METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 9, No. 3, pp. 377–392, September 2002

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LIMITING BEHAVIORS FOR BROWNIAN MOTION REFLECTED ON BROWNIAN MOTION *

XIA CHEN† AND WENBO V. LI‡

Abstract. Suppose that $g(t)$ and W_t are independent Brownian motions starting from $g(0) = W_0 = 0$. Consider the Brownian motion Y_t reflected on $g(t)$, obtained from W_t by the means of the Skorohod lemma. The upper and lower limiting behaviors of Y_t are presented. The upper tail estimate on exit time is computed via principal eigenvalue.

1. Introduction. Brownian motion reflected on Brownian motion appeared in recent papers by Soucaliuc, Toth and Werner (2000), Burdzy, Chen and Sylvester (2000) and Burdzy and Nualart (2002) in their study of reflected Bronian motion and corresponding heat equation in domains with space-time boundaries. In this paper, we study the upper and lower limiting behaviors of Brownian Motion reflected on Brownian Motion. Our starting point is the following beautiful result of Burdzy and Nualart (2002).

Suppose that $g(t)$ and W_t are independent real Brownian motions starting from $g(0) = W_0 = 0$. Consider the Brownian motion Y_t reflected on $g(t)$, obtained from W_t by the mean of the Skorohod lemma. Here g should be thought of as a "fixed Brownian path." Then

$$-Y_t = (W_t + C_t)/\sqrt{2}, \quad t \geq 0$$

where C_t is a 3-dimensional Bessel process independent of W_t and starting from 0. A process with the same distribution as $\{(W_t + C_t)/\sqrt{2}, t \geq 0\}$ is called a BMB-process in Burdzy and Nualart (2002) and many useful properties are given.

The main goal of this paper is to present some "global" results for BMB-process and a natural generalization. Namely

THEOREM 1.1. *Let $X(t)$, $X(0) = 0$, be a d-dimensional ($d \geq 1$) Bessel process independent of W. Then*

$$\limsup_{t\to\infty} \frac{1}{\sqrt{t \log\log t}} \big(W(t) + X(t)\big) = 2 \quad a.s. \tag{1.1}$$

$$\liminf_{t\to\infty} \frac{1}{\sqrt{t \log\log t}} \big(W(t) + X(t)\big) = -\sqrt{2} \quad a.s. \tag{1.2}$$

and for $d \neq 2$

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} \sup_{s\leq t} \big|W(s) + X(s)\big| = \frac{\pi}{2} \quad a.s. \tag{1.3}$$

In particular, we have for Y_t, Brownian motion reflected on Brownian motion,

$$\liminf_{t\to\infty} \frac{1}{\sqrt{t \log\log t}} Y(t) = -\sqrt{2} \quad a.s. \tag{1.4}$$

*Received July 11, 2001; accepted for publication January 10, 2003.
†Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA (xchen@math.utk.edu). Supported in part by NSF Grant DMS-0102238.
‡Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA (wli@math.udel.edu). Supported in part by NSF Grant DMS-0204513.

$$\limsup_{t\to\infty} \frac{1}{\sqrt{t\log\log t}} Y(t) = 1 \qquad a.s. \tag{1.5}$$

and

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} \sup_{s\le t} \big|Y(s)\big| = \frac{\pi}{\sqrt{8}} \quad a.s. \tag{1.6}$$

It is interesting to see that the behaviors in (1.4) and (1.6) for Y is exactly the same as those for W. Furthermore, due to the time reversibility, their behaviors near time zero are also the same.

Next we outline some of the tools we used. As it can be seen in the next section, the main part of this work is to estimate the upper tail of the exit time from a suitable domain. The approach we follow is to reduce our problem to the principal eigenvalue of the Markov process $(W(t), X(t))$ killed upon the exit from the domain. This approach has been effectively utilized by Donsker and Varadhan (1975-1983) in their fundamental work on Large deviations for Markov processes and its applications, and by Pinsky (1985, 1995) and Rémillard (1994) in various problems involving estimates of exponential type. In Donsker and Varadhan (1975), the principal eigenvalue is represented in terms of the I-function in large deviation theory. In Berestycki, Nirenberg and Varadhan (1994), the existence of the principal eigenvalue is discussed in general setting. Our results require exact evaluation of the principal eigenvalue, which is beyond the general theory. Fortunately, the generators we deal with are self-adjoint, in which case the principal eigenvalue can be written as a computable quadratic variation. Some techniques we use here are partially inspired by the work of Rémillard (1994). To be more precise, we state our main probability estimate.

THEOREM 1.2. *Let W and X be given as in Theorem 1.1. Then for $d \neq 2$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big\{ \sup_{s\le t} |W(s) - X(s)| \le 1 \big\} = -\frac{\pi^2}{4}. \tag{1.7}$$

Note that $\sup_{s\le t} |W(s) - X(s)|$ and $\sup_{s\le t} |W(s) + X(s)|$ have the same distribution and we use minus sign for convenience in our proofs. Furthermore, as it can be seen in the next section, we have the variation formula in the case $d = 2$, but could not evaluate it explicitly. We strongly believe that both (1.3) and (1.7) hold for $d = 2$.

There are two ways to view the estimate in (1.7). The first can be stated as

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big\{ \tau_\Gamma \ge t \big\} = -\frac{\pi^2}{4} \tag{1.8}$$

where τ_Γ is the first exit time of $(d + 1)$-dimensional Brownian motion from the unbounded domain

$$\Gamma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : -1 < y - |x|_2 < 1\}. \tag{1.9}$$

In this setup, the generator is the half Laplacian on $\mathbb{R}^{d+1}$ and the domain is the part between two parallel right corns. Our approaches detailed in the next section work in this setting as $d = 1$. Other related interesting problems and techniques on the first exit times of higher dimensional Brownian motion from unbounded domains can be found in Bañuelos, DeBlassie and Smits (2001), Li (2002).

The second way can be stated as

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big\{\tau_G\ge t\big\}=-\frac{\pi^2}{4} \tag{1.10}$$

where τ_G is the first exit time of the diffusion process $(X(t),W(t))$ from the unbounded domain

$$G=\{(x,y)\in\mathbb{R}^+\times\mathbb{R}:|y-x|<1\}. \tag{1.11}$$

As $d\ge 2$, the generator is

$$Lf(x,y)=\frac{1}{2}\triangle f(x,y)+\frac{d-1}{2x}\frac{\partial}{\partial x}f(x,y),\qquad (x,y)\in\mathbb{R}^+\times\mathbb{R}. \tag{1.12}$$

These are two ways we handle the problem in the next section.

Next we make some simple observations. We assume throughout this paper that $W(t),W_j(t)$, $j=1,2,\cdots d$ are independent standard Brownian motions and thus we can use the representation

$$X(t)=\left(\sum_{j=1}^{d}W_j(t)^2\right)^{1/2}=|B_d(t)|_2$$

where $B_d(t)=(W_1(t),\cdots,W_d(t))\in\mathbb{R}^d$ is the standard d-dimensional Brownian motion. It is well known and follows from rotation invariant that as process,

$$\{(W(t),W_1(t))\,:\,t\ge 0\}=\left\{\left(\frac{W(t)+W_1(t)}{\sqrt{2}},\frac{W(t)-W_1(t)}{\sqrt{2}}\right)\,:\,t\ge 0\right\}$$

in law and thus as process,

$$\{W(t)+|W_1(t)|\,:\,t\ge 0\}=\{\sqrt{2}\max(W(t),W_1(t))\,:\,t\ge 0\}$$

in law by using $2\max(a,b)=a+b+|a-b|$. This allows us to obtain the following sharp lower bound in the case $d=1$:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le s\le t}|W(s)+|W_1(s)||\le 1\right)\\
&=\mathbb{P}\left(\sqrt{2}\sup_{0\le s\le t}|\max(W(s),W_1(s))|\le 1\right)\\
&\ge\mathbb{P}\left(-1\le\sqrt{2}W(s)\le 1,\,\sqrt{2}W_1(s)\le 1\,\forall 0\le s\le t\right)\\
&=\mathbb{P}\left(\sqrt{2}\sup_{0\le s\le t}|W(s)|\le 1\right)\cdot\mathbb{P}\left(\sqrt{2}\sup_{0\le s\le t}W_1(s)\le 1\right).
\end{aligned}$$

For $d\ge 1$, an easy upper and lower bounds for the probability estimate in Theorem 1.2 can be found by using the well known estimates

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\sup_{0\le s\le t}X(s)\le 1\right)=-j_\nu^2/2 \tag{1.13}$$

where j_ν is the smallest positive zero of the Bessel function J_ν, $\nu=(d-2)/2$, and $j_{-1/2}=\pi/2$. The above estimate can be obtained either from the exact distribution result due to Ciesielski and Taylor (1962) or from a general principle eigenvalue approach detailed in Donsker and Varadhan (1976). Now by using the simple fact that

$$\mathbb{P}\left(\sup_{0\le s\le t}|W(s)+X(s)|\le 1\right)\le \mathbb{P}\left(\sup_{0\le s\le t}|W(s)|\le 1\right)$$

via Anderson's inequality, and

$$\mathbb{P}\left(\sup_{0\le s\le t}|W(s)+X(s)|\le 1\right)\ge \mathbb{P}\left(\sup_{0\le s\le t}|W(s)|\le \lambda\right)\cdot \mathbb{P}\left(\sup_{0\le s\le t}|X(s)|\le 1-\lambda\right)$$

for $\lambda = j_{-1/2}^{2/3}/(j_{-1/2}^{2/3}+j_\nu^{2/3})$, we have

$$-(j_{-1/2}^{2/3}+j_\nu^{2/3})^3/2\le \lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\sup_{0\le s\le t}|W(s)+X(s)|\le 1\right)\le -j_{-1/2}^2/2=-\pi^2/8.$$

In particular, combining the above estimate with Theroem 1.2, we see that for $-1<\nu<-1/2$, $j_\nu\ge (2^{1/3}-1)^{3/2}J_{-1/2}=(2^{1/3}-1)^{3/2}\pi/2$.

Finally, we mention the following heuristic argument which is suggestive but seems impossible to produce a rigorous upper or lower bound. We observe that for fixed $s\ge 0$,

$$W(s)+X(s)=W(s)+\sup_{|x|_2=1}\sum_{j=1}^d x_jW_j(s)=\sup_{|x|_2=1}\left(W(s)+\sum_{j=1}^d x_jW_j(s)\right)$$

and for fixed $x\in\mathbb{R}^d$ with $|x|_2=1$,

$$W(s)+\sum_{j=1}^d x_jW_j(s)=\sqrt{1+|x|_2^2}\,\widehat{W}(s)=\sqrt{2}\widehat{W}(s)$$

in distribution where $\widehat{W}$ is a standard Brownian motion. Jointly, our Theorem 1.2 implies

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\sup_{0\le s\le t}|W(s)+X(s)|\le 1\right)=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\sup_{0\le s\le t}|\sqrt{2}W(s)|\le 1\right)=-\frac{\pi^2}{4}$$

where the last equality follows from (1.13).

The rest of the paper is organized as follows. In Section 2, we present the proof of Theorem 1.2 viewed as the large deviations of the first exit time of the diffusion process $(X(t),W(t))$ from the unbounded domain G in the case $d\ge 2$, and the diffusion process $(W_1(t),W(t))$ from the unbounded domain Γ in the case $d=1$. They are necessary for the proofs of Theorem 1.1 and important in their own. In Section 3, we give the proof of Theorem 1.1 which also requires some large deviation estimates.

2. The first exit time and principal eigenvalue. Recall that G is given in (1.11). We now prove that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big\{\max_{s\le t}|W(s)-X(s)|<1\big\}=-\frac{1}{2}\inf_{|f|_\pi=1,\ f\in C_0^\infty(G)}\int_G|\nabla f(x,y)|^2x^{d-1}dxdy \tag{2.1}$$

where, by the notations we shall introduce below, $C_0^\infty(G)$ is the class of continuous functions $f(x,y)$ on $\bar{G}$ which are infinitely differentiable and $f(x,y)=0$ if $y-x=\pm 1$.

We first deal with the case $d \geq 2$. Consider the diffusion process $(X(t), W(t))$ with state space $\mathbf{R}^+ \times \mathbf{R}$ and generator given in (1.12). It is easy to see that as a linear operator on the Hilbert space $L^2(\mathbf{R}^+ \times \mathbf{R}, \pi)$, L is self adjoint, where π is the measure on $\mathbf{R}^+ \times \mathbf{R}$ given by

$$\pi(dx, dy) = x^{d-1} dxdy$$

and

$$< Lf, f >_\pi = -\frac{1}{2} \int_{\mathbf{R}^+ \times \mathbf{R}} |\nabla f(x,y)|^2 x^{d-1} dxdy$$

if f is smooth enough. Write

$$|f|_\pi = \Big(\int_{\mathbf{R}^+ \times \mathbf{R}} f^2(x,y) x^{d-1} dxdy \Big)^{1/2}.$$

For an open domain D (with respect to the relative Euclidian topology on $\mathbf{R}^+ \times \mathbf{R}$) in the space $\mathbf{R}^+ \times \mathbf{R}$ we define

$$\tau_D = \inf\{t \geq 0; \quad (X(t), W(t)) \notin D\}.$$

Define the semigroup T_t $(t \geq 0)$ by

$$T_t f(x,y) = \mathbb{E}_{(x,y)} \Big(f(X(t), W(t)) I_{\{\tau_D \geq t\}} \Big).$$

Note that $\mathbb{P}_{(x,y)}\{\tau_D = 0\} = 0$ for each $(x,y) \in D$. We have $T_0 = id$. Let $C_0^\infty(D)$ be the class of of functions f continuous on $\bar{D}$, infinitely differentiable in D and $f(\partial D) = 0$. Notice that $\mathbf{R}^+ \times \mathbf{R}$ is a whole space in our setting. In other words a open set D in $\mathbf{R}^+ \times \mathbf{R}$ may contain the vertical line segment $\{(x,y); \ x = 0 \ \textit{and} \ -1 < y < 1\}$, in which case the line segment should not be viewed as a part of ∂D. By a trivial extension, all functions f in $C_0^\infty(D)$ can be viewed as the functions defined on $\mathbf{R}^+ \times \mathbf{R}$ with $f = 0$ outside D. For each $f \in C_0^\infty(D)$,

$$f(X_{\tau_D \wedge t}, W_{\tau_D \wedge t}) - \int_0^{\tau_D \wedge t} Lf(X(s), W(s)) ds$$

is a martingale. Hence, using the fact that $f(\partial D) = 0$,

$$\frac{1}{t}\Big(T_t f(x,y) - f(x,y) \Big) = \frac{1}{t} \int_0^t \mathbb{E}_{(x,y)} \big(Lf(X(s), W(s)) I_{\{\tau_D \geq s\}} \big) ds \longrightarrow Lf(x,y)$$

as $t \to 0$ and $(x,y) \in D$. Thus the generator L_D of T_t coincides with L on $C_0^\infty(D)$. Since $C_0^\infty(D)$ is a core we have $T_t = e^{tL_D}$ on $C_0^\infty(D)$.

To prove the lower bound we need only to show

$$\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P}\big\{ \sup_{s \leq t} |W(s) - X(s)| < 1 \big\} \geq -\frac{1}{2} \int_G |\nabla f(x,y)|^2 x^{d-1} dxdy \tag{2.2}$$

for every $f \in C_0^\infty(G)$ with $|f|_\pi = 1$ and $K = \overline{\textit{support}(f)} \subset G$ being compact.

We fix an open domain D in $\mathbf{R}^+ \times \mathbf{R}$ with compact closure such that $K \subset D \subset \bar{D} \subset G$. and $(0,0) \in D$. Let $p_D\big(t;(x,y)\big)$ be the density (with respect to the Lebesgue measure) of the measure

$$\mu(A) = P\{\tau_D \ge t; (X(t), W(t)) \in A\}.$$

Then combining results of Azencott (1984) and Leandre (1987, e.g. Theorem 11.3), we have
$\inf_{(x,y)\in K} p_D\big(t_o;(x,y)\big) > 0$ for some $t_o > 0$, see also Stroock and Varadhan (1979).

By Markov property, for $t \ge t_o$, and every f as above

$$\begin{aligned}
&P\{\tau_D \ge t\} = \mathbb{E}_{(0,0)}\Big[I_{\{\tau_D\ge t_o\}}P_{(X(t_o),W(t_o))}\{\tau_D \ge t-t_o\}\Big]\\
&= \int \mathbb{P}_{(x,y)}\{\tau_D \ge t-t_o\}p_D\big(t_o,(x,y)\big)dxdy\\
&\ge |f|_\infty^{-1}\sup\{x^{d-1};(x,y)\in K\}^{-1}\inf_{(x,y)\in K} p_D\big(t_o,(x,y)\big)\int f(x,y)P_{(x,y)}\{\tau_D\ge t-t_o\}x^{d-1}dxdy\\
&\ge |f|_\infty^{-2}\sup\{x^{d-1};(x,y)\in K\}^{-1}\inf_{(x,y)\in K} p_D\big(t_o,(x,y)\big)\int fT_{t-t_o}fx^{d-1}dxdy\\
&= c\int fe^{(t-t_o)L_D}fx^{d-1}dxdy = c < f, e^{(t-t_o)L_D}f >_\pi
\end{aligned}$$

where the third step follows from

$$\begin{aligned}
&\mathbb{P}_{(x,y)}\{\tau_D \ge t-t_o\} = \mathbb{E}_{(x,y)}\big(I_{\{\tau_D\ge t-t_o\}}\big)\\
&\ge |f|_\infty^{-1}\mathbb{E}_{(x,y)}\Big(f(X_t,W_t)I_{\{\tau_D\ge t-t_o\}}\Big) = |f|_\infty^{-1}T_{t-t_o}f(x,y)
\end{aligned}$$

and the constant c does not depend on t.

We now consider the spectral structure of the self adjoint operator L_D. By Jensen's inequality

$$\begin{aligned}
&< f, e^{(t-t_o)L_D}f >_\pi = \int_{-\infty}^0 e^{(t-t_o)\lambda}E_f(d\lambda) \ge \exp\Big\{(t-t_o)\int_{-\infty}^0 \lambda E_f(d\lambda)\Big\}\\
&= \exp\Big\{(t-t_o) < L_D f, f >_\pi\Big\} = \exp\Big\{-(t-t_o)\frac{1}{2}\int_G |\nabla f(x,y)|^2 x^{d-1}dxdy\Big\}
\end{aligned}$$

Here we have used the fact that the spectral measure $E_f(d\lambda)$ is a probability measure due to

$$\int_{-\infty}^0 E_f(d\lambda) = |f|_\pi^2 = 1.$$

Hence (2.1) holds.

On the other hand, using the fact that $\mathbb{P}\big\{\max_{s\le t} X(s) \ge t^2\big\} \le e^{-\delta t^2}$ for any $\delta > 0$ and t large, we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big\{\max_{s\le t}|W(s)-X(s)| < 1\big\} \le \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big\{\tau_{D_t} \ge t\big\}$$

where $D_t = \{(x,y)\in\mathbf{R}^+\times\mathbf{R}: \ 0 \le x < t^2 \text{ and } |y-x| < 1\}$.

Next we observe

$$\mathbb{P}\big\{\tau_{D_t} \ge t\big\} = \mathbb{E}_{(0,0)}\Big[I_{\{\tau_{D_t}\ge 1\}}\mathbb{P}_{(X(1),W(1))}\{\tau_{D_t} \ge t-1\}\Big]$$

$$\le \mathbb{E}_{(0,0)}\Big[I_{\{((X(1),W(1))\in D_t\}}\mathbb{P}_{(X(1),W(1))}\{\tau_{D_t}\ge t-1\}\Big]$$
$$= C\int_{D_t}\mathbb{P}_{(x,y)}\{\tau_{D_t}\ge t-1\}x^{d-1}\exp\Big\{-\frac{x^2+y^2}{2}\Big\}dxdy$$
$$\le C\int_{D_t}\mathbb{P}_{(x,y)}\{\tau_{D_t}\ge t-1\}x^{d-1}dxdy.$$

Given $\epsilon>0$, let D_t^ϵ be the ϵ-neighborhood of D_t in $\mathbf{R}^+\times\mathbf{R}$ and choose a $f_0\in C_0^\infty(D_t^\epsilon)$ such that $f_0\ge 0$ is bounded and $f_0\ge 1$ in D_t. Then

$$\int_{D_t}\mathbb{P}_{(x,y)}\{\tau_{D_t}\ge t-1\}x^{d-1}dxdy$$
$$\le \int_{D_t} f_0(x,y)\mathbb{E}_{(x,y)}\Big[f_0\big(X(t-1),W(t-1)\big)I_{\{\tau_{D_t}\ge t-1\}}\Big]x^{d-1}dxdy$$
$$\le \int_{D_t^\epsilon} f_0(x,y)\mathbb{E}_{(x,y)}\Big[f_0\big(X(t-1),W(t-1)\big)I_{\{\tau_{D_t^\epsilon}\ge t-1\}}\Big]x^{d-1}dxdy$$
$$=< f_0, e^{(t-1)L_{D_t^\epsilon}}f_0>_\pi\le |f_0|_\pi^2\exp\Big\{(t-1)\sup_{|f|_\pi=1,\ f\in C_0^\infty(D_t^\epsilon)}< f, L_{D_t^\epsilon}f>_\pi\Big\}$$
$$\le C_1t^2\exp\Big\{-(t-1)\frac{1}{2}\inf_{|f|_\pi=1,\ f\in C_0^\infty(D_t^\epsilon)}\int_{D_t^\epsilon}|\nabla f(x,y)|^2x^{d-1}dxdy\Big\}$$
$$\le C_1t^2\exp\Big\{-(t-1)\frac{1}{2}\inf_{|f|_\pi=1,\ f\in C_0^\infty(G^\epsilon)}\int_{G^\epsilon}|\nabla f(x,y)|^2x^{d-1}dxdy\Big\}.$$

Hence we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big\{\max_{s\le t}|W(s)-X(s)|<1\big\}$$
$$\le -\frac{1}{2}\inf_{|f|_\pi=1,\ f\in C_0^\infty(G^\epsilon)}\int_{G^\epsilon}|\nabla f(x,y)|^2x^{d-1}dxdy.$$

Letting $\epsilon\longrightarrow 0$ we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big\{\max_{s\le t}|W(s)-X(s)|<1\big\}\le -\frac{1}{2}\inf_{|f|_\pi=1,\ f\in C_0^\infty(G)}\int_G|\nabla f(x,y)|^2x^{d-1}dxdy. \tag{2.3}$$

Therefore, (2.1) follows from (2.2) and (2.3).

The case $d=1$ is slightly different since the operator given by (1.12) is no longer the generator of $(X(t),W(t))$ (p.416, Revuz-Yor (1991)). In this case one can write $X(t)=|W_1(t)|$ where $W_1(t)$ is another 1-dimensional Brownian motion independent of $W(t)$. Notice that the two dimensional Brownian motion $(W_1(t),W(t))$ has $(1/2)\Delta$ as its generator. By the argument we proceed in the case $d\ge 2$,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big\{\max_{s\le t}|W(s)-X(s)|<1\big\}=-\frac{1}{2}\inf_{|f|_\pi=1,\ f\in C_0^\infty(\Gamma)}\int_\Gamma|\nabla f(x,y)|^2dxdy$$

where Γ is given by (1.9) (with $d=1$, of course). A simple argument shows that the infimum can be taken only for those $f\in C_0^\infty(G_1)$ satisfying $f(-x,y)=f(x,y)$ for all $(x,y)\in\Gamma$. Therefore,

$$\inf_{|f|_\pi=1,\ f\in C_0^\infty(G_1)}\int_\Gamma|\nabla f(x,y)|^2dxdy=\inf_{|f|_\pi=1,\ f\in C_0^\infty(G)}\int_G|\nabla f(x,y)|^2dxdy$$

Hence (2.1) remains valid in the case $d = 1$.

It remains to show that

$$\inf_{|f|_\pi=1,\ f\in C_0^\infty(G)} \int_G |\nabla f(x,y)|^2 x^{d-1} dxdy = \frac{\pi^2}{2}. \tag{2.4}$$

Given a $\epsilon > 0$, we take

$$f_\epsilon(x,y) = C_\epsilon e^{-\epsilon x} \cos \frac{\pi}{2}(y-x)$$

where the constant $C_\epsilon > 0$ is determined by $|f_\epsilon|_\pi = 1$. One can easily check that

$$\int_G |\nabla f_\epsilon(x,y)|^2 x^{d-1} dxdy = \frac{\pi^2}{2} + \epsilon^2$$

by using $|f_\epsilon|_\pi^2 = 1$. Since ϵ is arbitrary, we have proved

$$\inf_{|f|_\pi=1,\ f\in C_0^\infty(G)} \int_G |\nabla f(x,y)|^2 x^{d-1} dxdy \le \frac{\pi^2}{2}. \tag{2.5}$$

On the other hand, by the substitution $g(x,y) = x^{(d-1)/2} f(x,y)$ we have that for each $f \in C_0^\infty(G)$,

$$|\nabla f(x,y)|^2 = x^{-(d-1)}|\nabla g(x,y)|^2 + \frac{(d-1)^2}{4} x^{-(d+1)} g^2(x,y) - (d-1)x^{-d} g(x,y)\frac{\partial}{\partial x} g(x,y).$$

For $d \ge 3$, we have by definition of g that $\lim_{x\to 0^+} x^{-1} g^2(x,y) = 0$. Note that

$$\begin{aligned}&\int_G x^{-1} g(x,y)\frac{\partial}{\partial x} g(x,y) dxdy \\ &= \int_{-1}^1 dy \int_0^{y+1} x^{-1} g(x,y)\frac{\partial}{\partial x} g(x,y) dx + \int_1^\infty dy \int_{y-1}^{y+1} x^{-1} g(x,y)\frac{\partial}{\partial x} g(x,y) dx.\end{aligned}$$

By using integration by parts we see that

$$\begin{aligned}&\int_0^{y+1} x^{-1} g(x,y)\frac{\partial}{\partial x} g(x,y) dx \\ &= \frac{1}{2} x^{-1} g^2(x,y)\Big|_{x=0}^{x=y+1} + \frac{1}{2}\int_0^{y+1} x^{-2} g^2(x,y) dx \\ &= \frac{1}{2}\int_0^{y+1} x^{-2} g^2(x,y) dx.\end{aligned}$$

Similarly,

$$\int_{y-1}^{y+1} x^{-1} g(x,y)\frac{\partial}{\partial x} g(x,y) dx = \frac{1}{2}\int_{y-1}^{y+1} x^{-2} g^2(x,y) dx.$$

Therefore,

$$\int_G x^{-1} g(x,y)\frac{\partial}{\partial x} g(x,y) dxdy = \frac{1}{2}\int_G x^{-2} g^2(x,y) dxdy.$$

Combining above calculation together, we have

$$\begin{aligned}
&\int_G |\nabla f(x,y)|^2 x^{d-1} dxdy \\
&= \int_G |\nabla g(x,y)|^2 dxdy + \frac{(d-1)^2}{4}\int_G x^{-2} g^2(x,y)dxdy - (d-1)\int_G x^{-1} g(x,y)\frac{\partial}{\partial x} g(x,y)dxdy \\
&= \int_G |\nabla g(x,y)|^2 dxdy + \Big(\frac{(d-1)^2}{4} - \frac{d-1}{2}\Big)\int_G x^{-2} g^2(x,y)dxdy.
\end{aligned}$$

And thus for $d \geq 3$,

$$\int_G |\nabla f(x,y)|^2 x^{d-1} dxdy \geq \int_G |\nabla g(x,y)|^2 dxdy.$$

Hence we obtain

$$\begin{aligned}
&\inf_{|f|_\pi = 1,\ f\in C_0^\infty(G)} \int_G |\nabla f(x,y)|^2 x^{d-1} dxdy \\
&\geq \inf\Big\{\int_G |\nabla g(x,y)|^2 dxdy;\quad g\in C_0^\infty(G) \text{ and } \int_G g^2(x,y)dxdy = 1\Big\}. \qquad (2.6)
\end{aligned}$$

In the case $d = 1$, (2.6) is automatically holds with equality.

Next we consider the problem over a larger domain with some symmetry. Let $\widetilde{G} = \{(x,y)\in \mathbf{R}^2;\quad |y-x| < 1\}$. By symmetry between x and y,

$$\begin{aligned}
&\inf\Big\{\int_G |\nabla g(x,y)|^2 dxdy;\quad g\in C_0^\infty(G) \text{ and } \int_G g^2(x,y)dxdy = 1\Big\} \\
&= \inf\Big\{\int_{\widetilde{G}} |\nabla g(x,y)|^2 dxdy;\quad g\in C_0^\infty(\widetilde{G}) \text{ and } \int_{\widetilde{G}} g^2(x,y)dxdy = 1\Big\} \\
&\geq 2\inf\Big\{\int_{\widetilde{G}} \Big(\frac{\partial}{\partial x} g(x,y)\Big)^2 dxdy;\quad g\in C_0^\infty(\widetilde{G}) \text{ and } \int_{\widetilde{G}} g^2(x,y)dxdy = 1\Big\} \\
&= 2\inf\Big\{\int_{-\infty}^{\infty} dy \int_{y-1}^{y+1} \Big(\frac{\partial}{\partial x} g(x,y)\Big)^2 dx;\quad g\in C_0^\infty(\widetilde{G}) \text{ and } \int_{\widetilde{G}} g^2(x,y)dxdy = 1\Big\}
\end{aligned} \qquad (2.7)$$

where the first equality in (2.7) needs to be justified. Let

$$G' = \{(x,y)\in \mathbf{R}^2;\quad (-x,-y)\in G\}.$$

Then for each $g\in C_0^\infty(\widetilde{G})$

$$\begin{aligned}
&\int_{\widetilde{G}} |\nabla g(x,y)|^2 dxdy \\
&= \int_G |\nabla g(x,y)|^2 dxdy + \int_{G'} |\nabla g(x,y)|^2 dxdy \\
&= \int_G |\nabla g(x,y)|^2 dxdy + \int_G |\nabla \bar{g}(x,y)|^2 dxdy
\end{aligned}$$

where $\bar{g}(x,y) = g(-x,-y)$. Clearly, $\bar{g}\in C_0^\infty(G)$. Let

$$\lambda_G = \inf\Big\{\int_G |\nabla g(x,y)|^2 dxdy;\quad g\in C_0^\infty(G) \text{ and } \int_G g^2(x,y)dxdy = 1\Big\}.$$

Then

$$\int_G |\nabla g(x,y)|^2 dxdy \geq \lambda_G \int_G g^2(x,y)dxdy$$

for every $g \in C_0^\infty(G)$. Therefore,

$$\int_{\widetilde{G}} |\nabla g(x,y)|^2 dxdy \geq \lambda_G \Big\{ \int_G g^2(x,y)dxdy + \int_G \bar{g}^2(x,y)dxdy \Big\}$$
$$= \lambda_G \Big\{ \int_G g^2(x,y)dxdy + \int_{G'} g^2(x,y)dxdy \Big\} = \lambda_G$$

if $\int_{\widetilde{G}} g^2(x,y)dxdy = 1$.

On the other hand, if $g \in C_0^\infty(G)$ and $\int_G g^2(x,y)dxdy = 1$ satisfies

$$\int_G |\nabla g(x,y)|^2 dxdy < \epsilon + \lambda_G$$

we define $\tilde{g}(x,y) \in C_0^\infty(\widetilde{G})$ by $\tilde{g}(x,y) = g(x,y)/\sqrt{2}$ if $(x,y) \in G$ and $\tilde{g}(x,y) = g(-x,-y)/\sqrt{2}$ if $(x,y) \in G'$. Then

$$\int_{\widetilde{G}} g^2(x,y)dxdy = 1$$

and

$$\int_{\widetilde{G}} |\nabla \tilde{g}(x,y)|^2 dxdy = \int_G |\nabla g(x,y)|^2 dxdy < \epsilon + \lambda_G.$$

So

$$\inf \Big\{ \int_{\widetilde{G}} |\nabla g(x,y)|^2 dxdy; \quad g \in C_0^\infty(\widetilde{G}) \text{ and } \int_{\widetilde{G}} g^2(x,y)dxdy = 1 \Big\} \leq \lambda_G$$

and we finish the justification of (2.7).

Back to the estimate of the lower bound for (2.4). We start with a well known variational identity (see, c.f., Strassen (1964))

$$\inf \Big\{ \int_{-1}^1 |h'(x)|^2 dx; \quad h \in C_0^\infty(-1,1) \textit{ and } \int_{-1}^1 |h(x)|^2 dx = 1 \Big\} = \frac{\pi^2}{4}.$$

Under a simple substitution, this gives that for any $y \in \mathbf{R}$ and any $h \in C_0^\infty(y-1, y+1)$,

$$\int_{y-1}^{y+1} |h'(x)|^2 dx \geq \frac{\pi^2}{4} \int_{y-1}^{y+1} |h(x)|^2 dx.$$

Hence we have that for any $g \in C_0^\infty(\widetilde{G})$ with $\int_{\widetilde{G}} g^2(x,y)dxdy = 1$,

$$\int_{-\infty}^\infty dy \int_{y-1}^{y+1} \Big(\frac{\partial}{\partial x} g(x,y) \Big)^2 dx \geq \frac{\pi^2}{4} \int_{-\infty}^\infty dy \int_{y-1}^{y+1} g^2(x,y)dx = \frac{\pi^2}{4}.$$

In view of (2.7),

$$\inf_{|f|_\pi = 1,\ f \in C_0^\infty(G)} \int_G |\nabla f(x,y)|^2 x^{d-1} dxdy \geq \frac{\pi^2}{2}. \tag{2.8}$$

Finally, (2.4) follows from (2.5) and (2.8) and we finish the proof of Theorem 1.2.

We end this section with the following comment: From the proof of (2.8) one can see that except the case $d = 1$, the infimum can not be reached. This may suggest that the following eigenvalue problem

$$\frac{1}{2}\triangle f(x,y) + \frac{d-1}{2x}\frac{\partial}{\partial x} f(x,y) = -\frac{\pi^2}{2} f(x,y) \quad (x,y) \in G \quad \text{and} \quad f(\partial G) = 0$$

does not have a solution which is reasonably smooth in G (Recall that G contains the line segment $x = 0$ with $-1 < y < 1$) unless $d = 1$, in which case the function

$$f(x,y) = \cos\frac{\pi}{2}(y - x)$$

solves the equation. In the case $d = 5$, one can see that the function

$$f(x,y) = x^{-1}\cos\frac{\pi}{2}(y - x)$$

solves above eigenvalue problem but f fails to be continuous at $x = 0$ with $-1 < y < 1$.

3. Limiting behaviors. Let $B_d(t)$ be a d-dimensional Brownian motion independent of $W(t)$. Since $X \overset{d}{=} |B_d|_2$, we replace $X(t)$ by $|B_d(t)|_2$ in this section.

We first prove (1.3). By Theorem 1.2 and Borel-Cantelli Lemma, one can easily show that

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} \max_{s\le t} \big|W(s) + |B_d(s)|_2\big| \ge \frac{\pi}{2} \quad a.s.$$

On the other hand, let $t_k = k^k$ and let $\lambda > \pi/2$ be fixed. By Theorem 1.2,

$$\begin{aligned}
&\mathbb{P}\Big\{ \max_{t_k\le s\le t_{k+1}} \big|\big(W(s) - W(t_k)\big) + |B_d(s) - B_d(t_k)|_2\big| \le \lambda\sqrt{\frac{t_{k+1}}{\log\log t_{k+1}}}\Big\} \\
&= \mathbb{P}\Big\{ \max_{s\le t_{k+1}-t_k} \big|W(s) + |B_d(s)|_2\big| \le \lambda\sqrt{\frac{t_{k+1}}{\log\log t_{k+1}}}\Big\} \\
&\ge \mathbb{P}\Big\{ \max_{s\le 1} \big|W(s) + |B_d(s)|_2\big| \le \lambda\sqrt{\frac{1}{\log\log t_{k+1}}}\Big\} \\
&= \exp\Big\{ -\frac{\pi^2}{4(\lambda^2 + o(1))}\log\log t_{k+1}\Big\}.
\end{aligned}$$

Consequently

$$\sum_k \mathbb{P}\Big\{ \max_{t_k\le s\le t_{k+1}} \big|\big(W(s) - W(t_k)\big) + |B_d(s) - B_d(t_k)|_2\big| \le \lambda\sqrt{\frac{t_{k+1}}{\log\log t_{k+1}}}\Big\} = \infty.$$

Thus it follows from Borel-Cantelli lemma that,

$$\liminf_{k\to\infty} \sqrt{\frac{\log\log t_{k+1}}{t_{k+1}}} \max_{t_k\le s\le t_{k+1}} \big|\big(W(s) - W(t_k)\big) + |B_d(s) - B_d(t_k)|\big| \le \lambda \quad a.s.$$

Letting $\lambda \longrightarrow \pi/2$ yields

$$\liminf_{k\to\infty} \sqrt{\frac{\log\log t_{k+1}}{t_{k+1}}} \max_{t_k\le s\le t_{k+1}} \big|\big(W(s)-W(t_k)\big)+|B_d(s)-B_d(t_k)|\big| \le \frac{\pi}{2} \quad a.s.$$

Next note that

$$\sqrt{2t_k\log\log t_k} = o\bigg(\sqrt{\frac{t_{k+1}}{\log\log t_{k+1}}}\bigg)$$

and thus by the standard laws of iterated logarithm,

$$\lim_{k\to\infty} \sqrt{\frac{\log\log t_{k+1}}{t_{k+1}}} \max_{s\le t_k} |W(s)| = 0 \quad a.s.$$

and

$$\lim_{k\to\infty} \sqrt{\frac{\log\log t_{k+1}}{t_{k+1}}} \max_{s\le t_k} |B_d(s)|_2 = 0 \quad a.s.$$

Therefore the upper bound of (1.3) in Theorem 1.1 follows from

$$\begin{aligned} &\max_{s\le t_{k+1}} \big|W(s)+|B_d(s)|_2\big| \\ \le &\max_{t_k\le s\le t_{k+1}} \big|\big(W(s)-W(t_k)\big)+|B_d(s)-B_d(t_k)|_2\big| + \max_{s\le t_k}|W(s)| + \max_{s\le t_k}|B_d(s)|_2. \end{aligned}$$

We finished the proof of (1.3) in Theorem 1.1.

To prove (1.1) and (1.2) in Theorem 1.1, we need some upper tail estimates. First we show that

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^{-2}\log\mathbb{P}\Big\{\max_{t\le 1}\big(W(t)+|B_d(t)|_2\big)\ge\lambda\Big\} \\ =&\lim_{\lambda\to\infty} \lambda^{-2}\log\mathbb{P}\Big\{W(1)+|B_d(1)|_2\ge\lambda\Big\} = -\frac{1}{4} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^{-2}\log\mathbb{P}\Big\{\max_{t\le 1}\big(-W(t)-|B_d(t)|_2\big)\ge\lambda\Big\} \\ =&\lim_{\lambda\to\infty} \lambda^{-2}\log\mathbb{P}\Big\{-W(1)-|B_d(1)|_2\ge\lambda\Big\} = -\frac{1}{2}. \end{aligned} \tag{3.2}$$

Consider (W, B_d) as a Gaussian random element in $C\big\{[0,1], \mathbf{R}^{d+1}\big\}$. It follows from standard large deviation estimate,

$$\lim_{\lambda\to\infty} \lambda^{-2}\log\mathbb{P}\big\{\lambda^{-1}(W,B_d)\in F\big\} \le -\inf_{(x,y)\in F} I(x,y)$$

for each close set $F \in C\big\{[0,1], \mathbf{R}^{d+1}\big\}$ and

$$\lim_{\lambda\to\infty} \lambda^{-2}\log\mathbb{P}\big\{\lambda^{-1}(W,B_d)\in G\big\} \ge -\inf_{(x,y)\in G} I(x,y)$$

for each open set $G \in C\big\{[0,1], \mathbf{R}^{d+1}\big\}$, where

$$I(x,y) = \frac{1}{2}\int_0^1 \Big\{|\dot{y}(t)|^2 + \sum_{j=1}^d |\dot{x_j}(t)|^2\Big\}.$$

Thus by contraction principle we only need to show

$$\inf_{\max_{t\le 1}\big(y(t)+|x(t)|_2\big)=1} I(x,y) = \inf_{y(1)+|x(1)|_2=1} I(x,y) = \frac{1}{4} \tag{3.3}$$

and

$$\inf_{\max_{t\le 1}\big(-y(t)-|x(t)|_2\big)=1} I(x,y) = \inf_{-y(1)-|x(1)|_2=1} I(x,y) = \frac{1}{2}. \tag{3.4}$$

Note that

$$\max_{t\le 1}\big(y(t)+|x(t)|_2\big) \le \max_{t\le 1}\sqrt{2}\Big(|y(t)|^2+\sum_{j=1}^{d}|x_j(t)|^2\Big)^{1/2} \le 2\sqrt{I(x,y)}.$$

On the other hand,

$$y(1)+|x(1)|_2 = 2\sqrt{I(x,y)}$$

by taking $y(t) = x_1(t) = \cdots = x_d(t) = ct$ for some positive constant. Hence (3.3) holds.

Similarly, we have

$$-y(1)-||x(1)|| \le -y(t) \le \Big(\int_0^1 |\dot{y}(t)|^2\Big)^{1/2} \le \sqrt{I(x,y)}$$

and,

$$-y(1)-||x(1)|| = \sqrt{I(x,y)}$$

by taking $y(t) = -ct$ and $x_1(t) = \cdots = x_d(t) = 0$. Hence (3.4) holds.

For any $r > 1$, the upper bounds in (3.1) gives that

$$\sum_k \mathbb{P}\Big\{\max_{s\le r^k}\big(W(s)+|B_d(s)|_2\big) \ge (2+\epsilon)\sqrt{r^k\log\log r^k}\Big\} < \infty$$

for any $r > 1$. So Borel-Cantelli Lemma gives that

$$\limsup_{k\to\infty}\frac{1}{\sqrt{r^k\log\log r^k}}\max_{s\le r^k}\big(W(s)+|B_d(s)|_2\big) \le 2 \qquad a.s.$$

By making $r > 1$ arbitrarily close to 1 we obtain the upper bound of (1.1) by a standard deterministic estimate. Notice that the obvious relation $W(t)+X(t) \ge W(t)$ and the law of the iterated logarithm for Brownian motions give the lower bounds for (1.2). By the lower bounds in (3.1) and (3.2), given $\epsilon > 0$,

$$\sum_k \mathbb{P}\Big\{\big(W(r^{k+1})-W(r^k)\big)+|B_d(r^{k+1})-B_d(r^k)|_2 \ge (2-\epsilon)\sqrt{r^{k+1}\log\log r^{k+1}}\Big\} = \infty$$

$$\sum_k \mathbb{P}\Big\{\big(W(r^{k+1})-W(r^k)\big)+|B_d(r^{k+1})-B_d(r^k)|_2 \le -(\sqrt{2}-\epsilon)\sqrt{r^{k+1}\log\log r^{k+1}}\Big\} = \infty$$

for sufficiently large $r > 1$. By independence and Borel-Cantelli Lemma,

$$\limsup_{k\to\infty} \frac{1}{\sqrt{r^{k+1}\log\log r^{k+1}}}\Big[\big(W(r^{k+1})-W(r^k)\big)+|B_d(r^{k+1})-B_d(r^k)|_2\Big] \ge 2-\epsilon \qquad a.s.$$

$$\liminf_{k\to\infty} \frac{1}{\sqrt{r^{k+1}\log\log r^{k+1}}}\Big[\big(W(r^{k+1})-W(r^k)\big)+|B_d(r^{k+1})-B_d(r^k)|_2\Big] \le -\sqrt{2}+\epsilon \qquad a.s.$$

From the classic law of the iterated logarithm,

$$\limsup_{k\to\infty} \frac{1}{\sqrt{r^{k+1}\log\log r^{k+1}}}\big(|W(r^k)| + |B_d(r^k)|_2\big) \le 2\sqrt{2}r^{-1/2}$$

Since $r > 1$ can be arbitrarily large and $\epsilon > 0$ can be arbitrarily small, we have the lower bound of (1.1) and the upper bound of (1.2).

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