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A normal comparison inequality and its applications

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Abstract. Let $\xi = (\xi_i, 1 \le i \le n)$ and $\eta = (\eta_i, 1 \le i \le n)$ be standard normal random variables with covariance matrices $R^1 = (r_{ij}^1)$ and $R^0 = (r_{ij}^0)$, respectively. Slepian's lemma says that if $r_{ij}^1 \ge r_{ij}^0$ for $1 \le i, j \le n$, the lower bound $\mathbb{P}(\xi_i \le u \text{ for } 1 \le i \le n)/\mathbb{P}(\eta_i \le u \text{ for } 1 \le i \le n)$ is at least 1. In this paper an upper bound is given. The usefulness of the upper bound is justified with three concrete applications: (i) the new law of the iterated logarithm of Erdős and Révész, (ii) the probability that a random polynomial does not have a real zero and (iii) the random pursuit problem for fractional Brownian particles. In particular, a conjecture of Kesten (1992) on the random pursuit problem for Brownian particles is confirmed, which leads to estimates of principal eigenvalues.

1. Introduction

It is well known now that Slepian's inequality (lemma) and its variations provide a very useful tool in the theory of Gaussian processes and probability in Banach spaces. Very nice discussions with various applications can be found in Ledoux and Talagrand (1991) and Lifshits (1995). The simplest form of Slepian's lemma for centered Gaussian vectors $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ states that for any x,

$$\mathbb{P}\Big(\max_{1 \le i \le n} X_i \le x\Big) \le \mathbb{P}\Big(\max_{1 \le i \le n} Y_i \le x\Big). \tag{1.1}$$

if $\mathbb{E}\, X_i^2 = \mathbb{E}\, Y_i^2$ and $\mathbb{E}\, X_i X_j \le \mathbb{E}\, Y_i Y_j$ for all $i, j = 1, 2, ..., n$. An interesting and useful extension of Slepian's inequality, involving min-max, etc, can be found in Gordon (1985) with applications to local structure of finite-dimensional Banach spaces.

Another well-known and useful extension in (1.2) provides also estimate in the 'reverse' direction. To state it and fix the notation for the rest of this paper, let $\xi_1, \xi_2, \cdots, \xi_n$ be standard normal random variables with covariance matrix $R^1 = (r_{ij}^1)$, and $\eta_1, \eta_2, \cdots, \eta_n$ standard normal random variables with covariance matrix $R^0 = (r_{ij}^0)$. Put $\rho_{ij} = \max(|r_{ij}^1|, |r_{ij}^0|)$. Then as stated in Leadbetter, Lindgren and

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Rootzén (1983), page 81, and based on early works of Slepian (1962), Berman (1964, 1971), Cramér and Leadbetter (1967),

$$\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\Big) - \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\Big) \le \frac{1}{2\pi} \sum_{1\le i<j\le n} (r_{ij}^1 - r_{ij}^0)^+ (1-\rho_{ij}^2)^{-1/2} \exp\Big(-\frac{(u_i^2+u_j^2)}{2(1+\rho_{ij})}\Big), \quad (1.2)$$

for any real numbers u_i, $i = 1, 2, \cdots, n$.

The above normal comparison inequality plays a very important role in the extreme value theory. In a typical situation, one has $r_{ij}^1 \ge r_{ij}^0$ for every $1 \le i < j \le n$ and wants to estimate the probability of $\mathbb{P}(\xi_j \le u_j$ for $j = 1, \cdots, n)$, knowing the probability $\mathbb{P}(\eta_j \le u_j$ for $j = 1, \cdots, n)$. Then, (1.2) also implies zero as a lower bound for the left hand side of (1.2). However, the upper bound in (1.2) may be useless when the error bound is not close to zero.

The main aim of this paper is to give two refinements of the upper bound in (1.2) driven by three applications. The first application is the determination of constants in the so-called law of the iterated logarithm of Erdős and Révész (1990), based on the work of Shao (1994). The second application is an estimate of the decay exponent of the probability that a random polynomial does not have a real zero, which improves the one in Dembo, Poonen, Shao and Zeitouni (2000). The third and the most significant application is to the random pursuit problem for fractional Brownian particles. In particular, a conjecture of Kesten (1992) on the random pursuit problem for Brownian particles is confirmed. From exit probability point of view, the result is about the growth rate of the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on a subset of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ as dimension $n \to \infty$. From a large deviation point of view, the result is about the growth rate of an I-function on the number of Brownian particles. All results in our applications are new and we expect more to follow in the future.

The rest of the paper is arranged as follows. Precise statements of main results and their consequences are given in Section 2. Their proofs are delayed to Section 4. Three applications are given in Section 3 with proofs.

2. Comparison inequalities

We use the notations introduced before. Our first result shows that the headache term $(1-\rho_{ij}^2)^{-1/2}$ in (1.2) can be removed.

Theorem 2.1. *We have*

$$\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\Big) - \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\Big) \le \frac{1}{2\pi} \sum_{1\le i<j\le n} (\arcsin(r_{ij}^1) - \arcsin(r_{ij}^0))^+ \exp\Big(-\frac{(u_i^2+u_j^2)}{2(1+\rho_{ij})}\Big), \quad (2.3)$$

for any real numbers u_i, $i = 1, 2, \cdots, n$.

Since $\arcsin(x) \le x\pi/2$ for $0 \le x \le 1$, a direct consequence of Theorem 2.1 is

Corollary 2.1. *Let $\xi_1, \xi_2, \cdots, \xi_n$ be standard normal variables with $Cov(\xi_i, \xi_j) = r_{ij}$. Then*

$$|\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\Big) - \prod_{j=1}^{n}\mathbb{P}(\xi_j \le u_j)| \le \frac{1}{4}\sum_{1\le i<j\le n}|r_{ij}|\exp\Big(-\frac{u_i^2+u_j^2}{2(1+|r_{ij}|)}\Big)$$

for any real numbers u_i, $i = 1, 2, \cdots, n$.

By using Corollary 2.1 instead of (1.2), one can remove unnecessary conditions in several results in Leadbetter, Lindgren and Rootzén (1983). For example, the condition "$\sup_{n\ge1}|r_n| < 1$" in Lemmas 4.3.1 and Lemma 4.4.1 there can be taken away.

Our next theorem gives a sharper bound which is especially appealing when u_i is not too large, and is the main contribution of this paper. All applications in the next section depend on it.

Theorem 2.2. *Let $n \ge 3$. and let $(\xi_j, 1 \le j \le n)$ and $(\eta_j, 1 \le j \le n)$ be standard normal random variables with covariance matrices $R^1 = (r_{ij}^1)$ and $R^0 = (r_{ij}^0)$, respectively. Assume*

$$r_{ij}^1 \ge r_{ij}^0 \ge 0 \text{ for all } 1 \le i, j \le n \tag{2.4}$$

Then

$$\begin{aligned}
&\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\Big)\\
&\le \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\Big)\\
&\le \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\Big)\exp\Big\{\sum_{1\le i<j\le n}\ln\Big(\frac{\pi-2\arcsin(r_{ij}^0)}{\pi-2\arcsin(r_{ij}^1)}\Big)\exp\Big(-\frac{(u_i^2+u_j^2)}{2(1+r_{ij}^1)}\Big)\Big\}
\end{aligned} \tag{2.5}$$

for any $u_i \ge 0$, $i = 1, 2, \cdots, n$ satisfying

$$(r_{ki}^l - r_{ij}^l r_{kj}^l)u_i + (r_{kj}^l - r_{ij}^l r_{ki}^l)u_j \ge 0 \text{ for } l = 0, 1 \text{ and for all } 1 \le i, j, k \le n \tag{2.6}$$

If $u_i = u \ge 0$ for every i, then

$$(r_{ki}^l - r_{ij}^l r_{kj}^l)u_i + (r_{kj}^l - r_{ij}^l r_{ki}^l)u_j = u(r_{ki}^l + r_{kj}^l)(1 - r_{ij}^l) \ge 0$$

for all $1 \le i, j, k \le n$. So, (2.6) is satisfied. Thus, we have

Corollary 2.2. *Let $n \geq 3$. and let $(\xi_j, 1 \leq j \leq n)$ be standard normal random variables with covariance matrix $R = (r_{ij})$. Assume that $r_{ij} \geq 0$. Then*

$$
\begin{aligned}
&\mathbb{P}\Big(\bigcap_{j=1}^{m}\{\xi_j \leq u\}\Big)\mathbb{P}\Big(\bigcap_{m<j\leq n}\{\xi_j \leq u\}\Big) \\
&\leq \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \leq u\}\Big) \\
&\leq \mathbb{P}\Big(\bigcap_{j=1}^{m}\{\xi_j \leq u\}\Big)\mathbb{P}\Big(\bigcap_{m<j\leq n}\{\xi_j \leq u\}\Big). \\
&\quad \exp\Big\{\sum_{i=1}^{m}\sum_{j=m+1}^{n}\ln\Big(\frac{\pi}{\pi - 2\arcsin(r_{ij})}\Big)\exp\Big(-u^2/r_{ij}\Big)\Big\}
\end{aligned}
\tag{2.7}
$$

for $1 \leq m \leq n-1$ and $u \geq 0$, and

$$
\begin{aligned}
\mathbb{P}(\xi_1 \leq u)^n &\leq \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \leq u\}\Big) \\
&\leq \mathbb{P}(\xi_1 \leq u)^n \cdot \exp\Big\{\sum_{1\leq i<j\leq n}\ln\Big(\frac{\pi}{\pi - 2\arcsin(r_{ij})}\Big)\exp\Big(-u^2/r_{ij}\Big)\Big\}
\end{aligned}
\tag{2.8}
$$

for $u \geq 0$.

We remark that if $r_{ki}^l \geq r_{ij}^l r_{kj}^l$ holds for every $1 \leq i, j, k \leq n$ and $l = 0, 1$, then (2.6) is also satisfied. Moreover, it follows from the proof of Theorem 2.2 (see (4.5) in Section 4) that the right hand side of (2.5) can be replaced by

$$
\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u_j\}\Big)\exp\Big\{\frac{2}{\pi}\sum_{1\leq i<j\leq n}(\arcsin(r_{ij}^1) - \arcsin(r_{ij}^0))\exp\Big(-\frac{u_i^2 + u_j^2}{2(1 + r_{ij}^1)}\Big)\Big\}
$$

Hence, we have

Corollary 2.3. *Let $\{X(t), t \geq 0\}$ be the Ornstein-Uhlenbeck process, i.e., a stationary Gaussian process with covariance function*

$$
\rho(t, s) := \mathbb{E}\,(X(t)X(s)) = e^{-|t-s|/2}.
$$

Let

$$
A_k = \{X(t_{k,l}) \leq x_{k,l} \ \textit{for}\ l = 0, \cdots, m_k\}
$$

with all $t_{k,l}$ distinct and $x_{k,l} \geq 0$. Then

$$
\begin{aligned}
&\mathbb{P}\Big(\bigcap_{k=1}^{n} A_k\Big) \\
&\leq \prod_{k=1}^{n}\mathbb{P}(A_k)\exp\Big\{\sum_{1\leq i<j\leq n}\sum_{u=0}^{m_j}\sum_{v=0}^{m_i}\rho(t_{i,v}, t_{j,u})\exp\Big(-\frac{x_{i,v}^2 + x_{j,u}^2}{2(1 + \rho(t_{i,v}, t_{j,u}))}\Big)\Big\}.
\end{aligned}
$$

The proof of Theorems 2.1 and 2.2 will be given in Section 4. It would be interesting to see if (2.5) remains true for any real numbers u_i without assuming condition (2.6). Note also the well known fact that in the setting of (1.1), Slepian's type inequality does not hold for two sided case with absolute value. See, for example, Tong (1980). But, it may be possible to have comparisons of two sided probabilities $\mathbb{P}\Big(\cap_{j=1}^m \{|\xi_j| \le u_j\}\Big)$ and $\mathbb{P}\Big(\cap_{j=1}^m \{|\eta_j| \le u_j\}\Big)$ with an additional exp-term similar to the one in (2.5). This could be a very useful tool for estimating small ball probabilities. See a recent survey of Li and Shao (2001a) on the subject. However, at this time, we are unable to find a right formulation together with interesting applications in this direction.

3. Applications

In this section we give three applications to demonstrate the usefulness of the inequality (2.5).

3.1. The law of the iterated logarithm of Erdős and Révész

Our first application is to the law of the iterated logarithm of Erdős and Révész. Let $\{W(t), t \ge 0\}$ be the standard Brownian motion and define

$$\xi_\delta(t) = \sup\{s : \ 0 \le s \le t, \ W(s) \ge \big(2(1-\delta)s\log_2 s\big)^{1/2}\},$$

for $t > 0$ and $0 \le \delta < 1$, where $\log_2(s) = \ln\ln(\max(e^2, s))$. Erdős and Révész (1990) obtained a new law of the iterated logarithm

$$\liminf_{t\to\infty} \frac{(\log_2 t)^{1/2}}{\log_3 t \cdot \log t} \ln \frac{\xi_0(t)}{t} = -C \ \ a.s.$$

for some constant C with $1/4 \le C \le 2^{14}$, where $\log_3(t) = \ln\ln\ln\max(e^3, t)$. The exact value $C = 3\sqrt{\pi}$ was found in Shao (1994). Moreover, it was proved there that

$$\liminf_{t\to\infty} (\log t)^{\delta-1}(\log_2 t)^{-1/2} \cdot \ln \frac{\xi_\delta(t)}{t} = -2\delta\sqrt{\frac{\pi}{1-\delta}} \ \ a.s. \tag{3.9}$$

for $0 < \delta \le 1/2$. It is natural to conjecture that (3.9) holds for all $0 < \delta < 1$. With the help of Corollary 2.3, we can give an affirmative answer to the conjecture.

Theorem 3.1. *The result* (3.9) *holds for every* $0 < \delta < 1$.

Proof. By using Corollary 2.3 instead of Lemma 2.4 in Shao (1994) (see Lemma 2.5 there), for each $0 < \delta < 1$ and $0 < \eta < 1/2$, there exists a constant $n = n(\delta, \eta)$ such that

$$\mathbb{P}\Big(\bigcap_{e^a \le s \le e^b} \Big\{\frac{W(s)}{\sqrt{2(1-\delta)s\log_2 s}} \le 1\Big\}\Big) \le 6\exp\Big(-\frac{1-2\eta}{2\delta}\sqrt{\frac{1-\delta}{\pi}}\,(\log a)^{1/2}(b^\delta - a^\delta)\Big) \tag{3.10}$$

for $n \le a + 2 \le b + a^{1-\delta}$. The remaining of the proof follows exactly the same lines of that of Theorem 1 in Shao (1994).

3.2. Probability that a random polynomial has no real root

Let $a_i, i \geq 0$ be i.i.d random variables with zero means and finite moment of all order and consider the random polynomial

$$f_n(x) = \sum_{i=0}^{n} a_i x^i, \quad -\infty < x < \infty.$$

Dembo, Poonen, Shao and Zeitouni (2000) proved that

$$\mathbb{P}(f_n \text{ has no real root}) = n^{-b+o(1)}$$

as $n \to \infty$ through even integers n. The constant b is specified as

$$b = -4 \lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\leq t\leq T} X(t) \leq 0\Big),$$

where $X(t)$ is a centered stationary Gaussian process with

$$\mathbb{E}\, X(s)X(t) = \frac{2e^{-|t-s|/2}}{1+e^{-|t-s|}}.$$

It is proved in Dembo, Poonen, Shao and Zeitouni (2000) that $0.4 \leq b \leq 2$. Their simulation suggest $b = 0.76 \pm 0.03$. Recently, Li and Shao (2001b) find two additional limiting representations for b and show that $b \leq 1$. As another application of Theorem 2.2, we have

Theorem 3.2.

$$0.5 < b \leq 1 \tag{3.11}$$

Proof. It suffices to show that

$$\mathbb{P}\Big(\max_{1\leq i\leq 3n} X(4i) \leq 0\Big) \leq \exp(-0.125(12n)) \tag{3.12}$$

for $n \geq 1$. Let $r(x) = 2e^{-2x}/(1+e^{-4x})$. By (2.5),

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1\leq i\leq 3n} X(4i) \leq 0\Big) \\
&\leq \mathbb{P}(\max_{1\leq i\leq 3} X(4i) \leq 0)^n \exp\Big\{ \sum_{1\leq i<j\leq n} \sum_{k=1}^{3} \sum_{l=1}^{3} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r(3(j-i)+k-l))}\Big)\Big\} \\
&\leq \mathbb{P}(\max_{1\leq i\leq 3} X(4i) \leq 0)^n \exp\Big\{3n \sum_{i=1}^{\infty} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r(3i))}\Big) \\
&\quad + 2n \sum_{i=1}^{\infty} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r(3i+1))}\Big) + 2n \sum_{i=1}^{\infty} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r(3i-1))}\Big) \\
&\quad + n \sum_{i=1}^{\infty} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r(3i+2))}\Big) + n \sum_{i=1}^{\infty} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r(3i-2))}\Big)\Big\} \\
&:= \mathbb{P}(\max_{1\leq i\leq 3} X(4i) \leq 0)^n \exp(\lambda 12n).
\end{aligned}$$

A direct calculation gives $\lambda = 0.02050...$. On the other hand, by David (1953)

$$\begin{aligned}\mathbb{P}(\max_{1\le i\le 3} X(4i) \le 0) &= \frac{1}{8} + \frac{1}{4\pi}(2\arcsin(r(1)) + \arcsin(r(2))) \\ &< 0.17074 < \exp(-0.1473 \cdot 12)\end{aligned}$$

Putting the above inequality together yields

$$\mathbb{P}\Big(\max_{1\le i\le 3n} X(4i) \le 0\Big) \le \exp(-(0.1473 - 0.0206)(12n)) \le \exp(-0.1267(12n)),$$

as desired.

3.3. Capture time of the fractional Brownian pursuit

A Gaussian process $\{B_\alpha(t),\ t \ge 0\}$ is called a fractional Brownian motion of order α, $0 < \alpha < 2$ if

$$B_\alpha(0) = 0, \mathbb{E}\, B_\alpha(t) = 0 \text{ and } \mathbb{E}\,(B_\alpha(t) - B_\alpha(s))^2 = |t-s|^\alpha$$

for all $t, s \ge 0$. Obviously, it becomes the Brownian motion when $\alpha = 1$.

Let $\{B_{k,\alpha}(t);\ t \ge 0\}(k = 0, 1, 2, \ldots, n)$ denote independent fractional Brownian motions of order $\alpha \in (0, 2)$ and set

$$\tau_{n,\alpha} = \inf\ \Big\{t > 0 : \max_{1\le k\le n} B_{k,\alpha}(t) = B_{0,\alpha}(t) + 1\Big\}.$$

The stopping time $\tau_{n,\alpha}$ can be viewed as the capture time in the random pursuit problem for the fractional Brownian particles; see Kesten (1992) and Li and Shao (2000a) for more details. A natural question is: when is $\mathbb{E}\,(\tau_{n,\alpha})$ finite? The question is the same as estimating the lower tail probability of $\max_{1\le k\le n}\sup_{0\le t\le 1}(B_{k,\alpha}(t) - B_{0,\alpha}(t))$. In fact, for any $s > 0$, by the fractional Brownian scaling,

$$\begin{aligned}\mathbb{P}(\tau_{n,\alpha} > s) &= \mathbb{P}\Big(\max_{1\le k\le n}\ \sup_{0\le t\le s}\ (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < 1\Big) \\ &= \mathbb{P}\Big(\max_{1\le k\le n}\ \sup_{0\le t\le 1}\ (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < s^{-\alpha/2}\Big).\end{aligned}$$

Li and Shao (2000b) show that

$$\mathbb{P}\Big(\max_{1\le k\le n}\ \sup_{0\le t\le 1}\ (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < x\Big) = x^{2\gamma_{n,\alpha}/\alpha + o(1)}$$

as $x \to 0$, where

$$\gamma_{n,\alpha} := -\lim_{T\to\infty}\frac{1}{T}\ln\mathbb{P}\Big(\sup_{0\le t\le T}\ \max_{1\le k\le n}(X_{k,\alpha}(t) - X_{0,\alpha}(t)) \le 0\Big) \tag{3.13}$$

and $X_{k,\alpha}(t) = e^{-t\alpha/2}B_{k,\alpha}(e^t)$, $k = 0, 1, \cdots, n$, are the fractional Ornstein-Uhlenbeck process of order α. In other word,

$$\mathbb{P}\Big(\tau_{n,\alpha} > t\Big) = t^{-\gamma_{n,\alpha}+o(1)}$$

as $t \to \infty$ for fixed n.

It is proved by Kesten (1992) that for the Brownian motion case, $\alpha = 1$, $\gamma_n = \gamma_{n,1}$ is of order $\ln n$ when n is large. More precisely, Kesten showed that

$$0 < \liminf_{n\to\infty} \gamma_n/\ln n \le \limsup_{n\to\infty} \gamma_n/\ln n \le 1/4$$

and conjectured the existence of $\lim_{n\to\infty} \gamma_n/\ln n$. Kesten's method is based on large deviation results for independent stationary Ornstein-Uhlenbeck processes, which is hardly applicable for the fractional Brownian motion. As another application of Theorem 2.2, our next theorem shows that $\gamma_{n,\alpha}$ is also of order $\ln n$, which in turn shows that $\mathbb{E}\,\tau_{n,\alpha}$ is finite when n is large.

In the Brownian motion case, $\alpha = 1$, with $\tau_n = \tau_{n,1}$, $\mathbb{E}\,\tau_5 < \infty$ and $\mathbb{E}\,\tau_3 = \infty$ are proved in Li and Shao (2000a) by using some distribution identities and the Faber-Krahn isoperimetric inequality. It is still a conjecture due to Bramson and Griffeath (1991) that $\mathbb{E}\,\tau_4 < \infty$. Their simulation suggested that $\gamma_4 \approx 1.032$. Other representations for $\gamma_n = \gamma_{n,1}$ are discussed after the following main result.

Theorem 3.3. *We have*

$$\frac{1}{d_\alpha} \le \liminf_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} \le \limsup_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} < \infty\,. \tag{3.14}$$

where $d_\alpha = 2\int_0^\infty (e^{x\alpha} + e^{-x\alpha} - (e^x - e^{-x})^\alpha)dx$. *Furthermore, for* $\gamma_n = \gamma_{n,1}$,

$$\lim_{n\to\infty} \frac{\gamma_n}{\ln n} = \frac{1}{4} \tag{3.15}$$

We need a few remarks about the significance of (3.15) in the setting of Brownian motion. First, $\tau_n = \tau_{n,1}$ equals the first exit time by the $(n+1)$-dimensional Brownian motion $(B_0(t), \cdots, B_n(t))$ from the "wedge"

$$W_{n+1} = \{x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} : x_i - x_0 < 0, 1 \le i \le n\} \tag{3.16}$$

starting at $b = (0, -1, \cdots, -1) \in \mathbb{R}^{n+1}$. DeBlassie (1987) (see also DeBlassie (1988) and Bañuelos and Smits (1997)) has shown that

$$\mathbb{P}\{\tau_n > t\} \sim c(b)t^{-\gamma_n}, \quad \text{as} \quad t \to \infty, \tag{3.17}$$

where

$$\gamma_n = \frac{1}{2}\left(\sqrt{\lambda_1(W_{n+1}) + ((n-1)/2)^2} - (n-1)/2\right) \tag{3.18}$$

and $\lambda_1(W_{n+1})$ is the first (principal) eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on the subset $W_{n+1} \cap \mathbb{S}^n$ of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Second, by using distribution identities developed in Li and Shao (2000a), together with the first exit time approach above, we also have a somewhat easier (one-dimension less) representation

$$\gamma_n = \frac{1}{2}\left(\sqrt{\lambda_1(D_n) + ((n-2)/2)^2} - (n-2)/2\right) \tag{3.19}$$

and $\lambda_1(D_n)$ is the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on the subset $D_n \cap \mathbb{S}^{n-1}$ of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, where

$$D_n = \bigcap_{i=1}^{n} \left\{ x = (x_k) \in \mathbb{R}^n : \sum_{k=1}^{i} a_{i,k} x_k \le 0 \right\}$$

with $a_{i,k} = ((k+1)k)^{-1/2}$ for $1 \le k \le i-1$ and $i \ge 2$, and $a_{i,i} = ((i+1)/i)^{1/2}$, $i \ge 1$. Geometrically, D_n can be viewed as the polar set of the unique n unit vectors (up to rotation) in $\mathbb{R}^n$ with 60 degree angles between each other. Note that it seems very difficult to find or estimate $\lambda_1(W_{n+1})$ and $\lambda_1(D_n)$ (hence γ_n) by analytic method for n large. On the other hand, the probabilistic approach of proving (3.15) implies

$$\lambda_1(W_{n+1}) \sim \lambda_1(D_n) \sim \frac{1}{2} n \ln n \quad \text{as} \quad n \to \infty$$

by (3.18) and (3.19). In essence, our probability estimate obtained by using the new normal comparison inequality provides a way of estimating eigenvalues as dimension increase. This way of estimating the first eigenvalue for the Dirichlet problem seems very powerful and more works in this direction will be given in Li and Shao (2001c). Finally, we conjecture that

$$\lim_{n \to \infty} \frac{\gamma_{n,\alpha}}{\ln n} = \frac{1}{d_\alpha}$$

based on (3.14) and (3.15).

Before the detailed proof of Theorem 3.3, we need two lemmas. The first is due to Shao (1999) and is very useful in various contexts. See Li and Shao (2000b) for some sharp estimates of lower tail probabilities for Gaussian processes, including fractional Brownian sheets.

Lemma 3.1. *Let $\xi = (\xi_1, ..., \xi_m)$ be distributed according to $N(0, \Sigma_\xi)$, and $\eta = (\eta_1, ..., \eta_m)$ according to $N(0, \Sigma_\eta)$. If $\Sigma_\eta - \Sigma_\xi$ is positive semidefinite, then*

$$\forall\, C \subset \mathbb{R}^m, \ \mathbb{P}\Big(\xi \in C\Big) \le (\det(\Sigma_\eta)/\det(\Sigma_\xi))^{1/2} \mathbb{P}(\eta \in C)$$

Our next lemma is a consequence of the (strong) locally nondeterminism for fractional Brownian motion.

Lemma 3.2. *Let $X_\alpha(t) = e^{-t\alpha/2} B_\alpha(e^t)$ and $0 < t_1 < t_2 < \cdots < t_m$. Then there exists a constant $K_\alpha > 0$ such that*

$$\det\left(\mathbb{E}\, X_\alpha(t_i) X_\alpha(t_j)\right)_{1 \le i,j \le m} \ge K_\alpha^{m-1} \prod_{i=2}^{m} (1 - e^{-(t_i - t_{i-1})})^\alpha.$$

Furthermore, in the case of Ornstein-Uhlenbeck process, $\alpha = 1$, we have equality above with $K_1 = 1$.

Proof. We have for $2 \le i \le m$,

$$\mathrm{Var}(B_\alpha(t_i) \mid B_\alpha(t_j), 1 \le j < i) \ge K_\alpha (t_i - t_{i-1})^\alpha$$

given in Monrad and Rootzén (1995) with a nice direct proof and related references to locally nondeterminism. Thus

$$\begin{aligned}
\det \left(\mathbb{E}\, X_\alpha(t_i) X_\alpha(t_j)\right)_{1 \le i,j \le m} &= \mathrm{Var}(X_\alpha(t_1)) \prod_{i=2}^{m} \mathrm{Var}\left(X_\alpha(t_i) \mid X_\alpha(t_j), 1 \le j < i\right) \\
&= \prod_{i=2}^{m} \mathrm{Var}\left(e^{-t_i\alpha/2} X_\alpha(e^{t_i}) | e^{-t_j\alpha/2} X_\alpha(e^{t_j}), 1 \le j < i\right) \\
&\ge K_\alpha^{m-1} \prod_{i=2}^{m} e^{-t_i\alpha} (e^{t_i} - e^{t_{i-1}})^\alpha \\
&= K_\alpha^{m-1} \prod_{i=2}^{m} (1 - e^{-(t_i - t_{i-1})})^\alpha .
\end{aligned}$$

In the case of Ornstein-Uhlenbeck process, $\alpha = 1$, the determinant can be evaluated directly by transformations $\mathrm{row}(i-1) - e^{t_i - t_{i-1}} \mathrm{row}(i)$ for $2 \le i \le m$.

Proof of Theorem 3.3. The right hand side of (3.14) is proved in Li and Shao (2000b). So we only deal with the left hand side. Let $r(t) = \mathbb{E}\, X_{0,\alpha}(t) X_{0,\alpha}(0)$ for $t \ge 0$. It is easy to see that

$$\begin{aligned}
r(t) &= \frac{1}{2}\left(e^{t\alpha/2} + e^{-t\alpha/2} - (e^{t/2} - e^{-t/2})^\alpha\right) \\
&= \frac{1}{2}\left(e^{-t\alpha/2} + \alpha e^{-t(2-\alpha)/2} + O(e^{-t(4-\alpha)/2})\right) \qquad (3.20)
\end{aligned}$$

as $t \to \infty$. For $0 < \theta < 1/2$, let

$$\lambda_\theta := \lambda_{\theta,\alpha} = 1 + 2 \sum_{i=1}^{\infty} r(\theta i)$$

One can easily verify that

$$\lim_{\theta \to 0} \frac{(1-\theta)^2}{\theta \lambda_\theta} = \frac{1}{d_\alpha}.$$

Thus it suffices to show that for any $0 < \theta < 1/2$, there exists $K_{\theta,\alpha} > 0$ such that

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1 \le i \le m} \max_{1 \le k \le n} (X_{k,\alpha}(i\theta) - X_{0,\alpha}(i\theta)) \le 0\Big) \\
&\quad \le K_{\theta,\alpha}^m \Big(\exp\Big(-\frac{(1-\theta)^2}{\theta\lambda_\theta}(m\theta)\ln n\Big) + \exp\Big(-(m\theta)n^{\theta^2}\Big)\Big) \qquad (3.21)
\end{aligned}$$

for n sufficiently large uniformly in $m \ge 1$. Here and throughout this section, we use letter $K_{\theta,\alpha}$ for various positive constants which may be different from line to line.

Let $\xi = (\xi_1, \cdots, \xi_m)$ and $\eta = (\eta_1, \cdots, \eta_m)$, where $\xi_i = X_{0,\alpha}(i\theta)$ and η_i are i.i.d. normal random variables with mean zero and variance λ_θ which are independent of $(X_{k,\alpha})_{k=1}^n$. Then, $\Sigma_\eta - \Sigma_\xi$ is dominant principal diagonal matrix and hence positive semidefinite. Applying Lemma 3.1 and Lemma 3.2 yields

$$
\begin{aligned}
&\mathbb{P}\Big(\max_{1\le i\le m}\max_{1\le k\le n}(X_{k,\alpha}(i\theta) - X_{0,\alpha}(i\theta)) \le 0\Big)\\
&= \mathbb{E}\Big(\mathbb{P}\Big(\max_{1\le i\le m}\max_{1\le k\le n}(X_{k,\alpha}(i\theta) - X_{0,\alpha}(i\theta)) \le 0 | X_{k,\alpha}(i\theta), 1\le k\le n, 1\le i\le m\Big)\Big)\\
&\le \mathbb{E}\Big(\Big(\det(\Sigma_\eta)/\det(\Sigma_\xi)\Big)^{1/2}\mathbb{P}\Big(\max_{1\le i\le m}\max_{1\le k\le n}(X_{k,\alpha}(i\theta) - \eta_i)\\
&\le 0 \mid X_{k,\alpha}(i\theta), 1\le k\le n, 1\le i\le m\Big)\Big)\\
&= \Big(\det(\Sigma_\eta)/\det(\Sigma_\xi)\Big)^{1/2}\mathbb{P}\Big(\max_{1\le i\le m}\max_{1\le k\le n}(X_{k,\alpha}(i\theta) - \eta_i) \le 0\Big)\\
&\le K_{\theta,\alpha}^{m/2} J_{n,m},
\end{aligned}
\tag{3.22}
$$

where

$$
J_{n,m} = \mathbb{P}\Big(\max_{1\le i\le m}\max_{1\le k\le n}(X_{k,\alpha}(i\theta) - \eta_i) \le 0\Big).
$$

Let $a = (2(1-\theta)\ln n)^{1/2}, l = \sum_{i=1}^m I\{\eta_i \le a\}$ and $\{i_1, i_2, \cdots, i_l\} = \{j : \eta_j \le a\}$ with $i_1 < i_2 < \cdots < i_l$. Then

$$
\begin{aligned}
J_{n,m} &\le \mathbb{P}(l \le \theta m) + \mathbb{P}\Big(\max_{1\le i\le m}\max_{1\le k\le n}(X_{k,\alpha}(i\theta) - \eta_i) \le 0, l \ge \theta m\Big)\\
&\le \mathbb{P}(l \le \theta m) + \mathbb{P}\Big(\max_{1\le j\le l}\max_{1\le k\le n} X_{k,\alpha}(i_j\theta) \le a, l > \theta m\Big)\\
&\le \mathbb{P}(l\le\theta m) + \mathbb{E}\Big\{\mathbb{P}\Big(\max_{1\le j\le l}\max_{1\le k\le n} X_{k,\alpha}(i_j\theta) \le a \mid l > \theta m \text{ and } i_j, 1\le j\le l\Big)\Big\}\\
&= \mathbb{P}(l \le \theta m) + \mathbb{E}\Big\{\mathbb{P}\Big(\max_{1\le j\le l} X_\alpha(i_j\theta) \le a \mid l > \theta m \text{ and } i_j, 1 \le j \le l\Big)^n\Big\}.
\end{aligned}
\tag{3.23}
$$

Noting that $\mathbb{P}(\eta_i \ge a) \le \exp(-a^2/(2\lambda_\theta)) = \exp(-(1-\theta)(\ln n)/\lambda_\theta)$ and $\sum_{i=1}^m I\{\eta_i > a\}$ is a binomial random variable with parameters m and $\mathbb{P}(\eta_1 \ge a)$, we have (see, for example, (2.8) in Shao (1997))

$$
\begin{aligned}
\mathbb{P}(l \le \theta m) &= \mathbb{P}\Big(\sum_{i=1}^m I\{\eta_i > a\} \ge (1-\theta)m\Big)\\
&\le (6\mathbb{P}(\eta_1 \ge a))^{(1-\theta)m} \le 6^m \exp\Big(-\frac{(1-\theta)^2}{\lambda_\theta} m \ln n\Big).
\end{aligned}
$$

For given l and i_j, $1 \le j \le l$, by (3.20)

$$
\sum_{1\le j<k\le l} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r((i_k - i_j)\theta))}\Big)\exp(-\frac{a^2}{1+r(\theta)})
$$

$$\le \sum_{1\le j\le l}\sum_{k=j+1}^{\infty} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r((i_k - i_j)\theta))}\Big)\exp(-\frac{a^2}{1+r(\theta)})$$
$$\le \sum_{1\le j\le l}\sum_{i=1}^{\infty} \ln\Big(\frac{\pi}{\pi - 2\arcsin(r(i\theta))}\Big)\exp(-\frac{a^2}{1+r(\theta)})$$
$$= K_{\theta,\alpha} l \exp(-\frac{a^2}{1+r(\theta)})$$

for some finite $K_{\theta,\alpha}$. Applying Corollary 2.2 yields for given l and i_j, $1 \le j \le l$, and sufficiently large n

$$\begin{aligned}\mathbb{P}\Big(\max_{1\le j\le l} X_\alpha(i_j\theta) \le a\Big) &\le \exp\Big(K_{\theta,\alpha} l \exp(-\frac{a^2}{1+r(\theta)})\Big)\cdot \mathbb{P}^l(Z\le a)\\ &\le \exp\Big(K_{\theta,\alpha} l \exp(-\frac{a^2}{1+r(\theta)}) - l\cdot\mathbb{P}(Z>a)\Big)\\ &\le \exp\Big(K_{\theta,\alpha}\, l\cdot n^{-2(1-\theta)/(1+r(\theta))} - l\cdot n^{-(1-\theta)}\Big)\\ &\le \exp(-l\cdot n^{-(1-\theta^2)})\end{aligned}$$

for sufficiently large n. Therefore

$$\mathbb{E}\Big\{\mathbb{P}\Big(\max_{1\le j\le l} X_1(i_l\theta)\le a \mid l > m\theta \text{ and } i_j, 1\le j\le l\Big)^n\Big\} \le \exp\Big(-\theta m n^{\theta^2}\Big). \tag{3.24}$$

This proves (3.21), by (3.22) – (3.24).

4. Proof of Theorems 2.2 and 2.1

Proof of Theorem 2.2. Let

$$R^h = h\,R^1 + (1-h)\,R^0 = (r_{ij}^h) \ \text{ for } \ 0\le h\le 1.$$

Let $\boldsymbol{\zeta} = (\zeta_1,\cdots,\zeta_n) := (\zeta_1^h,\cdots,\zeta_n^h)$ be normal random variables with covariance matrix R^h, f_h be the density function of $\boldsymbol{\zeta}$ and

$$F(h) = \int_{-\infty}^{\mathbf{u}} f_h(y_1,\cdots,y_n)d\mathbf{y},$$

where $\mathbf{u} = (u_1,\cdots,u_n)$ and $d\mathbf{y} = dy_1\cdots dy_n$. Then

$$F(1) = \mathbb{P}\Big(\bigcap_{j=1}^n\{\xi_j\le u_j\}\Big) \text{ and } F(0) = \mathbb{P}\Big(\bigcap_{j=1}^n\{\eta_j\le u_j\}\Big).$$

Put

$$g(h) = \exp\Big\{\sum_{1\le i<j\le n} \ln\Big(\frac{\pi - 2\arcsin(r_{ij}^0)}{\pi - 2\arcsin(r_{ij}^h)}\Big)\exp\Big(-\frac{(u_i^2+u_j^2)}{2(1+r_{ij}^1)}\Big)\Big\}.$$

It suffices to show that $F(h)/g(h)$ is non-increasing, or equivalently,

$$g(h)F'(h) \le g'(h)F(h) \quad \text{for} \quad 0 \le h \le 1. \tag{4.1}$$

It is known and easy to see that (cf. [13] , p.82)

$$F'(h) = \sum_{1\le i<j\le n} (r_{ij}^1 - r_{ij}^0) \int_{-\infty}^{\mathbf{u}'} f_h(y_i = u_i, y_j = u_j)\, d\,\mathbf{y}', \tag{4.2}$$

where $f_h(y_i = u_i, y_j = u_j)$ denotes the function of $n-2$ variables formed by putting $y_i = u_i,\ y_j = u_j$, and the integration is over the remaining variables. Noting that

$$g'(h)/g(h) = \sum_{1\le i<j\le n} \frac{2(r_{ij}^1 - r_{ij}^0)}{(\pi - 2\arcsin(r_{ij}^h))(1-(r_{ij}^h)^2)^{1/2}} \exp\Big(-\frac{u_i^2+u_j^2}{2(1+r_{ij}^1)}\Big),$$

we only need to prove that

$$\begin{aligned}&\int_{-\infty}^{\mathbf{u}'} f_h(y_i = u_i, y_j = u_j) d\mathbf{y}'\\ &\le \frac{2}{(\pi - 2\arcsin(r_{ij}^h))(1-(r_{ij}^h)^2)^{1/2}} \exp\Big(-\frac{u_i^2+u_j^2}{2(1+r_{ij}^1)}\Big)\mathbb{P}\Big(\bigcap_{l=1}^n \{\zeta_l \le u_l\}\Big)\end{aligned} \tag{4.3}$$

for $1 \le i < j \le n$.

Let $\phi(x, y; r)$ be the standard bivariate normal density with correlation coefficient r. Then, we can write

$$\int_{-\infty}^{\mathbf{u}'} f_h(y_i = u_i, y_j = u_j)\, d\,\mathbf{y}' = \phi(u_i, u_j; r_{ij}^h)\, \mathbb{P}(\boldsymbol{\zeta}' \le \mathbf{u}' \mid \zeta_i = u_i, \zeta_j = u_j). \tag{4.4}$$

For the sake of simplicity, we work only with $i = 1,\ j = 2$. Since $(r_{i1}^h - r_{12}^h r_{i2}^h)u_1 + (r_{i2}^h - r_{12}^h r_{i1}^h)u_2$ is a concave function of h for $k = 3, \cdots, n$, condition (2.6) implies that

$$(r_{k1}^h - r_{12}^h r_{k2}^h)u_1 + (r_{k2}^h - r_{12}^h r_{k1}^h)u_2 \ge 0$$

for $0 \le h \le 1$. Noting that

$$\{\zeta_k - \frac{r_{k1}^h - r_{12}^h r_{k2}^h}{1-(r_{12}^h)^2}\zeta_1 - \frac{r_{k2}^h - r_{12}^h r_{k1}^h}{1-(r_{12}^h)^2}\zeta_2,\ k = 3, \cdots, n\} \ \text{and}\ \{\zeta_1,\ \zeta_2\}$$

are independent, we have

$$\begin{aligned}&\mathbb{P}\Big(\bigcap_{j=3}^n \{\zeta_j \le u_j\} \mid \zeta_1 = u_1,\ \zeta_2 = u_2\Big)\\ &= \mathbb{P}\Big(\bigcap_{j=3}^n \Big\{\zeta_j - \frac{r_{j1}^h - r_{12}^h r_{j2}^h}{1-(r_{12}^h)^2}\zeta_1 - \frac{r_{j2}^h - r_{12}^h r_{j1}^h}{1-(r_{12}^h)^2}\zeta_2 \le u_j - \frac{r_{j1}^h - r_{12}^h r_{j2}^h}{1-(r_{12}^h)^2}u_1 - \frac{r_{j2}^h - r_{12}^h r_{j1}^h}{1-(r_{12}^h)^2}u_2\Big\}\Big)\end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{P}\Big(\bigcap_{j=3}^{n}\Big\{\zeta_j \le u_j + \frac{1}{1-(r_{12}^h)^2}\Big((r_{j1}^h - r_{12}^h r_{j2}^h)\zeta_1 + (r_{j2}^h - r_{12}^h r_{j1}^h)\zeta_2\Big)\Big\}\Big) \\
&= \frac{\mathbb{P}\Big(\zeta_1 - r_{12}^h\zeta_2 \le 0, \zeta_2 - r_{12}^h\zeta_1 \le 0, \zeta_j \le u_j + \frac{1}{1-(r_{12}^h)^2}\Big(r_{j1}^h(\zeta_1 - r_{12}^h\zeta_2) + r_{j2}^h(\zeta_2 - r_{12}^h\zeta_1)\Big), j = 3, \cdots, n\Big)}{\mathbb{P}\Big(\zeta_1 - r_{12}^h\zeta_2 \le 0, \zeta_2 - r_{12}^h\zeta_1 \le 0\Big)} \\
&\le \frac{\mathbb{P}\Big(\zeta_1 - r_{12}^h\zeta_2 \le 0, \zeta_2 - r_{12}^h\zeta_1 \le 0, \zeta_j \le u_j, j = 3, \cdots, n\Big)}{\mathbb{P}\Big(\zeta_1 - r_{12}^h\zeta_2 \le 0, \zeta_2 - r_{12}^h\zeta_1 \le 0\Big)} \\
&\le \frac{\mathbb{P}\Big(\zeta_1 \le 0, \zeta_2 \le 0, \zeta_j \le u_j, j = 3, \cdots, n\Big)}{\mathbb{P}\Big(\zeta_1 - r_{12}^h\zeta_2 \le 0, \zeta_2 - r_{12}^h\zeta_1 \le 0\Big)} \\
&\le \frac{\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\zeta_j \le u_j\}\Big)}{\mathbb{P}\Big(\zeta_1 - r_{12}^h\zeta_2 \le 0, \zeta_2 - r_{12}^h\zeta_1 \le 0\Big)}\,. \qquad (4.5)
\end{aligned}
$$

It is easy to see that (cf. [13], p.83)

$$
\phi(u_1, u_2; r_{12}^h) \le \frac{1}{2\pi(1-(r_{12}^h)^2)^{1/2}} \exp\Big(-\frac{u_1^2 + u_2^2}{2(1+r_{12}^1)}\Big). \qquad (4.6)
$$

Noting that $\mathrm{corr}(\zeta_1 - r_{12}^h\zeta_2, \zeta_2 - r_{12}^h\zeta_1) = -r_{12}^h$, we have

$$
\mathbb{P}(\zeta_1 - r_{12}^h\zeta_2 \le 0,\ \zeta_2 - r_{12}^h\zeta_1 \le 0) = \frac{\pi - 2\arcsin(r_{12}^h)}{4\pi}
$$

which can be found in David (1953).

Putting the above inequalities together yields (4.3) for $i = 1$ and $j = 2$. Similarly, (4.3) holds for general $1 \le i < j \le n$. This finishes the proof.

Proof of Theorem 2.1. The proof follows the same line as that of Theorem 4.2.1 in Leadbetter, Lindgren and Rootzén (1983) with a modification given in (4.6) and the fact that

$$
\begin{aligned}
\int_0^1 \frac{1}{(1-(r_{ij}^h)^2)^{1/2}} dh &= \int_0^1 \frac{1}{(1-(h(r_{ij}^1 - r_{ij}^0) + r_{ij}^0)^2)^{1/2}} dh \\
&= \frac{1}{r_{ij}^1 - r_{ij}^0} \int_{r_{ij}^0}^{r_{ij}^1} \frac{1}{(1-h^2)^{1/2}} dh \\
&= \frac{1}{r_{ij}^1 - r_{ij}^0}\Big(\arcsin(r_{ij}^1) - \arcsin(r_{ij}^0)\Big).
\end{aligned}
$$

We omit the details.

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