Journal of Theoretical Probability, Vol. 15, No. 4, October 2002 (© 2002)

A Functional LIL and Some Weighted Occupation Measure Results for Fractional Brownian Motion

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Received March 12, 2001; revised October 19, 2001

Weighted occupation measure results are obtained for fractional Brownian motion. Proofs depend on small ball probability estimates of the sup-norm for these processes, which are then used to obtain a functional law of the iterated logarithm. The occupation measure results are consequences of the law of the iterated logarithm.

KEY WORDS: Functional LIL; fractional Brownian motion; small ball probabilities; weighted occupation measures.

1. INTRODUCTION AND MAIN RESULTS

Let $\{B(t): -\infty < t < \infty\}$ be a sample continuous Brownian motion with $B(0) = 0$. For $0 < \gamma < 2$ and $t \geqslant 0$, define

$$B_\gamma(t) = \int_{\mathbb{R}^1} k_\gamma(t, x)\, dB(x), \tag{1.1}$$

where

$$k_\gamma(t, x) = \begin{cases} 0 & \text{if} \quad x > t \\ a_\gamma (t-x)^{(\gamma-1)/2} & \text{if} \quad 0 \leqslant x \leqslant t \\ a_\gamma \{|t-x|^{(\gamma-1)/2} - |x|^{(\gamma-1)/2}\} & \text{if} \quad -\infty < x < 0, \end{cases} \tag{1.2}$$

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and

$$a_\gamma = \left(\gamma^{-1} + \int_{-\infty}^{0} ((1-s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2})^2\, ds\right)^{-1/2}. \tag{1.3}$$

If $\gamma = 1$, we naturally interpret $k_\gamma(t, x)$ to be $I_{[0,t]}(x)$, and hence $B_1(t)$ is a sample continuous Brownian motion starting at zero at time zero. Otherwise, $\{B_\gamma(t): t \geqslant 0\}$ is γ-fractional Brownian motion, and we set

$$M_\gamma(t) = \sup_{0 \leqslant s \leqslant t} |B_\gamma(s)| \tag{1.4}$$

for $t \geqslant 0$. For $n \geqslant 1$ and $t \geqslant 0$, let

$$\eta_n(t) = M_\gamma(nt)/(c_\gamma n/LLn)^{\gamma/2}, \tag{1.5}$$

where the constant $0 < c_\gamma < \infty$ is given by

$$c_\gamma = -\lim_{\epsilon \to 0^+} \epsilon^{2/\gamma} \log P(M_\gamma(1) \leqslant \epsilon). \tag{1.6}$$

The existence of c_γ in (1.6), with $0 < c_\gamma < \infty$, is given in Li and Linde[(5)] and independently in Shao.[(11)] The constant $c_1 = \pi^2/8$. For other values of γ, c_γ is unknown, but certain estimates are in Shao.[(10)] It was shown earlier in Monrad and Rootzen,[(8)] and Shao,[(9)] that the lim inf (lim sup) of the right hand term in (1.6) is strictly positive (finite), but here we need (1.6) as stated. Of course, $\eta_n(\cdot)$ in (1.5) depends on γ, but we suppress that to simplify the notation. Throughout, $\log x$ denotes the natural logarithm of x and $LLx = \max(1, \log\log x)$ for $x \geqslant 1$.

Our main objective here is to prove a functional LIL for the sequence $\{\eta_n\}$ based on fractional Brownian motions, and then to apply this result to obtain some weighted occupation measure results for these processes. The proof of our functional LIL follows the broad outlines of the companion result for stable process in Chen *et al.*,[(1)] but is completely different in the details. Once the functional LIL is established, the occupation measure results follow in a fashion similar to that established in Chen *et al.*[(1)]

The functional LIL is motivated by a result in Wichura[(13)] for Brownian motion. Wichura's proof depended on diffusion process techniques, and differs from what we do here, and also from what was done in Chen *et al.*[(1)] This is not a matter of choice, but of necessity at this point in time.

Now we turn to notation for our functional LIL. We let $\mathscr{M}$ be the space of functions $f: [0, \infty) \to [0, \infty]$ such that $f(0) = 0$, f is right continuous on $(0, \infty)$, nondecreasing and $\lim_{t \to +\infty} f(t) = \infty$. We endow $\mathscr{M}$

with the topology of weak convergence, that is, pointwise convergence at all continuity points of the limit function. As can be seen, for example, by the discussion in Chen *et al.*[1] (see pp. 259–263), the weak topology on $\mathscr{M}$ is metrizable, separable, and complete.

If $\{f_n\}$ is a sequence of points in $\mathscr{M}$, then $C(\{f_n\})$ denotes the cluster set of $\{f_n\}$, that is, all possible subsequential limits of $\{f_n\}$ in the weak topology. If $A \subseteq \mathscr{M}$, we write $\{f_n\} \twoheadrightarrow A$ if $\{f_n\}$ is relatively compact in $\mathscr{M}$, and $C(\{f_n\}) = A$ in the weak topology. A subset F of $\mathscr{M}$ is relatively compact in the weak topology if for every $\Gamma > 0$ there exists $t_0 = t_0(\Gamma)$ such that $t \geqslant t_0$ implies $\inf_{f \in F} f(t) \geqslant \Gamma$. This characterization of relative compactness is immediate from the discussion of the weak topology on $\mathscr{M}$ given in Chen *et al.*[1]

The functional LIL we obtain is our first theorem.

Theorem 1.1. Let $\{B_\gamma(t) : t \geqslant 0\}$ be a sample continuous γ-fractional Brownian motion with $B_\gamma(0) = 0$ and $0 < \gamma < 2$. Then

$$P(\{\eta_n\} \twoheadrightarrow K_\gamma) = 1, \tag{1.7}$$

where $\eta_n(\cdot)$ is given in (1.5) and

$$K_\gamma = \left\{ f \in \mathscr{M} : \int_0^\infty f^{-2/\gamma}(s)\, ds \leqslant 1 \right\}. \tag{1.8}$$

Corollary 1.1. Let $\{\eta_n\}$ be as in Theorem 1.1. Then

$$P(\liminf_n \eta_n(1) = 1) = 1. \tag{1.9}$$

Remarks.

(1) Theorem 1.1 is motivated by its companion result in Chen *et al.*[1] for symmetric stable processes having stationary independent increments. As their proofs reveal, the results obtained here, and in Chen *et al.*,[1] are about increasing processes with scaling properties. However, the proofs of the necessary probability estimates for the functional LIL vary considerably in the different settings, and the exact class of processes for which such results hold is not so clear to us at this time.

(2) The first passage time results for symmetric stable processes obtained in Corollary 1.2 of Chen *et al.*[1] have analogues for fractional Brownian motions. They can be derived in exactly the same fashion as Corollary 1.2. Wichura's approach for Brownian motion started with the

first passage time process, and derived a functional LIL from that setup. Our approach reverses that procedure. Additional details are contained in Corollary 1.2 of Chen *et al.*[1]

Our weighted occupation measure results for the fractional Brownian motions are obtained from Theorem 1.1 by applying suitable functionals. This is very much in the spirit of Donsker's invariance principle and Strassen's functional LIL and follows along the lines established in Chen *et al.*[1] For example, we know from Corollary 1.1 that with probability one $\liminf_n \eta_n(1) = 1$, but how fast does $\eta_n(\cdot)$ get away from the zero function, say over the interval $[0, 1]$, or how many samples $\eta_n(1)$, $n \leqslant t$, fall in the interval $[0, c]$, $c \geqslant 1$? One measure of these quantities is the weighted occupation measure

$$\Psi_c(t) = t^{-1} \int_0^t I_{[0,c]}(\eta_s(1)\, \theta(s/t))\, ds, \tag{1.10}$$

where $c \geqslant 1$, θ maps $(0, 1]$ into $[1, \infty)$ with $\theta(1) = 1$, $\eta_s(u) = M(su)/(c_\gamma s/LLs)^{\gamma/2}$ for $s > 0$, $u \geqslant 0$, and $\eta_0(u) = 0$ for all $u \geqslant 0$. We also assume

$$\theta(s) \text{ is non-increasing on } (0, 1] \text{ and } \lim_{s \to 0^+} \theta(s) = \infty, \tag{1.11}$$

and define the function

$$h(s) = \theta^{2/\gamma}(s) + \int_s^1 (\theta^{2/\gamma}(u)/u)\, du, \qquad 0 < s \leqslant 1. \tag{1.12}$$

Note that if θ is continuous on $(0, 1]$ and (1.11) holds with $\theta(1) = 1$, then $h(s)$ is strictly decreasing and continuous on $(0, 1]$. Furthermore, under these conditions it is easy to see that the range of $h(s)$ is all of $[1, \infty)$.

The functions $\theta(s) = (\log(e/s))^{\gamma/2}$ and $\theta(s) = s^{-\beta+\gamma/2}$, where $\beta > \gamma/2$, provide interesting weights which satisfy the conditions formulated in (1.11) and are continuous on $(0, 1]$. Now we can state our weighted occupation measure results.

Theorem 1.2. Let θ: $(0, 1] \to [1, \infty)$ such that $\theta(1) = 1$. In addition, assume (1.11) and that $h(s)$ defined as in (1.12) is strictly decreasing and continuous from $(0, 1]$ onto $[1, \infty)$. Then, with probability one

$$\limsup_{t \to \infty} \Psi_c(t) = 1 - s_c, \tag{1.13}$$

where $s = s_c$ is the unique solution to $h(s) = c^{2/\gamma}$, $c \geqslant 1$.

Remarks.

(1) Theorem 1.2 is proven is Section 4, and requires the parameter s in $\{\eta_s(\cdot)\}$ to converge to infinity continuously rather than through the integers. In particular, $\{\eta_s(\cdot)\}$ must satisfy (3.1), (3.2), (3.3) as $s \to \infty$, but this follows from easy modifications of Section 3 and the related material in Chen *et al.*[1]

(2) Theorem 1.2 and the examples which follow are remarkably similar to their analogues for the symmetric stable processes studied in Chen *et al.*[1] In fact, they only differ in the value of the small ball constants c_γ, and again one conjectures something more general must underly these results. Unlike Theorem 1.1, the proof of Theorem 1.2 and the verification of the examples follows along lines similar to that for their analogues for symmetric stable processes. On the other hand, in order to make precise an argument in the proof of Theorem 1.2 in Chen *et al.*,[1] we include some additional details. These, as well as other comments, are included in the proofs of Theorems 1.2 and 1.3.

Examples. If $\theta(s) = (\log(e/s))^{\gamma/2}$ on $(0, 1]$, then for $0 < s \leqslant 1$, $h(s) = 1 - 2\log s + (\log s)^2/2$. Solving $h(s) = c^{2/\gamma}$, $0 < s \leqslant 1$ and $c \geqslant 1$, we get $s_c = \exp\{2 - 2\sqrt{1 + (c^{2/\gamma} - 1)/2}\}$, and hence with probability one,

$$\begin{aligned}\limsup_{t\to\infty}\, t^{-1} \int_0^t I_{[0,c]}(\eta_s(1)(\log(et/s))^{\gamma/2})\, ds \\ = 1 - \exp\{2 - 2\sqrt{1 + (c^{2/\gamma} - 2)/2}\}\end{aligned}$$

for $c \geqslant 1$.

Let $\theta(u) = u^{-\beta + \gamma/2}$ where $\beta > \gamma/2$. Then $s_c = ((1 - c^{2/\gamma}(1 - 2\beta/\gamma))\,\gamma/2\beta)^{1/(1-\frac{2\beta}{\gamma})}$ and with probability one

$$\limsup_{t\to\infty} \Psi_c(t) = 1 - ((1 - c^{2/\gamma}(1 - 2\beta/\gamma))\,\gamma/2\beta)^{1/(1-\frac{2\beta}{\gamma})}.$$

Another gauge of the rate of escape is the quantity $t^{-1} \int_0^t I_{[0,c]}(\eta_t(s/t))\, ds$, which is similar to $\Psi_c(t)$ (as $t \to \infty$), provided $\theta(s) = s^{\gamma/2}$. With this choice of θ (1.11) fails, and $h(s) = 1$. Thus Theorem 1.2 is not applicable, but the techniques for its proof imply

$$\limsup_{t\to\infty}\, t^{-1} \int_0^t I_{[0,c]}(\eta_t(s/t))\, ds = \begin{cases} 1 & \text{if } \ c \geqslant 1 \\ c^{2/\gamma} & \text{if } \ 0 \leqslant c < 1. \end{cases} \tag{1.14}$$

The rate of escape with respect to L^p distances, $0 < p < \infty$, is

$$\liminf_{t \to \infty} \int_0^1 |\eta_t(u)|^p \, du = \inf_{f \in K_\gamma} \int_0^1 |f(u)|^p \, du = 1. \tag{1.15}$$

Since $\eta_t(\cdot)$ is increasing, the analogue of (1.15) for the sup-norm on $[0, 1]$ follows immediately from (1.9). The proof of (1.14), (1.9), and (1.15) follow much like their analogues in Chen *et al.*,[1] so we do not include the details.

Another class of examples follows. They were not considered in Chen *et al.*,[1] but seem worthwhile to be included here. To describe these results we let

$$H_c(t) = t^{-1} \int_0^t I_{[0, c]}(\eta_t(s/t)\, \theta(s/t))\, ds, \tag{1.16}$$

where θ: $(0, 1] \to [1, \infty)$, $\theta(1) = 1$, and $s\theta(s)^{2/\gamma}$ is non-increasing on $(0, 1]$. We also set

$$g(s) = s\theta(s)^{2/\gamma} + \int_s^1 \theta(u)^{2/\gamma} \, du \tag{1.17}$$

for $0 < s \leqslant 1$.

Although the functionals $\Psi_c(t)$ and $H_c(t)$ look quite different, the fact that

$$\eta_t(s/t)\, \theta(s/t) = \eta_s(1)(s/t)^{\gamma/2}\, \theta(s/t)(LLt/LLs)^{\gamma/2},$$

and that Lemma 4.1 below implies the factor $(LLt/LLs)^{\gamma/2}$ is negligible under certain circumstances, one conjectures that results about the weight $u^{\gamma/2}\theta(u)$ in $\Psi_c(t)$ should translate into results about the weight $\theta(u)$ in $H_c(t)$. Our next result makes this precise, and its proof follows along lines similar to those for Theorem 1.2.

Theorem 1.3. Let θ: $(0, 1] \to [1, \infty)$ satisfy $\theta(1) = 1$, and assume $s\theta^{2/\gamma}(s)$ is non-increasing in s on $(0, 1]$ with $\lim_{s \to 0^+} s\theta^{2/\gamma}(s) = \infty$. Furthermore, assume that $g(s)$ as defined in (1.17) is continuous and one-to-one from $(0, 1]$ onto $[1, \infty)$. Then with probability one

$$\limsup_{t \to \infty} H_c(t) = 1 - s_c, \tag{1.18}$$

where $s = s_c$ is the unique solution to $g(s) = c^{2/\gamma}$, $c \geqslant 1$.

Of course, Theorem 1.3 has examples similar to those for Theorem 1.2.

2. PROBABILITY ESTIMATES

The proof of Theorem 1.1 depends on the probability estimates for weighted sup-norms in this section. The form of these estimates parallels the analogues in Chen *et al.*[1] for symmetric stable processes with stationary independent increments, but the proofs are considerably different due to the lack of independent increments. For a survey of probability estimates of small balls and their applications see Li and Shao.[7] The first result is Theorem 1.1 of Li and Linde.[5]

Lemma 2.1. Let $\{B_\gamma(t): t \geqslant 0\}$ be as above, and for $t \geqslant 0$ define

$$W_\gamma(t) = \int_0^t k_\gamma(t, u)\, dB(u). \tag{2.1}$$

Then, for $0 < \gamma < 2$

$$\lim_{\epsilon \downarrow 0} \epsilon^{2/\gamma} \log P(M_\gamma(1) \leqslant \epsilon) = \lim_{\epsilon \downarrow 0} \epsilon^{2/\gamma} \log P(\sup_{0 \leqslant s \leqslant 1} |W_\gamma(s)| \leqslant \epsilon) = -c_\gamma, \tag{2.2}$$

where

$$0 < c_\gamma = -\inf_{\epsilon > 0} \epsilon^{2/\gamma} \log P(\sup_{0 \leqslant s \leqslant 1} |W_\gamma(s)| \leqslant \epsilon) < \infty.$$

Our first proposition provides a useful upper bound for the probabilities we need to estimate.

Proposition 2.2. Fix sequences $\{t_i\}_{i=1}^m$, $\{a_i\}_{i=1}^m$, and $\{b_i\}_{i=1}^m$ such that $0 = t_0 < t_1 < \cdots < t_m$ and $0 \leqslant a_i < b_i$ for $i = 1, 2,..., m$, and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_m$. Then

$$\overline{\lim_{\epsilon \downarrow 0}}\, \epsilon^{2/\gamma} \log P(a_i \epsilon \leqslant M_\gamma(t_i) \leqslant b_i \epsilon, 1 \leqslant i \leqslant m) \leqslant -c_\gamma \sum_{i=1}^m (t_i - t_{i-1})/b_i^{2/\gamma}. \tag{2.3}$$

Proof. Let $A_i = \{\sup_{t_{i-1} \leqslant s \leqslant t_i} |B_\gamma(s)| \leqslant b_i \epsilon\}$ for $1 \leqslant i \leqslant m$, and for $s \geqslant t \geqslant 0$ set

$$Y_\gamma(t, s) = Z_\gamma(s) + \int_0^t k_\gamma(s, u)\, dB(u), \tag{2.4}$$

where

$$Z_\gamma(s)=\int_{-\infty}^{0} k_\gamma(s,u)\,dB(u). \tag{2.5}$$

Then

$$P(a_i\epsilon \leqslant M_\gamma(t_i)\leqslant b_i\epsilon,\ 1\leqslant i\leqslant m)\leqslant P\left(\bigcap_{i=1}^{m} A_i\right), \tag{2.6}$$

and since

$$\int_{t_{m-1}}^{s} (s-u)^{(\gamma-1)/2}\,dB(u)$$

is independent of $\bigcap_{i=1}^{m-1} A_i$ and also $Y_\gamma(t_{m-1},\cdot)=y(\cdot)$ we have $P(\bigcap_{i=1}^{m} A_i)$ equal to

$$\int_{C[t_{m-1},t_m]} P\left(\bigcap_{i=1}^{m-1} A_i,\ \sup_{t_{m-1}\leqslant s\leqslant t_m}\left|\int_{t_{m-1}}^{s} k_\gamma(s,x)\,dB(x)+y(s)\right|\leqslant b_m\epsilon\right) dP_{Y_\gamma(t_{m-1},\cdot)}(y).$$

Hence by Anderson's inequality, and the independence mentioned previously

$$P\left(\bigcap_{i=1}^{m} A_i\right)\leqslant P\left(\bigcap_{i=1}^{m-1} A_i\right)P\left(\sup_{t_{m-1}\leqslant s\leqslant t_m}\left|\int_{t_{m-1}}^{s} k_\gamma(s,u)\,dB(u)\right|\leqslant b_m\epsilon\right).$$

Iterating the above, we see by setting $r=s-t_{i-1}$ in each term, that

$$\begin{aligned} P\left(\bigcap_{i=1}^{m} A_i\right) &\leqslant \prod_{i=1}^{m} P\left(\sup_{t_{i-1}\leqslant s\leqslant t_i}\left|\int_{t_{i-1}}^{s} k_\gamma(s,u)\,dB(u)\right|\leqslant b_i\epsilon\right)\\ &=\prod_{i=1}^{m} P\left(\sup_{0\leqslant r\leqslant t_i-t_{i-1}}\left|\int_{t_{i-1}}^{r+t_{i-1}} k_\gamma(r+t_{i-1},u)\,dB(u)\right|\leqslant b_i\epsilon\right)\\ &=\prod_{i=1}^{m} P\left(\sup_{0\leqslant r\leqslant t_i-t_{i-1}}\left|\int_{0}^{r} k_\gamma(r,v)\,dB(v)\right|\leqslant b_i\epsilon\right) \end{aligned} \tag{2.7}$$

by setting $v=u-t_{i-1}$ and using the homogeneous increments of Brownian motion.

Now the process $\{W_\gamma(ct): t \geqslant 0\}$ has the same distribution as $\{c^{\gamma/2} W_\gamma(t): t \geqslant 0\}$ for all $c > 0$, and hence setting $s = r/(t_i - t_{i-1})$ we get from (2.7) that

$$P\left(\bigcap_{i=1}^{m} A_i\right) \leqslant \prod_{i=1}^{m} P\left(\sup_{0 \leqslant s \leqslant 1} \left|\int_0^{(t_i - t_{i-1})s} k_\gamma(s(t_i - t_{i-1}), v)\, dB(v)\right| \leqslant b_i \epsilon\right)$$

$$= \prod_{i=1}^{m} P\left(\sup_{0 \leqslant s \leqslant 1} \left|\int_0^{s} k_\gamma(s, u)\, dB(u)\right| \leqslant b_i \epsilon / (t_i - t_{i-1})^{\gamma/2}\right). \tag{2.8}$$

Hence (2.1), (2.2), and (2.8) combine to imply

$$\overline{\lim_{\epsilon \downarrow 0}}\, \epsilon^{2/\gamma} \log P(a_i \epsilon \leqslant M_\gamma(t_i) \leqslant b_i \epsilon, 1 \leqslant i \leqslant m)$$

$$\leqslant \lim_{\epsilon \downarrow 0} \sum_{i=1}^{m} \epsilon^{2/\gamma} \log P\left(\sup_{0 \leqslant s \leqslant 1} \left|\int_0^{s} k_\gamma(s, u)\, dB(u)\right| \leqslant b_i \epsilon / (t_i - t_{i-1})^{\gamma/2}\right)$$

$$\leqslant -c_\gamma \sum_{i=1}^{m} \frac{(t_i - t_{i-1})}{b_i^{2/\gamma}}, \tag{2.9}$$

and the proposition is proven.

Remark. If we define

$$J_\gamma(t) = \sup_{0 \leqslant s \leqslant t} |W_\gamma(s)|, \tag{2.10}$$

then $J_\gamma(1)$ and $M_\gamma(1)$ are related by Lemma 2.1 and the previous argument easily implies the analogue of (2.3) for $J_\gamma(t)$, namely

$$\overline{\lim_{\epsilon \downarrow 0}}\, \epsilon^{2/\gamma} \log P(a_i \epsilon \leqslant J_\gamma(t_i) \leqslant b_i \epsilon, 1 \leqslant i \leqslant m) \leqslant -c_\gamma \sum_{i=1}^{m} \frac{(t_i - t_{i-1})}{b_i^{2/\gamma}}. \tag{2.11}$$

The necessary lower bound we require is the next proposition.

Proposition 2.3. Fix sequences $\{t_i\}_{i=1}^m$, $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$ such that $0 = t_0 < t_1 < \cdots < t_m$ and $0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_m < b_m$. Then,

$$\underline{\lim_{\epsilon \downarrow 0}}\, \epsilon^{2/\gamma} \log P(a_i \epsilon \leqslant M_\gamma(t_i) \leqslant b_i \epsilon, 1 \leqslant i \leqslant m) \geqslant -c_\gamma \sum_{i=1}^{m} (t_i - t_{i-1})/b_i^{2/\gamma}. \tag{2.12}$$

Proof. Let A_i, $1 \leqslant i \leqslant m$, $Y_\gamma(t, s)$, and $Z_\gamma(s)$ be defined as in the proof of Proposition 2.2. The first step of this proof is to obtain a lower bound for $P(\bigcap_{i=1}^m A_i)$.

Take $0 < \delta < b_1 = \min_{1 \leqslant i \leqslant m} b_i$. Then,

$$\begin{aligned} P\left(\bigcap_{i=1}^{m} A_i\right) &\geqslant P\left(\bigcap_{i=1}^{m} A_i, \sup_{t_{m-1} \leqslant s \leqslant t_m} |Y_\gamma(t_{m-1}, s)| \leqslant \delta\epsilon\right) \\ &\geqslant P\left(\bigcap_{i=1}^{m-1} A_i, \sup_{t_{m-1} \leqslant s \leqslant t_m} |Y_\gamma(t_{m-1}, s)| \leqslant \delta\epsilon, \right. \\ &\qquad \left. \sup_{t_{m-1} \leqslant s \leqslant t_m} \left| \int_{t_{m-1}}^{s} k_\gamma(s, u)\, dB(u) \right| \leqslant (b_m - \delta)\, \epsilon \right) \\ &= p_1 \cdot p_2, \end{aligned} \tag{2.13}$$

where

$$p_1 = P\left(\bigcap_{i=1}^{m-1} A_i, \sup_{t_{m-1} \leqslant s \leqslant t_m} |Y_\gamma(t_{m-1}, s)| \leqslant \delta\epsilon\right), \tag{2.14}$$

and

$$p_2 = P\left(\sup_{t_{m-1} \leqslant s \leqslant t_m} \left| \int_{t_{m-1}}^{s} k_\gamma(s, u)\, dB(u) \right| \leqslant (b_m - \delta)\, \epsilon \right). \tag{2.15}$$

Arguing as in (2.7) and (2.8) we have

$$p_2 = P(\sup_{0 \leqslant s \leqslant 1} |W_\gamma(s)| \leqslant (b_m - \delta)\, \epsilon / (t_m - t_{m-1})^{\gamma/2}). \tag{2.16}$$

To estimate p_1, we use the following lemma of Li,[4] which is a weakened form of the Gaussian correlation conjecture. A very short proof appears in Li and Shao.[6]

Lemma 2.4. Let μ be a centered Gaussian measure on separable Banach space E and assume A and B are symmetric μ-measurable convex subsets of E. Then for $0 < \lambda < 1$

$$\mu(A \cap B) \geqslant \mu(\lambda A)\, \mu((1 - \lambda^2)^{1/2} B). \tag{2.17}$$

Thus (2.17) implies for any $0<\lambda<1$ that

$$p_1 = P\left(\bigcap_{i=1}^{m-1} A_i, \sup_{0\leqslant s\leqslant t_m-t_{m-1}} \left|\int_{-\infty}^{t_{m-1}} k_\gamma(s+t_{m-1}, u)\, dB(u)\right| \leqslant \delta\epsilon\right)$$

$$\geqslant P\left(\bigcap_{i=1}^{m-1} \left\{\sup_{t_{i-1}\leqslant s\leqslant t_i} |B_\gamma(s)| \leqslant \lambda b_i \epsilon\right\}\right)$$

$$\cdot P\left(\sup_{0\leqslant s\leqslant t_m-t_{m-1}} \left|\int_{-\infty}^{t_{m-1}} k_\gamma(s+t_{m-1}, u)\, dB(u)\right| \leqslant (1-\lambda^2)^{1/2}\,\delta\epsilon\right). \quad (2.18)$$

Combining the estimate in (2.13) with (2.18), and iterating these estimates, implies

$$P\left(\bigcap_{i=1}^{m} A_i\right) \geqslant q_1 q_2, \quad (2.19)$$

where

$$q_1 = \prod_{i=1}^{m} P\left(\sup_{0\leqslant s\leqslant t_i-t_{i-1}} \left|\int_{-\infty}^{t_{i-1}} k_\gamma(s+t_{i-1}, u)\, dB(u)\right| \leqslant (1-\lambda^2)^{1/2}\,\lambda^{m-i}\delta\epsilon\right) \quad (2.20)$$

and

$$q_2 = \prod_{i=1}^{m} P\left(\sup_{0\leqslant s\leqslant 1} |W_\gamma(s)| \leqslant \lambda^{m-i}(b_i-\delta)\,\epsilon/(t_i-t_{i-1})^{\gamma/2}\right).$$

Applying Lemma 2.1 we see

$$\lim_{\epsilon\downarrow 0} \epsilon^{2/\gamma} \log q_2 = -c_\gamma \sum_{i=1}^{m} \frac{(t_i-t_{i-1})}{(\lambda^{m-i}(b_i-\delta))^{2/\gamma}},$$

and letting $\lambda\uparrow 1$, $\delta\downarrow 0$ we see $P(\bigcap_{i=1}^m A_i)$ has lower bound given by the right-hand side of (2.12) provided we show $\lim_{\epsilon\downarrow 0} \epsilon^{2/\gamma}\log q_1 = 0$.

Now $\log q_1$ is a finite sum of probabilities involving the process

$$R(s) = R_{a,b}(s) = \int_{-\infty}^{a} k_\gamma(a+s, u)\, dB(u) \quad (2.21)$$

where $0\leqslant s\leqslant b$. To study the small ball behavior of the centered Gaussian process $\{R(s)\colon 0\leqslant s\leqslant b\}$ we use the following result established in Talagrand[12] and formulated as given below by Ledoux,[3] p. 257.

Lemma 2.5. Let $\{X(t): t \in T\}$ be a centered Gaussian process. For every $\epsilon > 0$, let $N(T, d, \epsilon)$ denote the minimal number of balls of radius ϵ, under the (pseudo)-metric $d_X(s, t) = (E(X(t) - X(s))^2)^{1/2}$, that are necessary to cover T. Assume that $\psi(\epsilon)$ is defined on $[0, \infty)$ such that $N(T, d, \epsilon) \leqslant \psi(\epsilon)$, and such that for some constants $1 < c_1 \leqslant c_2 < \infty$, $c_1 \psi(\epsilon) \leqslant \psi(\epsilon/2) \leqslant c_2 \psi(\epsilon)$. Then, for some constant $K > 0$, for every $\epsilon > 0$,

$$P(\sup_{s, t \in T} |X(s) - X(t)| \leqslant \epsilon) \geqslant \exp\{-K\psi(\epsilon)\}. \tag{2.22}$$

With the help of Lemma 2.5 we now establish a result which applies to the process $\{R(t): 0 \leqslant t \leqslant b\}$.

Lemma 2.6. Fix $a \geqslant 0$, $b > 0$, $0 < \gamma < 2$, and for $0 \leqslant s \leqslant b$ define

$$R(s) = \int_{-\infty}^{a} k_\gamma(a+s, u)\, dB(u).$$

Then

$$\lim_{\epsilon \downarrow 0} \epsilon^{2/\gamma} \log P(\sup_{0 \leqslant s \leqslant b} |R(s)| \leqslant \epsilon) = 0. \tag{2.23}$$

Proof. For simplicity we assume $b = 1$. There are two cases to consider, and we recall $a \geqslant 0$.

Case 1. $\gamma = 1$ (and hence $a_\gamma = 1$). Then $k_\gamma(t, u) = I_{[0, t]}(u)$ for $t, u \geqslant 0$ and for $0 \leqslant s \leqslant b$

$$R(s) = \int_0^a dB(u) = B(a).$$

Hence in this case (2.23) is obvious.

Case 2. $0 < \gamma < 2$, $\gamma \neq 1$. If $0 < s < t \leqslant 1$, then

$$\begin{aligned} d^2(s, t) &= E\left(\left(\int_{-\infty}^{a} (k_\gamma(a+t, u) - k_\gamma(a+s, u))\, dB(u)\right)^2\right) \\ &= \int_{-\infty}^{a} (k_\gamma(a+t, u) - k_\gamma(a+s, u))^2\, du \\ &= a_\gamma^2 \int_{-\infty}^{a} ((a+t-u)^{(\gamma-1)/2} - (a+s-u)^{\gamma-1)/2})^2\, du \end{aligned}$$

$$= a_\gamma^2 \int_0^\infty ((t+v)^{(\gamma-1)/2} - (s+v)^{(\gamma-1)/2})^2 \, dv$$

$$= a_\gamma^2 \int_s^\infty [((t-s)+r)^{(\gamma-1)/2} - r^{(\gamma-1)/2}]^2 \, dr. \tag{2.24}$$

Now by the mean value theorem

$$|(t-s+r)^{(\gamma-1)/2} - r^{(\gamma-1)/2}| \leqslant |t-s| \, r^{(\gamma-3)/2}$$

for all $r > 0$, and hence for $\min(s, t) > 0$ we have

$$d(s, t) \leqslant a_\gamma \, |t-s| \, (\min(s, t))^{(\gamma-2)/2} \, (2-\gamma)^{-1/2}. \tag{2.25}$$

When $s = 0$, we have from the above that

$$d^2(0, t) = a_\gamma^2 t^\gamma \int_0^\infty [(1+u)^{(\gamma-1)/2} - (u)^{(\gamma-1)/2}]^2 \, du \leqslant C^2 t^\gamma,$$

where C is some constant depending only on γ.

For any $\epsilon > 0$ small, set $N(\epsilon) = \min\{n\colon t_n > 1\}$ where $t_0 = (\epsilon/C)^{2/\gamma}$ so that $d(0, t_0) \leqslant \epsilon$, and $\{t_n\colon n \geqslant 1\}$ are such that

$$a_\gamma (2-\gamma)^{-1/2} \, (t_i - t_{i-1}) \, t_{i-1}^{-(2-\gamma)/2} = \epsilon$$

for $i \geqslant 1$. Taking $t_{i-1} \leqslant 1$, we see for $1 \leqslant i \leqslant N(\epsilon)$,

$$t_i = t_{i-1}(1 + (2-\gamma)^{1/2} \, \epsilon t_{i-1}^{-\gamma/2}/a_\gamma) \geqslant t_{i-1}(1 + (2-\gamma)^{1/2} \, \epsilon/a_\gamma).$$

Thus by iterating these estimates we see

$$\begin{aligned} 1 \geqslant t_{N(\epsilon)-1} &\geqslant t_0(1 + (2-\gamma)^{1/2} \, \epsilon/a_\gamma)^{N(\epsilon)-1} \\ &= (\epsilon/C)^{2/\gamma} \, (1 + (2-\gamma)^{1/2} \, \epsilon/a_\gamma)^{N(\epsilon)-1}, \end{aligned}$$

which implies $N(\epsilon) \leqslant C\epsilon^{-1} \log \frac{1}{\epsilon}$. Here C depends only on γ, but may vary from line-to-line. Hence using t_i, $0 \leqslant i \leqslant N(\epsilon) - 1$, as centers, we obtain

$$N([0, 1], d, \epsilon) \leqslant N(\epsilon) \leqslant C\epsilon^{-1} \log \frac{1}{\epsilon}.$$

Applying Lemma 2.5 we thus have $K > 0$ such that

$$\log P\Big(\sup_{s, t \in [0, 1]} |R(s) - R(t)| \leqslant \epsilon\Big) \geqslant -K\epsilon^{-1} \log \frac{1}{\epsilon}.$$

Thus

$$\begin{aligned}\log P(\sup_{0\leqslant s\leqslant 1}|R(s)|\leqslant\epsilon)&\geqslant\log P\left(\sup_{0\leqslant s\leqslant 1}|R(s)-R(0)|\leqslant\frac{\epsilon}{2},|R(0)|\leqslant\frac{\epsilon}{2}\right)\\&\geqslant\log\left(P\left(\sup_{0\leqslant s\leqslant 1}|R(s)-R(0)|\leqslant\frac{\epsilon}{2}\right)P\left(|R(0)|\leqslant\frac{\epsilon}{2}\right)\right)\\&\geqslant\log P\left(\sup_{0\leqslant s,t\leqslant 1}|R(s)-R(t)|\leqslant\frac{\epsilon}{2}\right)+C\log\epsilon,\end{aligned}$$

where the second inequality is an application of Sidak's inequality. Hence again (2.23) is obvious as $2>\gamma$. Thus, except for the assumption $b=1$, (2.23) holds. However, for $0<b<\infty$, the proof is the same, only constants change, and hence the proof of the lower bound for $P(\bigcap_{i=1}^m A_i)$ is now complete.

To finish the proof of (2.12), and hence the proof of Proposition 2.3, we observe that

$$\begin{aligned}&P(a_i\epsilon\leqslant M_\gamma(t_i)\leqslant b_i\epsilon, i=1,\dots,m)\\&\quad\geqslant P\left(\bigcap_{i=1}^m A_i\right)-\sum_{j=1}^m P(M_\gamma(t_j)<a_j\epsilon, M_\gamma(t_i)\leqslant b_i\epsilon, 1\leqslant i\leqslant m, i\neq j).\end{aligned}\tag{2.26}$$

Then for $\epsilon>0$ sufficiently small, the upper bound in (2.8) and the lower bound for $P(\bigcap_{i=1}^m A_i)$ implies

$$\begin{aligned}&P\left(M_\gamma(t_j)\leqslant a_j\epsilon,\bigcap_{\substack{i=1\\i\neq j}}^m A_i\right)\Big/P\left(\bigcap_{i=1}^m A_i\right)\\&\quad\leqslant\exp\left\{-\epsilon^{-2/\gamma}\left((c_\gamma-\delta)\left[\sum_{\substack{i=1\\i\neq j}}^m(t_i-t_{i-1})/b_i^{2/\gamma}+(t_j-t_{j-1})/a_j^{2/\gamma}\right]\right.\right.\\&\qquad\left.+\epsilon^{-2/\gamma}(c_\gamma+\delta)\sum_{i=1}^m(t_i-t_{i-1})/b_i^{2/\gamma}\right\}\\&\quad=\exp\left\{\delta\epsilon^{-2/\gamma}\left(2\sum_{\substack{i=1\\i\neq j}}^m\frac{(t_i-t_{i-1})}{b_i^{2/\gamma}}+(t_j-t_{j-1})((1/a_j^{2/\gamma})+(1/b_j^{2/\gamma}))\right)\right.\\&\qquad\left.-\epsilon^{-2/\gamma}c_\gamma(t_j-t_{j-1})\left(\frac{1}{a_j^{2/\gamma}}-\frac{1}{b_j^{2/\gamma}}\right)\right\}.\end{aligned}$$

Now take $\delta > 0$ sufficiently small such that for $j = 1, \ldots, m$

$$c_\gamma (t_j - t_{j-1})(1/a_j^{2/\gamma} - 1/b_j^{2/\gamma}) > 2\delta \left(\sum_{i=1}^{m} (t_i - t_{i-1})/a_i^{2/\gamma} \right),$$

which implies

$$\lim_{\epsilon \downarrow 0} P\left(M_\gamma(t_j) \leqslant a_j \epsilon, \bigcap_{\substack{i=1 \\ i \neq j}}^{m} A_i \right) \Big/ P\left(\bigcap_{i=1}^{m} A_i \right) = 0 \tag{2.27}$$

for each $j = 1, \ldots, m$. Thus

$$\lim_{\epsilon \downarrow 0} \sum_{j=1}^{m} P\left(M_\gamma(t_j) < a_j \epsilon, \bigcap_{\substack{i=1 \\ i \neq j}}^{m} A_i \right) \Big/ P\left(\bigcap_{i=1}^{m} A_i \right) = 0,$$

and for $\epsilon > 0$ sufficiently small (2.26) then implies

$$P(a_i \epsilon \leqslant M_\gamma(t_i) \leqslant b_i \epsilon, 1 \leqslant i \leqslant m) \geqslant P\left(\bigcap_{i=1}^{m} A_i \right) \Big/ 2.$$

Thus the above implies

$$\begin{aligned} \underline{\lim}_{\epsilon \downarrow 0} \epsilon^{+2/\gamma} \log P(a_i \epsilon \leqslant M_\gamma(t_i) \leqslant b_i \epsilon, 1 \leqslant i \leqslant m) &\geqslant \underline{\lim}_{\epsilon \downarrow 0} \epsilon^{2/\gamma} \log P\left(\bigcap_{i=1}^{m} A_i \right) \\ &= -c_\gamma \sum_{i=1}^{m} (t_i - t_{i-1})/b_i^{2/\gamma}, \end{aligned}$$

and Proposition 2.3 is proven.

3. PROOF OF THEOREM 1.1

Given the probability estimates of Propositions 2.2 and 2.3, the proof of Theorem 1.1 follows once one establishes the following three facts. That is,

$$P(\{\eta_n\}) \text{ is relatively compact in } \mathscr{M} = 1, \tag{3.1}$$

$$P(C(\{\eta_n\}) \subseteq K_\gamma) = 1, \qquad \text{and} \tag{3.2}$$

$$P(K_\gamma \subseteq C(\{\eta_n\})) = 1. \tag{3.3}$$

The topology on $\mathscr{M}$ is given by weak convergence, and is discussed in detail in Chen *et al.*[1] As mentioned previously, the weak topology $\mathscr{M}$ is separable and metric, and a subset F of $\mathscr{M}$ is relatively compact if for every $\Gamma > 0$ there exists $t_0 = t_0(\Gamma)$ such that $t \geqslant t_0$ implies $\inf_{f \in F} f(t) \geqslant \Gamma$. The proofs of (3.1) and (3.2) follow as for their analogues in Chen *et al.*[1] Hence we do not include the details, but turn to (3.3), whose proof is different in this setting.

Proposition 3.1. $P(K_\gamma \subseteq C(\{\eta_n\})) = 1$.

Proof. Fix $f \in \mathscr{K}_\gamma$, with $\Lambda(f) = \int_0^\infty (f(t))^{-2/\gamma}\, dt < 1$. Then by the argument in Chen *et al.*,[1] it suffices to show for every weak neighborhood N_f of f that

$$P(\eta_n \in N_f \text{ i.o.}) = 1. \tag{3.4}$$

Let $t_f^* = \sup\{t\colon f(t) < \infty\}$. Then $t_f^* = \infty$ or $0 \leqslant t_f^* < \infty$. If $t_f^* = \infty$, then a typical weak neighborhood of f is of the form $N = \bigcap_{j=1}^{\ell} \Gamma_j$ where $0 < t_1 < \cdots < t_\ell$,

$$\Gamma_j = \{g\colon f(t_j) - \theta < g(t_j) < f(t_j) + \theta\}, \tag{3.5}$$

and $\theta > 0$. If $0 < t_f^* < \infty$, then a typical weak neighborhood of f is of the form

$$N_f = \left(\bigcap_{j=1}^{\ell} \Gamma_j\right) \cap \left(\bigcap_{k=1}^{s} R_{r+k}\right) \tag{3.6}$$

where $0 = t_0 < t_1 < \cdots < t_\ell < t_f^* \leqslant t_{r+1} < \cdots < t_{r+s}$, Γ_j is as in (3.5), and $R_{r+k} = \{g\colon f(t_{r+k}) > m_k\}$. When $t_f^* = 0$, then a typical neighborhood of f is of the form

$$N_f = \bigcap_{k=1}^{s} R_k \tag{3.7}$$

where $R_k = \{g\colon f(t_k) > m_k\}$.

Assuming $\Lambda(f) < 1$, we consider only the case $t_f^* = \infty$ (the other cases being much the same). Then $N_f = \bigcap_{j=1}^{\ell} \Gamma_j$ where Γ_j is as in (3.5). To verify (3.4) we take $n_k = k^k$. Then we define

$$\tilde{k}_\gamma(t, x) = b_\gamma\{|x-t|^{(\gamma-1)/2} - |x|^{(\gamma-1)/2}\}, \qquad -\infty < x < \infty, \quad t \geqslant 0,$$

where

$$b_\gamma = \left(\int_{-\infty}^{\infty} (|1-s|^{(\gamma-1)/2} - |s|^{(\gamma-1)/2})^2 \, ds \right)^{-1/2},$$

and set

$$\tilde{B}_\gamma(t) = \int_{-\infty}^{\infty} \tilde{k}(t, x) \, dB(x), \qquad t \geqslant 0.$$

$\{\tilde{B}_\gamma(t) \colon t \geqslant 0\}$ is also γ-fractional Brownian motion, and we will use this representation to take advantage of some previous results in Kuelbs *et al.*[2]

We also define

$$Z_r(t) = \int_{\{d_{r-1} \leqslant |x| \leqslant d_r\}} \tilde{k}_\gamma(t, x) \, dB(x),$$

and

$$X_r(t) = \tilde{B}_\gamma(t) - Z_r(t)$$

for $t \geqslant 0$, $d_r = r^{r+(1-\phi)}$, $0 < \phi < 1$, $r \geqslant 1$.

Next define for $r \geqslant 1$

$$A_r = \left\{ \sup_{0 \leqslant s \leqslant t_i} |Z_r(n_r s)| / (c_\gamma n_r / LLn_r)^{\gamma/2} \in \left(f(t_i) - \frac{3\beta}{2}, f(t_i) + \frac{3\beta}{2} \right), 1 \leqslant i \leqslant \ell \right\}$$

$$B_r = \{ \sup_{0 \leqslant s \leqslant t_\ell} |X_r(n_r s)| / (c_\gamma n_r / LLn_r)^{\gamma/2} \geqslant \beta/2 \}$$

$$C_r = \{ \tilde{\eta}_{n_r}(t_i) \in (f(t_i) - \beta, f(t_i) + \beta), 1 \leqslant i \leqslant \ell \}$$

where

$$\tilde{\eta}_n(t) = \sup_{0 \leqslant s \leqslant t} |\tilde{B}_\gamma(ns)| / (c_\gamma n / LLn)^{\gamma/2}, \qquad t \geqslant 0.$$

Then

$$C_r \subseteq A_r \cup B_r, \tag{3.8}$$

and applying Proposition 2.3 with $M_\gamma(t) = \sup_{0 \leqslant s \leqslant t} |\tilde{B}_\gamma(s)|$ and rescaling we see

$$\underline{\lim}_{r} \ (LLn_r)^{-1} \log P(C_r) \geqslant -\sum_{i=1}^{\ell} \ (t_i - t_{i-1})/(f(t_i)+\beta)^{2/\gamma}$$

$$\geqslant -\int_0^\infty \ (f(s)+\beta)^{-2/\gamma} \, ds.$$

Since $\int_0^\infty f(s)^{-2/\gamma} \, ds < 1$ we can choose $\beta > 0$ sufficiently small that $\int_0^\infty (f(s)+\beta)^{-2/\gamma} \, ds < 1-\beta$ and hence for r sufficiently large

$$P(C_r) \geqslant \exp\{-(LLn_r)(1-\beta)\}. \tag{3.9}$$

Thus with $n_r = r^r$ we see $\sum_{r \geqslant 1} P(C_r) = \infty$.

Hence we have $\sum_{r \geqslant 1} P(A_r) = \infty$, provided we show $\sum_{r \geqslant 1} P(B_r) < \infty$. This will complete the proof since the Z_r's are independent, and the Borel–Cantelli lemma therefore implies

$$P(A_r \text{ i.o.}) = 1. \tag{3.10}$$

That is, if (3.10) holds, and $\sum_{r \geqslant 1} P(B_r) < \infty$, then $P(B_r \text{ i.o.}) = 0$ and hence with probability one we have

$$\overline{\lim_{r}} \sup_{0 \leqslant s \leqslant t_\ell} |X(n_r s)|/(c_\gamma n_r/LLn_r)^{\gamma/2} \leqslant \beta/2. \tag{3.11}$$

Then we have

$$P(\tilde{\eta}_{n_r}(t_i) \in (f(t_i) - 2\beta, f(t_i) + 2\beta), 1 \leqslant i \leqslant \ell, \text{ i.o. in } r) = 1$$

and since $\beta > 0$ is arbitrary we have (3.4).

Thus it remains to show $\sum_{r \geqslant 1} P(B_r) < \infty$ for all $\beta > 0$. Since $d_r = r^{r+(1-\phi)}$, $0 < \phi < 1$, $n_r = r^r$, we have

$$\{X_r(t): 0 \leqslant t \leqslant t_\ell n_r\} \stackrel{\mathscr{L}}{=} \{n_r^{\gamma/2} Y_r(t/n_r): 0 \leqslant t \leqslant t_\ell n_r\}$$

where $\{Y_r(t): 0 \leqslant t \leqslant t_\ell\}$ is given as in Lemma 3.4 of Kuelbs *et al.*[2] Note that defining $Y_r(\cdot)$ for $0 \leqslant t \leqslant t_\ell$ requires a slightly different proof, but this is without difficulty. Hence by Lemma 3.3 and Lemma 3.4 in Kuelbs *et al.*,[2] with $q = 0$, $\theta = \hat{\theta}$, $\delta = \hat{\delta}$ there, we obtain for some $c > 0$ that

$$P(B_r) = P(\sup_{0 \leqslant s \leqslant t_\ell} |Y_r(s)| > \beta((LLn_r)/c_\gamma)^{-\gamma/2})$$

$$\leqslant \frac{1}{\hat{\theta}} \exp\{-\hat{\theta}((cr^{-\hat{\delta}})^{-1} \beta^2((LLn_r)/c_\gamma)^{-\gamma}\}.$$

Hence $\sum_{r \geqslant 1} P(B_r) < \infty$ for all $\beta > 0$, and the proof of Proposition 3.1 is complete.

4. PROOF OF THEOREM 1.2 AND THEOREM 1.3

First we provide two lemmas which allow us to identify the left-hand terms in (1.13), (1.14), and (1.18). The proof of the first follows in similar fashion to its analogue in Chen *et al.*[1] The second makes precise an argument in the proof of Theorem 1.2 in Chen *et al.*,[1] which certainly is less than obvious in light of the details necessary to establish Lemma 4.2 below. In particular, it imposes slightly stronger assumptions on the function θ than those used previously.

Lemma 4.1. Let $F_c(f) = \int_0^1 I_{[0,c]}(f(u)\, r(u))\, du$, and

$$G_c(t) = \int_0^1 I_{[0,c]}\left(\eta_t(u)\, r(u)\left(\frac{LLtu}{LLt}\right)^{\gamma/2}\right) du,$$

where $r\colon (0,1] \to (0,\infty)$ is measurable. Then for $c > 0$, with probability one,

$$\limsup_{t \to \infty} G_c(t) \leqslant \sup_{f \in K_\gamma} F_c(f). \tag{4.1}$$

Furthermore, we have equality in (4.1) whenever $\sup_{f \in K_\gamma} F_x(f)$ is left continuous in x at c.

Lemma 4.2. Let g be real-valued, non-negative, and continuous on $(0,1]$ with $0 < g(1) < 1$. If $tg(t)$ is non-increasing on $(0,1]$ and $\lim_{t \downarrow 0} tg(t) > 1$, then

$$\sup_{f \in K} \int_0^1 I_{\{t:\, f(t) \geqslant g(t)\}}(x)\, dx = 1 - u_0, \tag{4.2}$$

where K is the set of non-negative, non-increasing, right-continuous functions f on $(0,1]$ with $\int_0^1 f(t)\, dt \leqslant 1$, and u_0 satisfies

$$u_0\, g(u_0) + \int_{u_0}^1 g(u)\, du = 1. \tag{4.3}$$

Proof. To simplify notation let $E_f = \{t\colon f(t) \geqslant g(t)\}$ and m be Lebesgue measure on $[0,1]$. Let $\{f_n\} \subseteq K$ be such that $\lim_n m(E_{f_n}) = \sup_{f \in K} m(E_f)$. Then by a standard diagonalization argument there exists a function h on $(0, 1]$ which is non-negative and right continuous, and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\lim_k f_{n_k}(t) = h(t)$$

at all continuity points of h. Furthermore, by Fatou's lemma $h \in K$. Since

$$\overline{\lim_k}\, I_{E_{f_{n_k}}}(x) = \overline{\lim_k}\, I_{[1,\infty)}(f_{n_k}(x)/g(x)) \leqslant I_{[1,\infty)}(h(x)/g(x))$$

except for possibly countably many points in $(0, 1]$, we thus have

$$\lim_k m(E_{f_{n_k}}) \leqslant \int_0^1 \overline{\lim_k}\, I_{E_{f_{n_k}}}(x)\, dx \leqslant \int_0^1 I_{E_h}(x)\, dx = m(E_h).$$

Thus h is a point in K where the supremum in (4.2) is achieved. Furthermore, we may assume $h = g$ on $E_h = \{t\colon h(t) \geqslant g(t)\}$ as h and g are right continuous on $(0, 1]$, and otherwise h would not be optimal. We do this throughout the remainder of the proof.

If $\lambda(s) = sg(s) + \int_s^1 g(u)\, du$, then λ is continuous, non-negative, and non-increasing in $(0, 1]$ with $\lambda(1) < 1$. Since $\int_0^1 g(u)\, du = \infty$, there is a unique point $u_0 \in (0, 1)$ such that $\lambda(u_0) = 1$. Hence (4.3) holds for a unique u_0.

Furthermore, the function f_0 defined by

$$f_0(s) = \begin{cases} g(u_0) & 0 < s \leqslant u_0 \\ g(s) & u_0 \leqslant s \leqslant 1 \end{cases}$$

is then in K and $m(E_{f_0}) = 1 - u_0$. Hence it remains to show $h = f_0$, where h is a point in K where the supremum in (4.2) is attained.

To do this we select $s_0 = \inf\{s > 0 : \int_0^s I_{E_h}(s)\, ds > 0\}$. Then by right continuity $h(s_0) = g(s_0)$. If $s_0 \geqslant u_0$, then we are done since $s_0 > u_0$ is impossible as $sg(s)$ is decreasing on $(0, 1]$, and $s_0 = u_0$ easily implies $h = f_0$. If $s_0 = 0$, then there exists $t_n \downarrow 0$ such that $h(t_n) = g(t_n)$, and

$$\int_0^{t_n} I_{E_h}(s)\, ds > 0.$$

However, since $\int_0^{t_n} h(s)\, ds \geqslant t_n h(t_n) = t_n g(t_n) > 1$ for large n, this is impossible as $\int_0^1 h(s)\, ds = 1$.

Hence $0 < s_0 < u_0$, and we let $T = \int_{s_0}^1 (g(s) - h(s))\, ds$. Then $T > 0$, or

$$\int_0^1 h(s)\, ds \geqslant s_0 g(s_0) + \int_{s_0}^1 g(s)\, ds > u_0 g(u_0) + \int_{u_0}^1 g(s)\, ds = 1.$$

The strict inequality follows in the above as $sg(s) \downarrow$ on $(0, 1]$ and $g(s) > 0$ on $(0, 1]$ with $0 < s_0 < u_0$.

Now pick s_1, $s_0 < s_1 \leqslant u_0$, such that $h(s_1) = g(s_1)$, $\int_{s_0}^{s_1} (g(s) - h(s))\, ds \leqslant T/4$, and $0 < \int_{s_0}^{s_1} h(s)\, I_{E_h}(s)\, ds \leqslant T/4$. Such a choice of s_1 is possible since $\int_{s_0}^{s_0+\delta} I_{E_h}(s)\, ds > 0$ for all $\delta > 0$, and hence s_1 can be taken arbitrarily close to s_0. Now pick s_2 such that $s_1 < s_2 \leqslant 1$ and

$$\int_{s_0}^{s_1} h(s)\, I_{E_h}(s)\, ds = \int_{s_1}^{s_2} (g-h)(s)\, I_{E_h^c}(s)\, ds. \tag{4.4}$$

Recall that since $h(s) \leqslant g(s)$ on $(0, 1]$, $E_h^c = \{t \in (0, 1] : h(t) < g(t)\}$. Thus we define

$$h_1(t) = \begin{cases} h(s_1) = g(s_1) & 0 < t \leqslant s_1 \\ g(t) & s_1 \leqslant t < s_2 \\ h(t) & s_2 \leqslant t \leqslant 1, \end{cases}$$

and define AS to be the area saved and AA the area added in comparing the area under h_1 to that under h. Thus

$$\begin{aligned} AS &= s_0\, g(s_0) - s_1\, g(s_1) + \int_{s_0}^{s_1} h(s)\, ds \\ &\geqslant s_0\, g(s_0) - s_1\, g(s) + \int_{s_0}^{s_1} h(s)\, I_{E_h}(s)\, ds \\ &\geqslant \int_{s_0}^{s_1} h(s)\, I_{E_h}(s)\, ds \\ &> g(s_1) \int_{s_0}^{s_1} I_{E_h}(s)\, ds, \end{aligned} \tag{4.5}$$

where the second inequality holds since $sg(s)$ is non-increasing on $(0, 1]$ and the strict inequality because g is strictly decreasing on $(0, 1]$ with $\int_{s_0}^{s_1} I_{E_h}(s)\, ds > 0$ by our choice of s_1. By our choice of s_2, the area added is

$$AA = \int_{s_1}^{s_2} (g(s) - h(s))\, I_{E_h^c}(s)\, ds = \int_{s_0}^{s_1} h(s)\, I_{E_h}(s)\, ds.$$

Thus $\int_0^1 h_1(s)\, ds \leqslant \int_0^1 h(s)\, ds = 1$, which implies $h_1 \in K$.

Furthermore, $m(E_{h_1}) > m(E_h)$, because

$$g(s_1)\int_{s_0}^{s_1} I_{E_h}(s)\,ds < \int_{s_0}^{s_1} hI_{E_h}(s)\,ds$$

$$= \int_{s_1}^{s_2} (g(s)-h(s))\,I_{E_h^c}(s)\,ds$$

$$< g(s_1)\int_{s_1}^{s_2} I_{E_h^c}(s)\,ds, \tag{4.6}$$

where the first inequality follows from (4.5), the equality by definition of s_2 in (4.4), and the last inequality since g is strictly decreasing on $(0, 1]$ with $\int_{s_1}^{s_2} I_{E_h^c}(s)\,ds > 0$. Thus (4.6) implies $\int_{s_0}^{s_1} I_{E_h}(s)\,ds < \int_{s_1}^{s_2} I_{E_h^c}(s)\,ds$, which implies $m(E_{h_1}) > m(E_h)$.

Therefore we have a contradiction to $m(E_h)$ being maximal, so $0 < s_0 < u_0$ is impossible. Thus the only possible choice is $s_0 = u_0$, and $h = f_0$ as claimed.

Proof of Theorem 1.2. Since $\eta_s(1) = \eta_t(s/t)(tLLs/sLLt)^{\gamma/2}$ for $s, t > 0$, letting $u = s/t$ implies $\Psi_c(t)$ as given in (1.10), satisfies

$$\Psi_c(t) = \int_0^1 I_{[0,\,c]}\left(\eta_t(u)\,u^{-\gamma/2}\theta(u)\left(\frac{LLtu}{LLt}\right)^{\gamma/2}\right)du.$$

Applying Lemma 4.1 with $r(u) = u^{-\gamma/2}\theta(u)$ implies $\limsup_{t\to\infty}\Psi_c(t) = \sup_{f\in K_\gamma} F_c(f)$ with probability one, provided $\sup_{f\in K_\gamma} F_x(f)$ is left continuous in x at c.

When $c > 1$, Lemma 4.2 implies

$$\sup_{f\in K_\gamma} F_c(f) = \sup_{f\in K_\gamma}\int_0^1 I_{[0,\,c]}(f(u)\,u^{-\gamma/2}\theta(u))\,du = 1 - s_c. \tag{4.7}$$

That is, if $f(u) > cu^{\gamma/2}/\theta(u)$ on $[0, 1]$ then $f^{-2/\gamma}(u) < \theta^{2/\gamma}(u)/(c^{2/\gamma}u)$ on $(0, 1]$. Letting $g(u) = \theta^{2/\gamma}(u)/(c^{2/\gamma}u)$ on $(0, 1]$, we see from (1.11) that g satisfies the conditions in Lemma 4.2. Hence from (4.2) we have

$$\sup_{f\in K_\gamma}\int_0^1 I_{\{t:\, f^{-2/\gamma}(t)\geqslant g(t)\}}(x)\,dx = 1 - s_c, \tag{4.8}$$

where s_c is as defined in the theorem. However, since

$$\{t: f^{-2/\gamma}(t) \geqslant g(t)\} = \{t: f(t)\,t^{-\gamma/2}\theta(t) \leqslant c\},$$

we have (4.7) holding.

Therefore for $c>1$, $\sup_{f\in K_\gamma} F_c(f)=1-s_c$, and since h as given in (1.12) is one-to-one and continuous from $(0, 1]$ onto $[1, \infty)$ with $h(1)=1$ we have s_c continuous in c for each $c>1$. Thus Lemma 4.1, (4.7), and (4.8) imply (1.13) for $c>1$. If $c=1$, then for $\delta>0$

$$0 \leqslant \sup_{f\in K_\gamma} F_1(f) \leqslant \sup_{f\in K_\gamma} F_{1+\delta}(f)=1-s_{1+\delta},$$

and since $\lim_{\delta\downarrow 0} s_{1+\delta}=s_1=1$, we have $\sup_{f\in K_\gamma} F_c(f)=0$ when $c=1$. Thus the upper bound in (4.1) implies with probability one that $\limsup_{t\to\infty} \Psi_c(t)\leqslant 0$ when $c=1$. However, this lim sup is clearly non-negative, so (1.13) holds even when $c=1$. Thus Theorem 1.2 is proven.

Proof of Theorem 1.3. Let $H_c(t)$ be given by (1.16) and set $F_c(f)=\int_0^1 I_{[0,c]}(f(u)\,\theta(u))\,du$. Then, as in the proof of Lemma 4.1, with probability one we have

$$\limsup_{t\to\infty} H_c(t)\leqslant \sup_{f\in K_\gamma} F_c(f). \tag{4.9}$$

Furthermore, we have equality in (4.9) whenever $\sup_{f\in K_\gamma} F_x(f)$ is left continuous in x at c.

When $c>1$, Lemma 4.2 implies $\sup_{f\in K_\gamma} F_c(f)=\sup_{f\in K_\gamma}\int_0^1 I_{[0,c]}$ $(f(u)\,\theta(u))\,du=1-s_c$, where s_c is defined as in Theorem 1.3. The proof of this can completed by applying Lemma 4.2 as was done in the proof of Theorem 1.2 above. Here the function $g(s)$ in Lemma 4.2 is taken to be $\theta(s)^{2/\gamma}/c^{2/\gamma}$. All the details are the same, and hence Theorem 1.3 follows.

ACKNOWLEDGMENTS

J.K. and W.V.L. are supported in part by NSF Grants.

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