constant, the fourth step follows from Lemma 7 of Jain and Pruitt (1984) (here the aperiodicity assumption is required) and the sixth from Lemma 1 in Chen (2000). Hence, (3.1) holds.

We now prove Proposition 3.1 without aperiodicity assumption. Let $0 < \lambda < 1$ be fixed and let $\{\delta_n\}_{n\geq 1}$ be a sequence of i.i.d. Bernoulli random variables with common law

$$\mathbb{P}\{\delta_1 = 0\} = 1 - \mathbb{P}\{\delta_1 = 1\} = \lambda.$$

We assume independence between $\{\delta_n\}_{n\geq 1}$ and $\{S_n\}_{n\geq 1}$. Define the renewal sequence $\{\tau_k\}_{k\geq 1}$:

$$\tau_1 = \inf\{n \geq 1;\quad \delta_n = 1\} \quad \text{and} \quad \tau_{k+1} = \inf\{n > \tau_k;\quad \delta_n = 1\}.$$

Then τ_1 has geometric distribution $\mathbb{P}\{\tau_1 = n\} = (1-\lambda)\lambda^{n-1}$, $n \geq 1$, and the sequence $\{S(\tau_k)\}_{k\geq 1}$ is a random walk whose i.i.d increments has the distribution same as that of $S(\tau_1)$. In particular, the random walk $\{S(\tau_k)\}_{k\geq 1}$ is aperiodic and symmetric with the variance equal to $\mathbb{E}\,|S(\tau_1)|^2 = \mathbb{E}\,\tau_1 \mathbb{E}\,S_1^2 = (1-\lambda)^{-1}\sigma^2$. Write $l'(n, x)$ for its local time. By what we have proved,

$$l'(n, [n^{1/2}\cdot])/\sqrt{n} \xrightarrow{d} \sqrt{1-\lambda}\,\sigma^{-1} L(1, \sqrt{1-\lambda}\,\sigma^{-1}\cdot) \qquad (n \to \infty) \tag{3.2}$$

in the space $\mathcal{L}^p(\mathbb{R})$.

Let $t(n) = \delta_1 + \cdots + \delta_n = \max\{k;\ \tau_k \leq n\}$ and notice that for each $x \in \mathbb{Z}$,

$$l'\big(t(n), x\big) = \sum_{k=1}^{t(n)} I_{\{S(\tau_k) = x\}} = \sum_{k=1}^{n} \delta_k I_{\{S(k) = x\}} \qquad n = 1, 2, \cdots.$$

By the law of large numbers, $t(n)/n \xrightarrow{p} 1 - \lambda$ as $n \to \infty$. From (3.2),

$$n^{-1/2} \sum_{k=1}^{n} \delta_k I_{\{S(k) = [n^{1/2}\cdot]\}} \xrightarrow{d} (1-\lambda)\sigma^{-1} L(1, \sqrt{1-\lambda}\,\sigma^{-1}\cdot) \qquad (n \to \infty).$$

Replacing $\{\delta_n\}$ by the Bernoulli sequence $\{1 - \delta_n\}$ gives

$$n^{-1/2} \sum_{k=1}^{n} (1-\delta_k) I_{\{S(k) = [n^{1/2}\cdot]\}} \xrightarrow{d} \lambda\sigma^{-1} L(1, \sqrt{\lambda}\,\sigma^{-1}\cdot) \qquad (n \to \infty).$$

Thus, the desired conclusion follows from the decomposition

$$l(n, [n^{1/2}\cdot]) = \sum_{k=1}^{n} \delta_k I_{\{S(k) = [n^{1/2}\cdot]\}} + \sum_{k=1}^{n} (1-\delta_k) I_{\{S(k) = [n^{1/2}\cdot]\}}$$

and then taking $\lambda \longrightarrow 0^+$. □

4. Exponential moment for rescaled occupation times

As we shall see in the next section, the upper bound for the moderate deviation stated in Theorem 1.3 can be established based on Theorem 1.1 and Theorem 1.2. The harder part is the lower bound. In this section we develop some tools for the lower bound, which can been viewed as a partial extension of (2.4) in the case of random walks. Since exponential moment estimation we establish below may be used in other applications, we state it in a more general form, which holds for multi-dimensional random walks. Limited to this section, $d \geq 1$ is an integer and $S(n) = \sum_{k=1}^{n} X_k$, $n = 1, 2, \cdots$ is an random walk with covariance matrix Γ generated by an $\mathbb{Z}^d$-valued symmetric i.i.d. sequence $\{X_n\}_{n\geq 1}$. We assume that the smallest group which supports $\{S(n)\}_{n\geq 1}$ is $\mathbb{Z}^d$.

Theorem 4.1. *For any bounded continuous function f on $\mathbb{R}^d$ and positive sequence $\{b_n\}$ satisfying (1.11),*

$$\liminf_{n\to\infty} b_n^{-1} \log \mathbb{E} \exp\left\{\frac{b_n}{n}\sum_{k=1}^{n} f\Big((b_n/n)^{1/2} S_k\Big)\right\}$$
$$\geq \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}\langle\nabla g(x), \Gamma\nabla g(x)\rangle dx\right\} \tag{4.1}$$

where $\mathcal{F}_d$ is the set of absolutely continuous functions g on $\mathbb{R}^d$ with

$$\int_{\mathbb{R}^d} |g(x)|^2 dx = 1 \quad \textit{and} \quad \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx < \infty.$$

Note that Theorem 4.1 holds also for random walks with continuous values under certain regularity conditions, and it is natural to expect that the correspondent upper bound holds. What we need in this paper is the lower bound (4.1).

Proof of Theorem 4.1. In view of how we go from the case of aperiodicity to the general case in the proof of Proposition 3.1, we may assume that $\{S_n\}_{n\geq 1}$ is aperiodic. Write $t_n = [n/b_n]$, $\gamma_n = [n/t_n]$ and thus $t_n\gamma_n \leq n < t_n(\gamma_n + 1)$. Therefore

$$\mathbb{E}\exp\left\{\frac{b_n}{n}\sum_{k=1}^{n} f\Big((b_n/n)^{1/2} S_k\Big)\right\}$$
$$\geq \mathbb{E}\exp\Big\{-2|f|_\infty\Big\}\mathbb{E}\exp\left\{\frac{b_n}{n}\sum_{k=t_n+1}^{\gamma_n t_n} f\Big((b_n/n)^{1/2} S_k\Big)\right\}.$$

Hence, we need only to show the lower bound (4.1) for

$$\liminf_{n\to\infty} b_n^{-1}\log\mathbb{E}\exp\left\{\frac{b_n}{n}\sum_{k=t_n+1}^{\gamma_n t_n} f\Big((b_n/n)^{1/2} S_k\Big)\right\}.$$

For each n, define the continuous linear operator Π_n on $\mathcal{L}^2(\mathbb{Z}^d)$ as

$$\Pi_n\xi(x) = \exp\Big\{\frac{b_n}{2n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}x\Big)\Big\}$$
$$\times\mathbb{E}_x\bigg(\exp\Big\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_k\Big) + \frac{b_n}{2n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_{t_n}\Big)\Big\}\xi(S_{t_n})\bigg)$$

where $x \in \mathbb{Z}^d$ and $\xi \in \mathcal{L}^2(\mathbb{Z}^d)$. Due to symmetry of the random walk, Π_n is *self-adjoint*: $\Pi_n^* = \Pi_n$, where Π_n^* is the dual operator of Π_n. Indeed, it is straightforward to find out that the linear operator T_n given by

$$T_n\xi(x) = \exp\Big\{\frac{b_n}{2n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}x\Big)\Big\}\mathbb{E}_x\bigg(\exp\Big\{\frac{b_n}{2n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_1\Big)\Big\}\xi(S_1)\bigg)$$

is self adjoint and $\Pi_n = T_n^{t_n}$.

Let g be a bounded function on $\mathbb{R}^d$ and assume that g is infinitely differentiable, supported by a finite box $[-M, M]^d$ with

$$\int_{\mathbb{R}^d} |g(x)|^2 dx = 1 \tag{4.2}$$

and write

$$\xi_n(x) = g\big((b_n/n)^{1/2}x\big)\cdot\bigg(\sum_{y\in\mathbb{Z}^d} g^2\big((b_n/n)^{1/2}y\big)\bigg)^{-1/2}, \qquad x \in \mathbb{Z}^d.$$

Then

$$\mathbb{E}\exp\Big\{\frac{b_n}{n}\Big(\sum_{k=t_n+1}^{\gamma_n t_n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_k\Big)\Big\}$$
$$= \sum_{x\in\mathbb{Z}^d} P_{t_n}(0,x)\mathbb{E}_x\exp\Big\{\frac{b_n}{n}\Big(\sum_{k=1}^{(\gamma_n-1)t_n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_k\Big)\Big\}$$
$$\geq \frac{1+o(1)}{\sup_y|g(y)|}\Big\{\sum_{y\in\mathbb{Z}^d} g^2\Big(\big(\frac{b_n}{n}\big)^{1/2}y\Big)\Big\}$$
$$\cdot\sum_{x\in\mathbb{Z}^d} P_{t_n}(0,x)\xi_n(x)\exp\Big\{f\Big(\big(\frac{b_n}{n}\big)^{1/2}x\Big)\Big\}$$
$$\times\mathbb{E}_x\bigg(\exp\Big\{\frac{b_n}{n}\Big(\sum_{k=1}^{(\gamma_n-1)t_n-1} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_k\Big)$$
$$+\frac{b_n}{2n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_{(\gamma_n-1)t_n}\Big)\Big\}\xi_n\big(S_{(\gamma_n-1)t_n}\big)\bigg)$$
$$= \frac{1+o(1)}{\sup_y|g(y)|}\Big\{\sum_{y\in\mathbb{Z}^d} g^2\Big(\big(\frac{b_n}{n}\big)^{1/2}y\Big)\Big\}\cdot\sum_{x\in\mathbb{Z}^d} P_{t_n}(0,x)\xi_n(x)\Pi_n^{\gamma_n-1}\xi_n(x)$$

where the last step follows from the Markov property. Note that

$$\sum_{y\in\mathbb{Z}^d} g^2\Big(\big(\frac{b_n}{n}\big)^{1/2}y\Big) \sim \Big(\frac{n}{b_n}\Big)^{d/2}\int_{\mathbb{R}^d}|g(x)|^2dx = \big(\frac{b_n}{n}\big)^{d/2}$$

as $n\to\infty$ and in the light of the Remark of Le Gall and Rosen (1991), p. 661, the aperiodicity of the random walk implies

$$\sup_{x\in\mathbb{Z}}\Big|t_n^{d/2}P_{t_n}(0,x)-(2\pi)^{-d/2}\det(G)^{-1/2}\exp\Big\{-(2t_n)^{-1}\langle x,\Gamma^{-1}x\rangle\Big\}\Big| \longrightarrow 0 \quad (n\to\infty).$$

Since $\xi_n(x)=0$ outside $[-M(n/b_n)^{1/2}, M(n/b_n)^{1/2}]^d$, there is a $\delta>0$ independent of n, such that

$$\mathbb{E}\exp\Big\{\frac{b_n}{n}\Big(\sum_{k=1}^{\gamma_n t_n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_k\Big)\Big\}\geq \delta\sum_{x\in\mathbb{Z}^d}\xi_n(x)\Pi_n^{\gamma_n-1}\xi_n(x)=\delta(\xi_n,\Pi_n^{\gamma_n-1}\xi_n).$$

Consider the spectral representation of Π_n:

$$(\xi_n,\Pi_n\xi_n)=\int_0^\infty \lambda\mu_{\xi_n}(d\lambda)$$

where μ_{ξ_n} is a probability measure on $\mathbb{R}^+$. By the mapping theorem,

$$(\xi_n,\Pi_n^{\gamma_n-1}\xi_n)=\int_0^\infty\lambda^{\gamma_n-1}\mu_{\xi_n}(d\lambda)\geq\Big(\int_0^\infty\lambda\mu_{\xi_n}(d\lambda)\Big)^{\gamma_n-1}=(\xi_n,\Pi_n\xi_n)^{\gamma_n-1}$$

where the second step follows from Jensen's inequality. Hence,

$$\liminf_{n\to\infty} b_n^{-1}\log\mathbb{E}\exp\Big\{\frac{b_n}{n}\Big(\sum_{k=1}^{\gamma_n t_n} f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_k\Big)\Big\}\geq\liminf_{n\to\infty}\log(\xi_n,\Pi_n\xi_n).$$

Next note that

$$\begin{aligned}(\xi_n,\Pi_n\xi_n) &= \Big(\sum_{y\in\mathbb{Z}^d} g^2\Big(\big(\frac{b_n}{n}\big)^{1/2}y\Big)\Big)^{-1}\cdot\sum_{x\in\mathbb{Z}^d} g\Big(\big(\frac{b_n}{n}\big)^{1/2}x\Big)\\ &\quad\times\exp\Big\{\frac{b_n}{2n}f\Big(\big(\frac{b_n}{n}\big)^{1/2}x\Big)\Big\}\mathbb{E}_x\Big(\exp\Big\{\frac{b_n}{n}\sum_{k=1}^{t_n-1}f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_k\Big)\\ &\quad+\frac{b_n}{2n}f\Big(\big(\frac{b_n}{n}\big)^{1/2}S_{t_n}\Big)\Big\}g\Big(\big(\frac{b_n}{n}\big)^{1/2}S_{t_n}\Big)\Big)\\ &= \big(1+o(1)\big)\Big(\frac{b_n}{n}\Big)^{d/2}\sum_{x\in\mathbb{Z}^d}g\Big(\big(\frac{b_n}{n}\big)^{1/2}x\Big)\\ &\quad\times\mathbb{E}\Big(\exp\Big\{\frac{b_n}{n}\sum_{k=1}^{t_n}f\Big(\big(\frac{b_n}{n}\big)^{1/2}(x+S_k)\Big)\Big\}g\Big(\big(\frac{b_n}{n}\big)^{1/2}(x+S_{t_n})\Big)\Big)\\ &\longrightarrow\int_{\mathbb{R}^d}g(x)\mathbb{E}_x\Big(\exp\Big\{\int_0^1 f\big(\widetilde{W}(s)\big)ds\Big\}g\big(\widetilde{W}(1)\big)\Big)dx \quad\text{as } n\to\infty\end{aligned}$$

where $\widetilde{W}(t)$ is a d-dimensional Lévy Gaussian process (Brownian motion) such that $\widetilde{W}(1)$ has the covariance matrix Γ.

Summarizing what we have so far, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} b_n^{-1}\log \mathbb{E}\exp\left\{\frac{b_n}{n}\left(\sum_{k=1}^{\gamma_n t_n} f\left(\left(\frac{b_n}{n}\right)^{1/2} S_k\right)\right\}\\
&\quad\ge \log\int_{\mathbb{R}^d} g(x)\mathbb{E}_x\left(\exp\left\{\int_0^1 f\big(\widetilde{W}(s)\big)ds\right\}g\big(\widetilde{W}(1)\big)\right)dx. \qquad (4.3)
\end{aligned}$$

What follows next is a standard treatment (see, e.g., Remillard (1998)) which is briefly described here. Let the semigroup of linear operators $\{T_t\}$ on $\mathcal{L}^2(\mathbb{R}^d)$ be defined as

$$T_t h(x) = \mathbb{E}_x \exp\left(\left\{\int_0^t f\big(\widetilde{W}(s)\big)ds\right\}h\big(\widetilde{W}(t)\big)\right), \quad h\in\mathcal{L}^2(\mathbb{R}^d), \quad t\ge 0.$$

The infinitesimal generator of T_t is

$$\mathcal{A}h(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2 h}{\partial x_i\partial x_j}(x) + f(x)h(x)$$

where a_{ij} $(1\le i, j\le d)$ are entries of the matrix Γ. Clearly, $\mathcal{A}$ is self-adjoint. Let

$$(g,\mathcal{A}g) = \int_{-\infty}^{\infty}\lambda\mu_g(d\lambda)$$

be the spectral representation of the quadratic form $(g,\mathcal{A}g)$, where μ_g is, in view of (4.2), a probability measure on $(-\infty,\infty)$. By Jensen's inequality,

$$\begin{aligned}
&\mathbb{E}_x\left(\exp\left\{\int_0^1 f\big(\widetilde{W}(s)\big)ds\right\}g\big(\widetilde{W}(1)\big)\right)dx = (g, T_1 g)\\
&= \int_{-\infty}^{\infty} e^{\lambda}\mu_g(d\lambda) \ge \exp\left\{\int_{-\infty}^{\infty}\lambda\mu_g(d\lambda)\right\} = \exp\big\{(g,\mathcal{A}g)\big\}\\
&= \exp\left\{\int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\right\}.
\end{aligned}$$

From (4.3) we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} b_n^{-1}\log \mathbb{E}\exp\left\{\frac{b_n}{n}\left(\sum_{k=1}^{\gamma_n t_n} f\left(\left(\frac{b_n}{n}\right)^{1/2} S_k\right)\right\}\\
&\quad\ge \int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx
\end{aligned}$$

for any bounded and infinitely smooth function g supported in a bounded region in $\mathbb{R}^d$, which also satisfies (4.2). Note that the set of such g is dense in $\mathcal{F}_d$. Taking supremum over g on the right hand side finishes the proof. □

5. Moderate deviation for intersection local times

In this section, we prove Theorem 1.3. We shall omit the details of the parts that are similar to the proof of Theorem 1.1 given in section 2 and put emphasis on places where a different treatment is needed. In the light of variation evaluation given in Lemma 7.2, we need only to establish

$$\lim_{n\to\infty} b_n^{-1} \log \mathbb{E} \exp\left\{\left(\frac{b_n}{n}\right)^{(mp+1)/2mp}\left(\sum_{x\in\mathbb{Z}}\prod_{j=1}^m l_j^p(n,x)\right)^{1/mp}\right\}$$
$$= \left(m\sigma^2\right)^{-(mp-1)/(mp+1)} \sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty}|g(x)|^{2mp}dx\right)^{1/mp}\right.$$
$$\left. -\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}. \tag{5.1}$$

We first deal with the lower bound. Note that

$$\sum_{x\in\mathbb{Z}}\prod_{j=1}^m l_j^p(n,x) = \int_{-\infty}^{\infty}\prod_{j=1}^m l_j^p(n,[x])dx$$
$$= (n/b_n)^{1/2}\int_{-\infty}^{\infty}\prod_{j=1}^m l_j^p\Big(n,\big[(n/b_n)^{1/2}x\big]\Big)dx.$$

So we only need to show the lower bound (5.1) for

$$\liminf_{n\to\infty} b_n^{-1}\log\mathbb{E}\exp\left\{(b_n/n)^{1/2}\left(\int_{-\infty}^{\infty}\prod_{j=1}^m l_j^p\Big(n,\big[(n/b_n)^{1/2}x\big]\Big)dx\right)^{1/mp}\right\}.$$

Similar to (2.8), by Lemma 11 of Jain and Pruitt (1984), given $\epsilon > 0$,

$$\lim_{\delta\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\left\{\sup_{|x-y|\le\delta}\Big|l\Big(n,\big[(n/b_n)^{1/2}x\big]\Big)\right.$$
$$\left.-l\Big(n,\big[(n/b_n)^{1/2}y\big]\Big)\Big|\ge\epsilon\sqrt{nb_n}\right\} = -\infty. \tag{5.2}$$

In fact, in the proof of the law of the iterated logarithm for local times, Jain and Pruitt obtained this result for much more general random walks in the case $b_n = \log\log n$. By carefully examining their proof one can see that it actually can be extended to the general b_n defined by (1.11). By the same argument given in section 2, we will have the lower bound if we can prove

$$\liminf_{n\to\infty} b_n^{-1}\log\mathbb{E}\exp\left\{\gamma(b_n/n)^{1/2}\left(\int_{-a}^{a}l^p\Big(n,\big[(n/b_n)^{1/2}x\big]\Big)dx\right)^{1/p}\right\}$$
$$\ge\sup_{g\in\mathcal{F}}\left\{\gamma\left(\int_{-a}^{a}|g(x)|^{2p}dx\right)^{1/p}-\frac{\sigma^2}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\} \tag{5.3}$$

for any $a > 0$ and $\gamma > 0$ in the case $m = 1$ and $p > 1$.

Observe that, for any bounded continuous function f supported on $[-a, a]$ satisfying (2.6),

$$\begin{aligned}
&\left(\int_{-a}^{a} l^p\Big(n, \big[(n/b_n)^{1/2}x\big]\Big)dx\right)^{1/p} \\
&\geq \int_{-\infty}^{\infty} f(x) l\Big(n, \big[(n/b_n)^{1/2}x\big]\Big)dx \\
&= (b_n/n)^{1/2}\int_{-\infty}^{\infty} f\Big((b_n/n)^{1/2}x\Big) l(n, [x])dx \\
&= (b_n/n)^{1/2}\Big\{o(n) + \sum_{x\in\mathbb{Z}} f\Big((b_n/n)^{1/2}x\Big) l(n, x)\Big\} \\
&= (b_n/n)^{1/2}\Big\{o(n) + \sum_{k=1}^{n} f\Big((b_n/n)^{1/2}S_k\Big)\Big\}
\end{aligned}$$

where the notation $o(n)$ should be viewed as a possibly random quantity bounded by $\alpha_n n$ for some deterministic positive sequence $\{\alpha_n\}$ with $\alpha_n \to 0$ as $n \to \infty$. Taking $d = 1$ in Theorem 4.1 gives

$$\begin{aligned}
&\liminf_{n\to\infty} b_n^{-1}\log\mathbb{E}\exp\left\{\gamma(b_n/n)^{1/2}\left(\int_{-a}^{a} l^p\Big(n, \big[(n/b_n)^{1/2}x\big]\Big)dx\right)^{1/p}\right\} \\
&\geq \sup_{g\in\mathcal{F}}\left\{\gamma\int_{-a}^{a} f(x)g^2(x)dx - \frac{\sigma^2}{2}\int_{-\infty}^{\infty}|g'(x)|dx\right\}.
\end{aligned}$$

We obtain (5.3) by taking supremum over f on the right hand side above.

Similar to the Brownian motion case in section 2, we only need to establish the upper bound necessary for (5.1) in the special case $m = 1$ and $p > 1$. That is, we need to show

$$\begin{aligned}
&\limsup_{n\to\infty} b_n^{-1}\log\mathbb{E}\exp\left\{\Big(\frac{b_n}{n}\Big)^{(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p(n, x)\Big)^{1/p}\right\} \\
&\leq \sigma^{-2(p-1)/(p+1)}\sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}. \quad (5.4)
\end{aligned}$$

We first prove the following upper tail estimate.

Lemma 5.1. *Let $p > 1$. Then for any $a, b > 0$ and any integer $n \geq 1$,*

$$\begin{aligned}
&\mathbb{P}\left\{\Big(\sum_{x\in\mathbb{Z}} l^p(n, x)\Big)^{1/p} \geq a + b\right\} \\
&\quad\leq \mathbb{P}\left\{\Big(\sum_{x\in\mathbb{Z}} l^p(n, x)\Big)^{1/p} \geq a - 1\right\}\mathbb{P}\left\{\Big(\sum_{x\in\mathbb{Z}} l^p(n, x)\Big)^{1/p} \geq b\right\}.
\end{aligned}$$

Proof. Define the stopping time

$$\tau = \inf\left\{k;\quad \Big(\sum_{x\in\mathbb{Z}} l^p(k, x)\Big)^{1/p} \geq a - 1\right\}.$$

Note that for any $k \geq 1$, $\left(\sum_{x\in\mathbb{Z}} l^p(k+1,x)\right)^{1/p} - \left(\sum_{x\in\mathbb{Z}} l^p(k,x)\right)^{1/p} \leq 1$ and hence $\sum_{x\in\mathbb{Z}} l^p(\tau,x) \leq a^p$. Consequently, for each $1 \leq k \leq n$,

$$\Big(\sum_{x\in\mathbb{Z}} \big(l(n,x)-l(k,x)\big)^p\Big)^{1/p} \geq \Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p} - \Big(\sum_{x\in\mathbb{Z}} l^p(k,x)\Big)^{1/p} \geq b$$

on the event $\left\{\tau = k, \left(\sum_{x\in\mathbb{Z}} l^p(n,x)\right)^{1/p} \geq a+b\right\}$. Therefore,

$$\begin{aligned}
&\mathbb{P}\Big\{\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p} \geq a+b\Big\} \\
&= \sum_{k=1}^n \mathbb{P}\Big\{\tau = k, \quad \Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p} \geq a+b\Big\} \\
&\leq \sum_{k=1}^n \mathbb{P}\Big\{\tau = k, \quad \Big(\sum_{x\in\mathbb{Z}} \big(l(n,x)-l(k,x)\big)^p\Big)^{1/p} \geq b\Big\} \\
&= \sum_{k=1}^n \mathbb{P}\{\tau = k\}\mathbb{P}\Big\{\Big(\sum_{x\in\mathbb{Z}} l^p(n-k,x)\Big)^{1/p} \geq b\Big\} \\
&\leq \mathbb{P}\Big\{\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p} \geq a-1\Big\}\mathbb{P}\Big\{\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p} \geq b\Big\}.
\end{aligned}$$

□

Here is how we utilize Lemma 5.1: Let $0 < \delta < 1$ be fixed. For any $\lambda \geq 2$ we have

$$\begin{aligned}
&\mathbb{P}\Big\{\Big(\sum_{x\in\mathbb{Z}} l^p([\delta n],x)\Big)^{1/p} \geq \lambda n^{(p+1)/2p}\Big\} \\
&\quad \leq \Big(\mathbb{P}\Big\{\Big(\sum_{x\in\mathbb{Z}} l^p([\delta n],x)\Big)^{1/p} \geq n^{(p+1)/2p} - 1\Big\}\Big)^{[\lambda]}.
\end{aligned}$$

On the other hand, taking $m = 1$ in Theorem 1.2 we have

$$n^{-(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p} \xrightarrow{d} \sigma^{-(p-1)/p}\Big(\int_{-\infty}^{\infty} L^p(1,x)dx\Big)^{1/p}. \tag{5.5}$$

Therefore, for any $\gamma > 0$, one can take $\delta > 0$ small enough so that

$$\sup_n \mathbb{P}\Big\{\Big(\sum_{x\in\mathbb{Z}} l^p([\delta n],x)\Big)^{1/p} \geq \lambda n^{(p+1)/2p}\Big\} \leq e^{-2\gamma\lambda}.$$

holds for all $\lambda > 0$ large. In particular

$$\sup_n \mathbb{E}\exp\Big\{\gamma n^{-(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p([\delta n],x)\Big)^{1/p}\Big\} < \infty.$$

By the triangular inequality and independence of increments, for large n,

$$\mathbb{E}\exp\Big\{\gamma n^{-(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p}\Big\}$$
$$\le\Big(\mathbb{E}\exp\Big\{\gamma n^{-(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p([\delta n],x)\Big)^{1/p}\Big\}\Big)^{2\delta^{-1}}.$$

We thus conclude that for any $\gamma > 0$,

$$\sup_n \mathbb{E}\exp\Big\{\gamma n^{-(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p}\Big\} < \infty.$$

This, together with (5.5), implies that for any $\gamma > 0$,

$$\lim_{n\to\infty}\mathbb{E}\exp\Big\{\gamma n^{-(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p}\Big\}$$
$$=\mathbb{E}\exp\Big\{\gamma\sigma^{-(p-1)/p}\Big(\int_{-\infty}^{\infty}L^p(1,x)dx\Big)^{1/p}\Big\}. \tag{5.6}$$

Let $\lambda > 0$ be fixed but arbitrary. Write $t_n = [\lambda n/b_n]$ and $\gamma_n = [n/t_n]$. Then $n \le t_n(\gamma_n + 1)$. Hence,

$$\mathbb{E}\exp\Big\{\Big(\frac{b_n}{n}\Big)^{(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p}\Big\}$$
$$\le\Big(\mathbb{E}\exp\Big\{\Big(b_n/n\Big)^{(p+1)/2p}\Big(\sum_{x\in\mathbb{Z}} l^p(t_n,x)\Big)^{1/p}\Big\}\Big)^{\gamma_n+1}.$$

From (5.6),

$$\limsup_{n\to\infty} b_n^{-1}\log\mathbb{E}\exp\Big\{\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p}\Big\}$$
$$\le\frac{1}{\lambda}\log\mathbb{E}\exp\Big\{\lambda^{(p+1)/2p}\sigma^{-(p-1)/p}\Big(\int_{-\infty}^{\infty}L^p(1,x)dx\Big)^{1/p}\Big\}.$$

Letting $\lambda \to \infty$ on the right hand side and taking $m = 1$ in (2.1) we have proved (5.4).

6. The law of the iterated logarithm

We only prove (1.14) as the proof of (1.15) is similar. Recall the constant $C_3(m, p)$ is given in (1.16). Let $t_k = \theta^k$ ($k \ge 1$) with $\theta > 1$. In view of (1.5), we have from Theorem 1 that

$$\sum_k \mathbb{P}\Big\{\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(t_k,x)dx \ge \gamma t_k^{(mp+1)/2}\big(\log\log t_k\big)^{(mp-1)/2}\Big\} < \infty \tag{6.1}$$

for any $\gamma > C_3(m, p)$. By the Borel-Cantelli lemma we obtain

$$\limsup_{k\to\infty} t_k^{-(mp+1)/2}\big(\log\log t_k\big)^{-(mp-1)/2}\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(t_k, x)dx \le C_3(m, p) \quad a.s.$$

By monotonicity, for any $t_k \le t \le t_{k+1}$,

$$\begin{aligned}
&t^{-(mp+1)/2}\big(\log\log t\big)^{-(mp-1)/2}\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(t, x)dx\\
&\quad\le \big(\theta^{\frac{mp+1}{2}} + o(1)\big)t_{k+1}^{-(mp+1)/2}\big(\log\log t_{k+1}\big)^{-(mp-1)/2}\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(t_{k+1}, x)dx
\end{aligned}$$

as $k \to \infty$. Therefore

$$\limsup_{t\to\infty} t^{-(mp+1)/2}\big(\log\log t\big)^{-(mp-1)/2}\int_{-\infty}^{\infty}\prod_{j=1}^m L_j^p(t, x)dx \le \theta^{\frac{mp+1}{2}} C_3(m, p)\, a.s.$$

which gives the upper bound by letting $\theta \to 1^+$.

To prove the lower bound, write $s_k = k^k$, $k \ge 1$ and notice that

$$\int_{-\infty}^{\infty}\prod_{j=1}^m \big[L_j(s_{k+1}, x) - L_j(s_k, x)\big]^p dx = \int_{-\infty}^{\infty}\prod_{j=1}^m L_{j,k}^p\big(s_{k+1} - s_k, x - W_j(s_k)\big)dx$$

where $L_{j,k}(t, x)$ is the local time of the Brownian motion $W_{j,k}(t)$:

$$W_{j,k}(t) = W_j(s_k + t) - W_j(s_k) \qquad t \ge 0, \quad k \ge 1, \quad 1 \le j \le m.$$

Hence,

$$\begin{aligned}
&\bigg|\bigg(\int_{-\infty}^{\infty}\prod_{j=1}^m \big[L_j(s_{k+1}, x) - L_j(s_k, x)\big]^p dx\bigg)^{1/mp} - \bigg(\int_{-\infty}^{\infty}\prod_{j=1}^m L_{j,k}^p\big(s_{k+1} - s_k, x\big)dx\bigg)^{1/mp}\bigg|\\
&\le \bigg(\int_{-\infty}^{\infty}\bigg|\prod_{j=1}^m L_{j,k}^{1/m}\big(s_{k+1} - s_k, x - W_j(s_k)\big) - \prod_{j=1}^m L_{j,k}^{1/m}\big(s_{k+1} - s_k, x\big)\bigg|^{mp} dx\bigg)^{1/mp}\\
&\le \sum_{j=1}^m\bigg(\prod_{i\ne j}\int_{-\infty}^{\infty} L_{i,k}^{mp}\big(s_{k+1} - s_k, x\big)dx\bigg)^{1/m^2p}\\
&\quad\times\bigg(\int_{-\infty}^{\infty}\Big|L_{j,k}^{1/m}\big(s_{k+1} - s_k, x - W_j(s_k)\big) - L_{j,k}^{1/m}\big(s_{k+1} - s_k, x\big)\Big|^{m^2p} dx\bigg)^{1/m^2p}\\
&\le 2^{1/m}\sum_{j=1}^m\bigg(\prod_{i\ne j}\int_{-\infty}^{\infty} L_{i,k}^{mp}\big(s_{k+1} - s_k, x\big)dx\bigg)^{1/m^2p}\\
&\quad\times\bigg(\int_{-\infty}^{\infty}\Big|L_{j,k}\big(s_{k+1} - s_k, x - W_j(s_k)\big) - L_{j,k}\big(s_{k+1} - s_k, x\big)\Big|^{mp} dx\bigg)^{1/m^2p}\\
&\le 2^{1/m}s_{k+1}^{1/m^2p}\sum_{j=1}^m\bigg(\prod_{i\ne j}\int_{-\infty}^{\infty} L_{i,k}^{mp}\big(s_{k+1}, x\big)dx\bigg)^{1/m^2p}\\
&\quad\times\sup_x\Big|L_{j,k}\big(s_{k+1} - s_k, x - W_j(s_k)\big) - L_{j,k}\big(s_{k+1} - s_k, x\big)\Big|^{(mp-1)/m^2p}
\end{aligned}$$

where the third step follows from the easy inequality

$$\left|a^{1/m} - b^{1/m}\right| \le 2^{1/m}|a-b|^{1/m} \qquad \forall a, b \ge 0 \text{ and } m \ge 1.$$

Let $m = 1$ and replace p by mp in (6.1). By the Borel-Cantelli lemma one has

$$\limsup_{k\to\infty} s_{k+1}^{-(mp+1)/2}\big(\log\log s_{k+1}\big)^{-(mp-1)/2} \int_{-\infty}^{\infty} L_{i,k}^{mp}\big(s_{k+1}, x\big)dx \le \beta_1 \quad a.s.$$

for each $1 \le i \le p$, where β_1 is a constant depending only on m and p. Note that as $k \to \infty$, $s_k \log\log s_k = o\big(s_{k+1}/\log\log s_{k+1}\big)$. By the classic law of the iterated logarithm for Brownian motions,

$$\lim_{k\to\infty} \Big(s_{k+1}^{-1}\log\log s_{k+1}\Big)^{1/2} |W_j(s_k)| = 0 \qquad a.s.$$

for every $1 \le j \le p$. Therefore,

$$\begin{aligned}
&\limsup_{k\to\infty} (2s_{k+1}\log\log s_{k+1})^{-1/2} \\
&\qquad \times \sup_{x\in\mathbb{R}} \Big|L_{j,k}\big(s_{k+1}-s_k, x - W_j(s_k)\big) - L_{j,k}\big(s_{k+1}-s_k, x\big)\Big| \\
&\quad \le \lim_{\delta\to 0^+} \limsup_{k\to\infty} (2s_{k+1}\log\log s_{k+1})^{-1/2} \\
&\qquad \times \sup_{|x-y|\le\delta(s_{k+1}/\log\log s_{k+1})^{1/2}} \Big|L_{j,k}\big(s_{k+1}-s_k, y\big) - L_{j,k}\big(s_{k+1}-s_k, x\big)\Big| \\
&\quad = 0 \quad a.s.
\end{aligned}$$

where the last step follows from the Borel-Cantelli lemma, and (2.8) that can be stated as, after a proper rescaling,

$$\begin{aligned}
&\lim_{\delta\to 0^+} \limsup_{t\to\infty} (\log\log t)^{-1} \\
&\quad \times \log\mathbb{P}\Big\{\sup_{|x-y|\le\delta(t/\log\log t)^{1/2}} |L(t,y) - L(t,x)| \ge (t/\log\log t)^{1/2}\Big\} = -\infty.
\end{aligned}$$

Combining what we have observed so far, we reach the conclusion that

$$\begin{aligned}
&\Bigg|\bigg(\int_{-\infty}^{\infty}\prod_{j=1}^{m}\big(L_j(s_{k+1},x) - L_j(s_k,x)\big)^p dx\bigg)^{1/mp} \\
&\qquad - \bigg(\int_{-\infty}^{\infty}\prod_{j=1}^{m} L_{j,k}^p\big(s_{k+1}-s_k, x\big)dx\bigg)^{1/mp}\Bigg| \\
&\quad = o\Big(s_{k+1}^{(mp+1)/2mp}\big(\log\log s_{k+1}\big)^{(mp-1)/2mp}\Big) \quad a.s. \quad (k\to\infty). \qquad (6.2)
\end{aligned}$$

On the other hand, by (1.7) in Theorem 1.1, for any $\gamma < C_3(m, p)$

$$
\begin{aligned}
&\sum_k \mathbb{P}\Big\{ \int_{-\infty}^{\infty} \prod_{j=1}^m L_{j,k}^p\big(s_{k+1} - s_k, x\big) dx \geq \gamma s_{k+1}^{(mp+1)/2}\big(\log\log s_{k+1}\big)^{(mp-1)/2}\Big\} \\
&\quad = \sum_k \mathbb{P}\Big\{ \int_{-\infty}^{\infty} \Big(\prod_{j=1}^m L_j\big(s_{k+1} - s_k, x\big)\Big)^p dx \\
&\quad \geq \gamma s_{k+1}^{(mp+1)/2}\big(\log\log s_{k+1}\big)^{(mp-1)/2}\Big\} = \infty
\end{aligned}
$$

By the Borel-Cantelli lemma and the independence of the sequence

$$
\int_{-\infty}^{\infty} \prod_{j=1}^m L_{j,k}^p\big(s_{k+1} - s_k, x\big) dx, \qquad k = 1, 2, \cdots
$$

we have

$$
\begin{aligned}
&\limsup_{k\to\infty} s_{k+1}^{-(mp+1)/2}\big(\log\log s_{k+1}\big)^{-(mp-1)/2} \\
&\quad \times \int_{-\infty}^{\infty} \prod_{j=1}^m L_{j,k}^p\big(s_{k+1} - s_k, x\big) dx \geq C_3(m, p) \quad a.s.
\end{aligned}
$$

In view of (6.2),

$$
\begin{aligned}
&\limsup_{k\to\infty} s_{k+1}^{-(mp+1)/2}\big(\log\log s_{k+1}\big)^{-(mp-1)/2} \\
&\quad \times \int_{-\infty}^{\infty} \prod_{j=1}^m \big(L_j(s_{k+1}, x) - L_j(s_k, x)\big)^p dx \geq C_3(m, p) \quad a.s.
\end{aligned}
$$

Note that

$$
\int_{-\infty}^{\infty} \prod_{j=1}^m L_j^p(s_{k+1}, x) dx \geq \int_{-\infty}^{\infty} \prod_{j=1}^m \big(L_j(s_{k+1}, x) - L_j(s_k, x)\big)^p dx, \qquad \forall k \geq 1.
$$

Hence,

$$
\limsup_{t\to\infty} t^{-(mp+1)/2}(\log\log t)^{-(mp-1)/2} \int_{-\infty}^{\infty} \prod_{j=1}^m L_j^p(t, x) dx \geq C_3(m, p) \qquad a.s.
$$

which finishes the proof of (1.14).

7. Two analytic lemmas

Recall that $\mathcal{F}$ denotes the set of absolutely continuous functions on $(-\infty, \infty)$ satisfying (2.2).

Lemma 7.1. *For any $p > 1$,*

$$\limsup_{M\to\infty} \sup_{g\in\mathcal{F}} \left\{ \left(\int_0^M \Big(\sum_{k\in\mathbb{Z}} g^2(x+kM) \Big)^p dx \right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |g'(x)|^2 dx \right\}$$
$$\leq \sup_{g\in\mathcal{F}} \left\{ \left(\int_{-\infty}^{\infty} |g(x)|^{2p} dx \right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |g'(x)|^2 dx \right\}.$$

Proof. Without loss of generality we may assume that the right hand side, call it J, is finite. Indeed, J is found explicitly in the next lemma. We only need to prove that given $\epsilon > 0$, there is a $M > 0$ such that for any $g \in \mathcal{F}$,

$$\left(\int_0^M \Big(\sum_{k\in\mathbb{Z}} g^2(x+kM) \Big)^p dx \right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |g'(x)|^2 dx$$
$$\leq \epsilon + (1-\epsilon)^{-(p-1)/(p+1)} J. \tag{7.1}$$

The hard part is that M has to be independent of g. We will determine the value of M later. Let $\bar{g}^2(x) = \sum_{k\in\mathbb{Z}} g^2(x+kM)$ with $\bar{g}(x) \geq 0$. Then $\int_0^M \bar{g}^2(x) = \int_{-\infty}^{\infty} g^2(x)dx = 1$ and $\bar{g}$ is absolutely continuous with $|\bar{g}'(x)|^2 \leq \sum_{k\in\mathbb{Z}} |g'(x+kM)|^2$ which is easy to see by direct computation of $\bar{g}'(x)$ and then using the Cauchy-Schwarz inequality. Consequently,

$$\left(\int_0^M \Big(\sum_{k\in\mathbb{Z}} g^2(x+kM) \Big)^p dx \right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |g'(x)|^2 dx$$
$$\leq \left(\int_0^M \bar{g}^{2p}(x) dx \right)^{1/p} - \frac{1}{2}\int_0^M |\bar{g}'(x)|^2 dx. \tag{7.2}$$

Next we need to construct a function $f \in \mathcal{F}$ which is equal to $\bar{g}$ on $[M^{1/2}, M - M^{1/2}]$ and is negligible in some suitable sense at the rest part of the real line as M gets large. Let $E = [0, M^{1/2}] \cup [M - M^{1/2}, M]$. By Lemma 3.4 in Donsker-Varadhan (1975), there is a real number a such that $\int_0^M \bar{g}^2(x-a)dx \leq 2M^{-1/2}$. We may assume $a = 0$, i.e,

$$\int_0^M \bar{g}^2(x)dx \leq 2M^{-1/2} \tag{7.3}$$

for otherwise we can replace $\bar{g}(\cdot)$ by $\bar{g}(\cdot + a)$. Define

$$\varphi(x) = \begin{cases} xM^{-1/2} & 0 \leq x \leq M^{1/2} \\ 1 & M^{-1/2} \leq x \leq M - M^{1/2} \\ M^{1/2} - xM^{-1/2} & M - M^{1/2} \leq x \leq M \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that

$$0 \leq \varphi(x) \leq 1, \quad |\varphi'(x)| \leq M^{-1/2}, \quad |\big(\varphi^2(x)\big)'| \leq 2M^{-1/2}, \qquad -\infty < x < \infty.$$

Define

$$f(x) = \bar{g}(x)\varphi(x) \cdot \left(\int_{-\infty}^{\infty} \bar{g}^2(x)\varphi^2(x)dx \right)^{-1/2}.$$

Clearly, $f \in \mathcal{F}$. Set $\alpha = \int_{-\infty}^{\infty} \bar{g}^2(x)\varphi^2(x)dx$. Then

$$\begin{aligned} |f'(x)|^2 &= \frac{1}{\alpha}\Big\{ |\bar{g}'(x)|^2\varphi^2(x) + \bar{g}^2(x)|\varphi'(x)|^2 + \frac{1}{2}\big(\bar{g}^2(x)\big)'\big(\varphi^2(x)\big)' \Big\} \\ &\le \frac{1}{\alpha} I_{[0,M]}(x)\Big\{ |\bar{g}'(x)|^2 + M^{-1}\bar{g}^2(x) + M^{-1/2}\big|\big(\bar{g}^2(x)\big)'\big| \Big\}. \end{aligned}$$

Note that

$$\begin{aligned} \int_0^M \big|\big(\bar{g}^2(x)\big)'\big|dx &\le 2\bigg(\int_0^M |\bar{g}'(x)|^2 dx \bigg)^{1/2} \bigg(\int_0^M \bar{g}^2(x)dx \bigg)^{1/2} \\ &= 2\bigg(\int_0^M |\bar{g}'(x)|^2 dx \bigg)^{1/2}. \end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} |f'(x)|^2 dx \le \frac{1}{\alpha}\bigg\{ \int_0^M |\bar{g}'(x)|^2 + M^{-1} + 2M^{-1/2}\bigg(\int_0^M |\bar{g}'(x)|^2 dx \bigg)^{1/2} \bigg\}. \tag{7.4}$$

On the other hand,

$$\begin{aligned} \bigg(\int_0^M \bar{g}^{2p}(x)dx \bigg)^{1/p} &\le \bigg(\alpha^p \int_{-\infty}^{\infty} |f(x)|^{2p}dx + \int_E |\bar{g}(x)|^{2p}dx \bigg)^{1/p} \\ &\le \alpha\bigg(\int_{-\infty}^{\infty} |f(x)|^{2p}dx \bigg)^{1/p} + \bigg(\int_E |\bar{g}(x)|^2 dx \bigg)^{1/p} \end{aligned} \tag{7.5}$$

and from (7.3)

$$\begin{aligned} \bigg(\int_E |\bar{g}(x)|^{2p}dx \bigg)^{1/p} &\le \sup_{0\le x\le M} |\bar{g}(x)|^{2/q}\bigg(\int_E |\bar{g}(x)|^2 dx \bigg)^{1/p} \\ &\le (2M^{-1/2})^{1/p} \sup_{0\le x\le M} |\bar{g}(x)|^{2/q}. \end{aligned}$$

Observe that for any $0 \le x \le M$, if $x + 1 \in [0, M]$ then

$$\begin{aligned} |\bar{g}(x)| &\le \int_x^{x+1} |\bar{g}(y)|dy + \int_x^{x+1}\bigg(\int_x^y |\bar{g}'(z)|dz \bigg)dy \\ &\le \bigg(\int_0^M \bar{g}^2(x)dx \bigg)^{1/2} + \bigg(\int_0^M |\bar{g}'(x)|^2 dx \bigg)^{1/2} \\ &\le 1 + \bigg(\int_0^M |\bar{g}'(x)|^2 dx \bigg)^{1/2}. \end{aligned}$$

Note that this inequality holds even $x + 1 \notin [0, M]$, in which case we integrate on $[x - 1, x]$ instead of $[x, x + 1]$ in the above estimate. Therefore,

$$\left(\int_E |\bar{g}(x)|^{2p} dx\right)^{1/p} \le (2M^{-1/2})^{1/p}\left\{1 + \left(\int_0^M |\bar{g}'(x)|^2 dx\right)^{1/2}\right\}^{2/q}. \quad (7.6)$$

By combining (7.4), (7.5), (7.6), we see

$$\begin{aligned}
&\left(\int_0^M \bar{g}^{2p}(x)dx\right)^{1/p} - \frac{1-\epsilon}{2}\int_0^M |\bar{g}'(x)|^2 dx - \left(\frac{1}{\sqrt{M}}\left(\int_0^M |\bar{g}'(x)|^2 dx\right)^{1/2}\right. \\
&\quad \left. + (2M^{-1/2})^{1/p}\left\{1 + \left(\int_0^M |\bar{g}'(x)|^2 dx\right)^{1/2}\right\}^{2/q}\right) \\
&\le \frac{1-\epsilon}{2M} + \alpha\left\{\left(\int_{-\infty}^{\infty} |f(x)|^{2p} dx\right)^{1/p} - \frac{1-\epsilon}{2}\int_{-\infty}^{\infty} |f'(x)|^2 dx\right\} \\
&\le \frac{1}{2M} + \sup_{f\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty} |f(x)|^{2p} dx\right)^{1/p} - \frac{1-\epsilon}{2}\int_{-\infty}^{\infty} |f'(x)|^2 dx\right\} \\
&= \frac{1}{2M} + \left(\frac{1}{1-\epsilon}\right)^{\frac{p-1}{p+1}} J
\end{aligned}$$

where the second inequality follows from the fact that $\alpha \le 1$ and the final step from the substitution

$$f(x) = \left(\frac{1}{1-\epsilon}\right)^{p/2(p+1)} h\left(\left(\frac{1}{1-\epsilon}\right)^{p/(p+1)} x\right).$$

Since $q > 1$, there is a sufficiently large $M = M(\epsilon) > 0$ such that $M^{-1} \le \epsilon$ and that

$$(x/M)^{1/2} + (2M^{-1/2})^{1/p}(1 + \sqrt{x})^{2/q} \le \epsilon(x + 1)/2$$

for all $x \ge 0$. Note that the choice of M is independent of the function g!. For such M,

$$\left(\int_0^M \bar{g}^{2p}(x)dx\right)^{1/p} - \frac{1}{2}\int_0^M |\bar{g}'(x)|^2 dx \le \epsilon + \left(\frac{1}{1-\epsilon}\right)^{\frac{p-1}{p+1}} J \quad (7.7)$$

Finally, (7.1) follows from (7.2) and (7.7). □

Lemma 7.2. *For any real number $p > 1$,*

$$\begin{aligned}
J &= \sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty} |g(x)|^{2p} dx\right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |g'(x)|^2 dx\right\} \\
&= p^{-2p/(p+1)}\left(\frac{\sqrt{2}}{(p-1)(p+1)} B\left(\frac{1}{p-1}, \frac{1}{2}\right)\right)^{-2(p-1)/(p+1)}.
\end{aligned}$$

Proof. The proof is partially inspired by Strassen (1964). By a close examination of the proof of Theorem 6.7 in Mansmann (1991), one can show that $J < \infty$ and there is a $f \in \mathcal{F}$ such that

$$J = \Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |f'(x)|^2 dx$$

with the properties

$$f(x) \geq 0, \quad f(-x) = f(x), \quad \forall x \quad \text{and} \quad f(x) \geq f(y) \quad \text{for } |x| \leq |y|. \qquad (7.8)$$

Let $W^{1,2}(\mathbb{R})$ be the Hilbert space

$$W^{1,2}(\mathbb{R}) = \Big\{g;\ g \text{ is absolutely continuous}, \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty, \int_{-\infty}^{\infty} |g'(x)|^2 dx < \infty\Big\}$$

with the Sobolev norm $|g|_{1,2}^2 = \int_{-\infty}^{\infty} |g(x)|^2 dx + \int_{-\infty}^{\infty} |g'(x)|^2 dx$. Applying the Lagrange multiplier gives that for any $g \in W^{1,2}(\mathbb{R})$

$$2\Big(\int_{-\infty}^{\infty} |f(x)|^{2p} dx\Big)^{-(p-1)/p} \int_{-\infty}^{\infty} f^{2p-1}(x)g(x)dx - \int_{-\infty}^{\infty} f'(x)g'(x)dx = 2\lambda \int_{-\infty}^{\infty} f(x)g(x)dx.$$

Note that as $|x| \to \infty$,

$$\Big|g(x)\int_0^x f(y)dy\Big| \leq |g(x)|\sqrt{|x|} \to 0$$

and therefore as $|x| \to \infty$,

$$\Big|g(x)\int_0^x f^{2p-1}(y)dy\Big| \leq f^{2p-2}(0)\Big|g(x)\int_0^x f(y)dy\Big| \to 0.$$

Hence, using integration by parts,

$$-2\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{-(p-1)/p} \int_{-\infty}^{\infty} g'(x)\int_0^x f^{2p-1}(y)dydx - \int_{-\infty}^{\infty} f'(x)g'(x)dx = -2\lambda\int_{-\infty}^{\infty} g'(x)\int_0^x f(y)dydx.$$

Thus for all x,

$$\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{-(p-1)/p} \int_0^x f^{2p-1}(y)dy + \frac{1}{2}f'(x) = \lambda\int_0^x f(y)dy.$$

Therefore, $f(x)$ has a continuous second derivative $f''(x)$ satisfying

$$\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{-(p-1)/p} f^{2p-1}(x) + \frac{1}{2}f''(x) = \lambda f(x) \qquad (7.9)$$

with $f'(0) = 0$. Multiplying both sides of (7.9) by $f(x)$ and integrating we obtain

$$\lambda = \Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |f'(x)|^2 dx = J. \qquad (7.10)$$

Multiplying both sides of (7.9) by $f'(x)$, integrating we have after simplification that

$$\big(f'(x)\big)^2 = 2\Big(\lambda f^2(x) - \frac{1}{p}\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{-(p-1)/p} f^{2p}(x) + C\Big)$$

where, using $f'(0) = 0$,

$$C = \frac{1}{p}\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{-(p-1)/p} f^{2p}(0) - \lambda f^2(0). \qquad (7.11)$$

Thus, by the fact that $f'(x) \le 0$ for all $x \ge 0$,

$$dx = -\frac{1}{\sqrt{2}}\Big(\lambda f^2(x) - p^{-1}\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{-(p-1)/p} f^{2p}(x) + C\Big)^{-1/2} \times df(x) \qquad \forall x \ge 0. \qquad (7.12)$$

Consequently, for all $x \ge 0$,

$$x = \frac{1}{\sqrt{2}}\int_{f(x)}^{f(0)} \Big(\lambda y^2 - p^{-1}\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{-(p-1)/p} y^{2p} + C\Big)^{-1/2} dy. \qquad (7.13)$$

Letting $x \to \infty$ we have $f(x) \to 0$ and the above relation implies $C = 0$. Hence by (7.11),

$$f(0) = \Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{1/2p} p^{1/(2p-2)} \lambda^{1/(2p-2)}. \qquad (7.14)$$

Combining (7.8), (7.12) and (7.14) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f^{2p}(x)dx &= 2\int_0^{\infty} f^{2p}(x)dx \\ &= \sqrt{2}\int_0^{f(0)} y^{2p}\Big(\lambda y^2 - p^{-1}\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{-(p-1)/p} y^{2p}\Big)^{-1/2} dy \\ &= f^{2p}(0)\sqrt{2/\lambda}\int_0^1 \frac{u^{2p}du}{\sqrt{u^2 - u^{2p}}} \\ &= \sqrt{2}p^{p/(p-1)}\lambda^{(p+1)/2(p-1)}\int_{-\infty}^{\infty} f^{2p}(x)dx\int_0^1 \frac{u^{2p}du}{\sqrt{u^2 - u^{2p}}}.\end{aligned}$$

Therefore,

$$1 = \sqrt{2}p^{p/(p-1)}\lambda^{(p+1)/2(p-1)}\int_0^1 \frac{u^{2p}du}{\sqrt{u^2 - u^{2p}}}.$$

Hence

$$\lambda = p^{-2p/(p+1)}\Big(\frac{\sqrt{2}}{(p-1)(p+1)} B\Big(\frac{1}{p-1}, \frac{1}{2}\Big)\Big)^{-2(p-1)/(p+1)}$$

and the desired conclusion follows from (7.10). □

Remark 7.3. It can be seen from the proof of Lemma 7.2 that the maximizer $f(x)$ is unique. In addition,

$$f(x) = K_p\varphi_p^{-1}(C_p|x|), \qquad x \in \mathbb{R}$$

where $\varphi_p^{-1}(x)$ $(x \in [0, \infty))$ is the inverse of the decreasing function φ_p:

$$\varphi_p(y) = \int_y^1 \frac{1}{\sqrt{u^2 - u^{2p}}}du, \qquad y \in (0, 1]$$

and,

$$C_p = 2^{1/(p+1)}p^{-p/(p+1)}\left(\frac{1}{(p-1)(p+1)}B\left(\frac{1}{p-1}, \frac{1}{2}\right)\right)^{-(p-1)/(p+1)}$$

$$K_p = 2^{1/2(p+1)}(p+1)^{(p-1)/2(p+1)}\left(\frac{p}{(p-1)}B\left(\frac{1}{p-1}, \frac{1}{2}\right)\right)^{-p/(p+1)}.$$

Indeed, by (7.14) a suitable integration substitution in (7.13) gives

$$x = C_p^{-1}\int_{f(x)/f(0)}^1 \frac{1}{\sqrt{u^2 - u^{2p}}}du, \qquad x \geq 0$$

which gives that $f(x) = f(0)\varphi_p^{-1}(C_p|x|)$ for all $x \in \mathbb{R}$. This, together with the constraint $\int_{-\infty}^{\infty} f^2(x)dx = 1$ gives $f(0) = K_p$.

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