Gaussian Processes: Inequalities, Small Ball Probabilities and Applications

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Contents

1 Introduction

A Gaussian measure μ on a real separable Banach space E equipped with its Borel σ-field $\mathcal{B}$ and with norm $\|\cdot\|$ is a Borel probability measure on $(E, \mathcal{B})$ such that the law of each continuous linear functional on E is Gaussian (normal distribution). The small ball probability (or small deviation) for the Gaussian measure studies the behaviour of

$$\log \mu(x : \|x\| \leq \varepsilon) \tag{1.1}$$

as $\varepsilon \to 0$, while the large deviation for the Gaussian measure studies the behaviour of

$$\log \mu(x : \|x\| \geq a)$$

as $a \to \infty$.

It is well-known that the large deviation result plays a fundamental role in studying the upper limits of Gaussian processes, such as the Strassen type law of the iterated logarithm. The theory on large deviation has been well developed during the last few decades; see, for example, Ledoux and Talagrand [LT91], Ledoux [L96] and Bogachev [Bog98] for Gaussian measures, Varadhan [V84] and Dembo and Zeitouni [DZ98] for the general theory of large deviations. However, the complexity of the small ball estimate is well-known, and there are only a few Gaussian measures for which the small ball probability can be determined completely. The small ball probability is a key step in studying the lower limits of the Gaussian process. It has been found that the small ball estimate has close connections with various approximation quantities of compact sets and operators, and has a variety of applications in studies of Hausdorff dimensions, rate of convergence in Strassen's law of the iterated logarithm, and empirical processes, just mentioning a few here.

Our aim in writing this exposition is to survey recent developments in the theory of Gaussian processes. In particular, our focus is on inequalities, small ball probabilities and their wide range of applications. The compromise attempted here is to provide a reasonable detailed view of the ideas and results that have already gained a firm hold, to make the treatment as unified as possible, and sacrifice some of the details that do not fit the scheme or tend to inflate the survey beyond reasonable limits. The price to pay is that such a selection is inevitably biased. The topics selected in this survey are not exhaustive and actually only reflect the tastes and interests of the authors. We also include a number of new results and simpler proofs, in particular in Section 4. The survey is the first to systematically study the existing techniques and applications which are spread over various areas. We hope that readers can use results summarized here in their own works and contribute to this exciting area of research.

We must say that we omitted a great deal of small ball problems for other important processes such as Markov processes (in particular stable processes, diffusions with scaling), polygonal processes from partial sums, etc. Probably the most general formulation of small ball problems is the following. Let E be a Polish space (i.e., a complete, separable metric space) and suppose that $\{\mu_\varepsilon : \varepsilon > 0\}$ is a family of probability measures on E with the properties that $\mu_\varepsilon \Longrightarrow \mu$ as $\varepsilon \to 0$, i.e., μ_ε tends weakly to the measure μ. If, for some bounded, convex set $A \subset E$, we have $\mu(A) > 0$ and $\mu_\varepsilon(\varepsilon A) \to 0$ as $\varepsilon \to 0$, then one can reasonable say that, as $\varepsilon \to 0$, the measures μ_ε "see" the small event εA. What is often an important and interesting problem is the determination of just how "small" the event εA is. That is, one wants to know the rate at which $\mu_\varepsilon(\varepsilon A)$ is tending to 0. In general, a detailed answer to this question is seldom available in the infinite dimensional setting. However, if one only asks about the exponential rate, the rate at which $\log \mu_\varepsilon(\varepsilon A) \to 0$, then one has a much better chance of finding a solution and one is studying the *small ball probabilities* of the family $\{\mu_\varepsilon : \varepsilon > 0\}$ associated with the ball-like set A. In the case where all measures μ_ε are the same and $\mu_\varepsilon = \mu$, $A = \{x \in E : \|x\| \le 1\}$, then we are in the setting of (1.1).

When we compare the above formulation with the general theory of large deviations, see page one of Deuschel and Stroock [DS89] for example, it is clear that the small ball probability deals with a sequence of measures *below* the non-trivial limiting measure μ and the large deviation is *above*. The following well known example helps to see the difference. Let X_i, $i \ge 1$, be i.i.d. random variables with $\mathbb{E}\, X_i = 0$, $\mathbb{E}\, X_i^2 = 1$ and $\mathbb{E}\, \exp(t_0|X_1|) < \infty$ for $t_0 > 0$, and let $S_n = \sum_{i=1}^n X_i$. Then as $n \to \infty$ and $x_n \to \infty$ with $x_n = o(\sqrt{n})$

$$\log \mathbb{P}\left(\frac{1}{\sqrt{n}} \max_{1\le i\le n} |S_i| \ge x_n\right) \sim -\frac{1}{2}x_n^2$$

and as $n \to \infty$ and $\varepsilon_n \to 0$, $\sqrt{n}\varepsilon_n \to \infty$

$$\log \mathbb{P}\left(\frac{1}{\sqrt{n}} \max_{1\le i\le n} |S_i| \le \varepsilon_n\right) \sim -\frac{\pi^2}{8}\varepsilon_n^{-2}.$$

That is why the small ball probability is sometimes called small deviation. We make no distinction between them. Of course, certain problems can be viewed from both points of view. In particular, the large deviation theory of Donsker and Varadhan for the occupation measure can be used to obtain small ball probabilities when the Markov processes and the norm used have the scaling property. A tip of the iceberg can be seen in Section 3.3, 7.10 and below.

The small ball probability can also be seen in many other topics. To see how various topics are related to small ball estimates, it is instructive to exam the Brownian motion $W(t)$ on $\mathbb{R}^1$ under the sup-norm. We have by scaling

$$\mathbb{P}\left(\sup_{0\le t\le 1} |W(t)| \le \varepsilon\right) = \mathbb{P}\left(\sup_{0\le t\le T} |W(t)| \le 1\right) = \mathbb{P}\,(\tau \ge T) \tag{1.2}$$

and

$$\log \mathbb{P}\left(\sup_{0\le t\le 1} |W(t)| \le \varepsilon\right) = \log \mathbb{P}\,(L(T,1) = 1, L(T,-1) = 0) \sim -\frac{\pi^2}{8}\cdot T \sim -\frac{\pi^2}{8}\frac{1}{\varepsilon^2} \tag{1.3}$$

as $\varepsilon \to 0$ and $T = \varepsilon^{-2} \to \infty$. Here

$$\tau = \inf\{s : |W(s)| \ge 1\}$$

is the first exit (or passage) time and

$$L(T,y) = \frac{1}{T}\int_0^T 1_{(-\infty,y]}(W_s)ds$$

is a distribution (occupation) function with a density function called local time. In (1.2), the first expression is clearly the small ball probability for the Wiener measure or small deviation from the "flat" trajectories or lower tail estimate for the positive random variable $\sup_{0\le t\le 1}|W(t)|$; the second and third expressions are related to the two sided boundary crossing probability and exit or escape time. In (1.3), the second expression can be viewed as a very special case of the asymptotic theory developed by Donsker and Varadhan. The value $\pi^2/8$ is the principle eigenvalue of the Laplacian over the domain $[-1,1]$.

We believe that a theory of small ball probabilities should be developed. The topics we cover here for Gaussian processes are part of the general theory. The organization of this paper is as follows.

Section 2 summarizes various inequalities for Gaussian measures or for Gaussian random variables. The emphasis is on comparison inequalities and correlation inequalities which play an important role in small ball estimates. In Section 3 we present small ball probabilities in the general setting. The links with metric entropy and Laplacian transforms are elaborated. Sections 4 and 5 pay special attention to Gaussian processes with index set in $\mathbb{R}$ and $\mathbb{R}^d$ respectively. In Section 6, we give exact values of small ball constants for certain special processes. Various applications are discussed in Section 7.

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2 Inequalities for Gaussian random elements

Inequalities are always one of the most important parts of a general theory. In this section, we present some fundamental inequalities for Gaussian measures or Gaussian random variables.

The density and distribution function of the standard Gaussian (normal) distribution on the real line $\mathbb{R}$ are

$$\phi(x) = (2\pi)^{-1} \exp\{-x^2/2\} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(t)dt. \tag{2.1}$$

Let γ_n denote the canonical Gaussian measure on $\mathbb{R}^n$ with density function

$$\phi_n(x) = (2\pi)^{-n/2} \exp(-|x|^2/2)$$

with respect to Lebesgue measure, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. We use μ to denote a centered Gaussian measure throughout. All results for γ_n on $\mathbb{R}^n$ in this paper can be used to determine the appropriate infinite dimensional analogue by a classic approximation argument presented in detail in Chapter 4 of [L96].

2.1 Isoperimetric inequalities

The following isoperimetric inequality is one of the most important properties of the Gaussian measure. It has played a fundamental role in topics such as integrability and upper tail behavior of Gaussian seminorms, deviation and regularity of Gaussian sample paths, and small ball probabilities.

Theorem 2.1 *For any Borel set A in $\mathbb{R}^n$ and a half space $H = \{x \in \mathbb{R}^n : \langle x, u\rangle \le a\}$ such that*

$$\gamma_n(A) \ge \gamma_n(H) = \Phi(a)$$

for some real number a and some unit vector $u \in \mathbb{R}^n$, we have for every $r \ge 0$

$$\gamma_n(A + rU) \ge \gamma_n(H + rU) = \Phi(a + r),$$

where U is the unit ball in $\mathbb{R}^n$ and $A + rU = \{a + ru : a \in A, u \in U\}$.

The result is due independently to Borell [B75] and Sudakov and Tsirelson [ST74]. The standard proof is based on the classic isoperimetric inequality on the sphere and the fact that the standard Gaussian distribution on $\mathbb{R}^n$ can be approximated by marginal distributions of uniform laws on spheres in much higher dimensions. The approximation procedure, so called Poincare limit, can be found in [L96], chapter 1. A direct proof based on the powerful Gaussian symmetrization techniques is given by Ehrhard [E83]. This also led him to a rather complete isoperimetric calculus in Gauss space, see [E84] and [E86]. In particular, he obtained the following remarkable Brunn-Minkowski type inequality with both sets A and B convex.

Theorem 2.2 *(Ehrhard's inequality) For any convex set A and Borel set B of $\mathbb{R}^n$, and $0 \le \lambda \le 1$,*

$$\Phi^{-1} \circ \gamma_n(\lambda A + (1-\lambda)B) \ge \lambda \Phi^{-1} \circ \gamma_n(A) + (1-\lambda)\Phi^{-1} \circ \mu(B) \tag{2.2}$$

where $\lambda A + (1-\lambda)B = \{\lambda a + (1-\lambda)b : a \in A, b \in B\}$.

The above case of one convex set and one Borel set is due to Latala [La96]. A special case was studied in [KL95]. Ehrhard's inequality is a delicate result, which implies the isoperimetric inequality for Gaussian measures and has some other interesting consequences as well; see, for example [E84], [Kw94] and [KwS93]. It is still an open problem to prove (2.2) for two arbitrary Borel sets and the result in $\mathbb{R}$ suffices to settle the conjecture. If the conjecture were true, it would improve upon the more classical so called log-concavity of Gaussian measures:

$$\log \mu(\lambda A + (1-\lambda)B) \ge \lambda \log \mu(A) + (1-\lambda) \log \mu(B). \tag{2.3}$$

A proof of (2.3) may be given using again the Poincare limit on the classical Brunn-Minkowski inequality on $\mathbb{R}^n$; see [LT91] for details.

Talagrand [T92, T93] has provided very sharp upper and lower estimates for $\gamma_n(A + rU)$ when r is large and A is convex symmetric. In particular, the estimates relate to the small ball problem and its link with metric entropy; see Section 7.3 for some consequences.

Other than using addition of sets as enlargement, multiplication to a set can also be considered. The following result is due to Landau and Shepp [LaS70].

Theorem 2.3 *For any convex set A in $\mathbb{R}^n$ and a half space $H = \{x \in \mathbb{R}^n : \langle x, u \rangle \le a\}$ such that $\gamma_n(A) \ge \gamma_n(H) = \Phi(a)$ for some $a \ge 0$ and some unit vector $u \in \mathbb{R}^n$, one has for every $r \ge 1$*

$$\gamma_n(rA) \ge \gamma_n(rH) = \Phi(ra),$$

where $rA = \{rx : x \in A\}$.

The proof is based on the Brunn-Minkowski inequality on the sphere without using the Poincare limit. An application of Theorem 2.3 is the exponential square integrability of the norm of a Gaussian measure.

For a symmetric convex set A, the following was conjectured by Shepp in 1969 (so called S-conjecture) and proved recently by Latała and Oleszkiewicz [LO99]. Early work and related problems can be found in [KwS93].

Theorem 2.4 *Let μ be a centered Gaussian measure on a separable Banach space E. If A is a symmetric, convex, closed subset of E and $S \subset E$ is a symmetric strip, i.e., $S = \{x \in E : |x^*(x)| \le 1\}$ for some $x^* \in E^*$, the dual space of E, such that $\mu(A) = \mu(S)$, then*

$$\mu(tA) \ge \mu(tS) \text{ for } t \ge 1$$

and

$$\mu(tA) \le \mu(tS) \text{ for } 0 \le t \le 1.$$

The proof uses both Theorem 2.1 and Theorem 2.2. A consequence of Theorem 2.4 is the following result, which gives the best constants in comparison of moments of Gaussian vectors.

Theorem 2.5 *If ξ_i are independent standard normal r.v. and x_i are vectors in some separable Banach space $(E, \|\cdot\|)$ such that the series $S = \sum x_i \xi_i$ is a.s. convergent, then*

$$(\mathbb{E}\,\|S\|^p)^{1/p} \le \frac{a_p}{a_q} (\mathbb{E}\,\|S\|^q)^{1/q}$$

for any $p \ge q > 0$, where $a_p = (\mathbb{E}\,|\xi_1|^p)^{1/p} = \sqrt{2}(\pi^{-1/2}\Gamma((p+1)/2))^{1/p}$.

2.2 Concentration and deviation inequalities

Here we only summarize some of the key estimates. We refer to Ledoux and Talagrand [LT91], Ledoux [L96] and Lifshits [Lif95] for more details and applications. Let f be Lipschitz function on $\mathbb{R}^n$ with Lipschitz norm given by

$$\|f\|_{Lip} = \sup\left\{|f(x) - f(y)|/|x-y| : x, y \in \mathbb{R}^n\right\}.$$

Denote further by M_f a median of f for μ and by $\mathbb{E}_f = \int f d\mu(x)$ for the expectation of f.

Theorem 2.6

$$\mu(|f - M_f| > t) \le \exp\{-t^2/2\,\|f\|^2_{Lip}\} \tag{2.4}$$

and

$$\mu(|f - \mathbb{E}_f| > t) \le 2\exp\{-t^2/2\,\|f\|^2_{Lip}\} \tag{2.5}$$

Another version of the above result can be stated as follows. Let $\{X_t, t \in T\}$ be a centered Gaussian process with

$$d(s,t) = (\mathbb{E}\,|X_s - X_t|^2)^{1/2}, \quad s, t \in T$$

and $\sigma^2 = \sup_{t\in T} \mathbb{E}\, X_t^2$.

Theorem 2.7 *For all $x > 0$, we have*

$$\mathbb{P}\Big(\sup_{t\in T} X_t - \mathbb{E}\sup_{t\in T} X_t \ge x\Big) \le \exp\Big(-\frac{x^2}{2\sigma^2}\Big).$$

A proof based on log-concavity and a connection to Wills functional are given in Vitale [Vi96, Vi99b].

Theorem 2.8 *Let $N(T, d; \varepsilon)$ denote the minimal number of open balls of radius ε for the metric d that are necessary to cover T. Then*

$$\mathbb{P}\Big(\sup_{t\in T} X_t \ge x + 6.5\int_0^{\sigma/2} (\log N(T, d; \varepsilon))^{1/2} d\varepsilon\Big) \le \exp\Big(-\frac{x^2}{2\sigma^2}\Big).$$

The above result is due to Dudley [Du67]. Among Fernique type upper bounds, the following inequality due to Berman [Be85] gives a sharp bound.

Theorem 2.9 *Let $\{X_t, t \in T\}$, $T \subset \mathbb{R}^d$ be a centered Gaussian process. Let*

$$\rho(\varepsilon) = \sup_{s,t\in T, |s-t|\le\varepsilon} d(s,t)$$

and

$$Q(\delta) = \int_0^\infty \rho(\delta e^{-y^2})dy.$$

Then for all $x > 0$

$$\mathbb{P}\Big(\sup_{t\in T} X_t > x\Big) \le C(Q^{-1}(1/x))^{-d}\exp\Big(-\frac{x^2}{2\sigma^2}\Big),$$

where Q^{-1} is the inverse function of Q and C is an absolute constant.

2.3 Comparison inequalities

Ledoux and Talagrand [LT91] and Lifshits [Lif95] have a very nice discussion on comparison inequalities for Gaussian random variables. We list below several main results, which are also useful in the small ball problems; see [LS99a] and [Li99b].

In this subsection, we let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ be independent centered Gaussian random variables. The following identity due to Piterbarg [Pit82] gives a basis for various modifications of comparison inequalities.

Theorem 2.10 *Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be a function with bounded second derivatives. Then*

$$\mathbb{E}\, f(X) - \mathbb{E}\, f(Y) = \frac{1}{2}\int_0^1 \sum_{1\le i,j\le n} (\mathbb{E}\, X_iX_j - \mathbb{E}\, Y_iY_j)\mathbb{E}\, \frac{\partial^2 f}{\partial x_i \partial x_j}((1-\varepsilon)^{1/2}X + \varepsilon^{1/2}Y)d\varepsilon.$$

From the above identity, one can easily derive the famous Slepian lemma given in [Sl62].

Theorem 2.11 *(Slepian's lemma) If $\mathbb{E}\, X_i^2 = \mathbb{E}\, Y_i^2$ and $\mathbb{E}\, X_iX_j \le \mathbb{E}\, Y_iY_j$ for all $i, j = 1, 2, ..., n$, then for any x,*

$$\mathbb{P}\left(\max_{1\le i\le n} X_i \le x\right) \le \mathbb{P}\left(\max_{1\le i\le n} Y_i \le x\right).$$

Other interesting and useful extensions of Slepian's inequality, involving min-max, etc, can be found in Gordon [Gor85]. The next result is due to Fernique [F75] and requires no condition on the diagonal. Some elaborated variants are given in Vitale [Vi99c].

Theorem 2.12 *If*

$$\mathbb{E}\,(X_i - X_j)^2 \ge \mathbb{E}\,(Y_i - Y_j)^2 \quad \text{for} \quad 1 \le i, j \le n$$

then

$$\mathbb{E}\max_{1\le i\le n} X_i \ge \mathbb{E}\max_{1\le i\le n} Y_i$$

and

$$\mathbb{E}\, f(\max_{i,j}(X_i - X_j)) \ge \mathbb{E}\, f(\max_{i,j}(Y_i - Y_j))$$

for every non-negative convex increasing function f on $\mathbb{R}^+$.

We end this subsection with Anderson's inequality given in [A55], while the second inequality below is due to [S99].

Theorem 2.13 *Let Σ_X and Σ_Y be the covariance matrices of X and Y, respectively. If $\Sigma_X - \Sigma_Y$ is positive semi-definite, then for any $a \in \mathbb{R}^n$, any convex symmetric set C in $\mathbb{R}^n$, and any arbitrary set A in $\mathbb{R}^n$*

$$\mathbb{P}(X \in C) \le \mathbb{P}(Y \in C),$$

$$\mathbb{P}(X \in A) \ge \left(\frac{\det(\Sigma_Y)}{\det(\Sigma_X)}\right)^{1/2} \mathbb{P}(Y \in A)$$

and

$$\mathbb{P}(X + ar \in C) \le \mathbb{P}(X \in C),$$

is a monotone decreasing function of r, $0 \le r \le 1$.

2.4 Correlation inequalities

The Gaussian correlation conjecture states that for any two symmetric convex sets A and B in a separable Banach space E and for any centered Gaussian measure μ on E,

$$\mu(A \cap B) \geq \mu(A)\mu(B). \tag{2.6}$$

For early history of the conjecture we refer to Das Gupta, Eaton, Olkin, Perlman, Savage, and Sobel [G-72], Tong [To80] and Schechtman, Schlumprecht and Zinn [SSZ98].

An equivalent formulation of the conjecture is as follows: If $(X_1, \dots, X_n)$ is a centered, Gaussian random vector, then

$$\mathbb{P}\left(\max_{1\leq i\leq n} |X_i| \leq 1\right) \geq \mathbb{P}\left(\max_{1\leq i\leq k} |X_i| \leq 1\right)\mathbb{P}\left(\max_{k+1\leq i\leq n} |X_i| \leq 1\right) \tag{2.7}$$

for each $1 \leq k < n$. Khatri [Kh67] and Sidak [Si67, Si68] have shown that (2.7) is true for $k = 1$. That is,

$$\mathbb{P}\Big(\max_{1\leq i\leq n} |X_i| \leq x\Big) \geq \mathbb{P}(|X_1| \leq x)\mathbb{P}\big(\max_{2\leq i\leq n} |X_i| \leq x\Big). \tag{2.8}$$

The Khatri-Sidak inequality has become one of the most powerful tools for lower bound estimates of small ball probabilities; see Section 3.4. The inequality (2.8) was extended to elliptically contoured distributions in [G-72]. The original proofs of Khatri and Sidak are very lengthy. Simpler proofs are given in [J70] and [SSZ98]. Here we give an alternative proof. We only need to show that for any symmetric and convex set A in $\mathbb{R}^{n-1}$,

$$\mathbb{P}\Big(|X_1| \leq x, (X_2, \dots, X_n) \in A\Big)/\mathbb{P}(|X_1| \leq x) := f(x)/g(x)$$

is a monotone decreasing function of x, $x > 0$. Let $\phi(x_1, x_2, \dots, x_n)$ be the joint density function of $X_1, X_2, ...X_n$, and $\phi_1(x)$ be the density function of X_1. It suffices to show that

$$\forall \;\; x \geq 0, \;\; g(x)f'(x) - f(x)g'(x) \leq 0 \tag{2.9}$$

Let $y = (x_2, \dots, x_n)$ and $Y = (X_2, \dots, X_n)$. Noting that

$$f'(x) = 2\int_{y\in A} f(x,y)dy = 2\phi_1(x)\mathbb{P}(Y \in A \mid X_1 = x) \tag{2.10}$$

and $g'(x) = 2\phi_1(x)$, we have

$$\begin{aligned}
& g(x)f'(x) - f(x)g'(x) \\
= \;& 2\phi_1(x)\Big(\mathbb{P}(|X_1| \leq x)\mathbb{P}(Y \in A \mid X_1 = x) - \mathbb{P}(|X_1| \leq x, Y \in A)\Big) \\
= \;& 2\phi_1(x)\mathbb{P}(|X_1| \leq x)\Big(\mathbb{P}(Y \in A \mid X_1 = x) - \mathbb{P}(Y \in A \mid |X_1| \leq x)\Big) \\
\leq \;& 0
\end{aligned}$$

by Anderson's inequality, as desired.

It is also known that the Gaussian correlation conjecture is true for other special cases. Pitt [P77] showed that (2.7) holds for $n = 4$ and $k = 2$. The recent paper [SSZ98] sheds new light on the conjecture. They show that the conjecture is true whenever the sets are symmetric and ellipsoid or the sets are not too large. Hargé [Ha98] proves that (2.6) holds if one set is symmetric ellipsoid and the other is simply symmetric convex. Vitale [Vi99a] proves that (2.6) holds for two classes of sets: Schur cylinders and barycentrically ordered sets. Shao [S99] shows that

$$\mathbb{P}\left(\max_{1\le i\le n}|X_i|\le 1\right)\ge 2^{-\min(k,n-k)}\mathbb{P}\left(\max_{1\le i\le k}|X_i|\le 1\right)\mathbb{P}\left(\max_{k+1\le i\le n}|X_i|\le 1\right). \tag{2.11}$$

Recently, Li [Li99a] presented a weak form of the correlation conjecture, which is a useful tool to prove the existence of small ball constants; see Section 3.3. The varying parameter λ plays a fundamental role in most of the applications we know so far.

Theorem 2.14 *Let μ be a centered Gaussian measure on a separable Banach space E. Then for any $0<\lambda<1$, any symmetric, convex sets A and B in E*

$$\mu(A\cap B)\mu(\lambda^2 A+(1-\lambda^2)B)\ge\mu(\lambda A)\mu((1-\lambda^2)^{1/2}B).$$

In particular,

$$\mu(A\cap B)\ge\mu(\lambda A)\mu((1-\lambda^2)^{1/2}B) \tag{2.12}$$

and

$$\mathbb{P}(X\in A,Y\in B)\ge\mathbb{P}\Big(X\in\lambda A\Big)\mathbb{P}\Big(Y\in(1-\lambda^2)^{1/2}B\Big) \tag{2.13}$$

for any centered jointly Gaussian vectors X and Y in E.

The proof follows along the arguments of Proposition 3 in [SSZ98], where the case $\lambda=1/\sqrt{2}$ was proved. Here we present a simple proof for (2.13) given in [LS99b]. Let $a=(1-\lambda^2)^{1/2}/\lambda$, and (X^*,Y^*) be an independent copy of (X,Y). It is easy to see that $X-aX^*$ and $Y+Y^*/a$ are independent. Thus, by Anderson's inequality

$$\begin{aligned}\mathbb{P}(X\in A,Y\in B) &\ge \mathbb{P}(X-aX^*\in A,Y+Y^*/a\in B)\\ &= \mathbb{P}(X-aX^*\in A)\mathbb{P}(Y+Y^*/a\in B)\\ &= \mathbb{P}\Big(X\in\lambda A\Big)\mathbb{P}\Big(Y\in(1-\lambda^2)^{1/2}B\Big),\end{aligned}$$

as desired. The main difference between the Khatri-Sidak inequality and Theorem 2.14 in the applications to small ball probabilities is that the former only provides the rate (up to a constant) and the latter can preserve the rate together with the constant.

For various other approaches related to the Gaussian correlation conjecture, see Hu [Hu97], Hitczenko, Kwapień, Li, Schechtman, Schlumprecht and Zinn [H-98], Szarek and Werner [SzW99], Lewis and Pritchard [LP99].

3 Small ball probabilities in general setting

In this section, we present some fundamental results in the general setting for the small ball probabilities of Gaussian processes and Gaussian measures. Throughout, we use the following notations. Let E^* be the topological dual of E with norm $\|\cdot\|$ and X be a centered E-valued Gaussian random vector with law $\mu=\mathcal{L}(X)$. It is well known that there is a unique Hilbert space $H_\mu\subseteq E$ (also called the reproducing Hilbert space generated by μ) such that μ is determined by considering the pair (E,H_μ) as an abstract Wiener space (see [Gr70]). The Hilbert space H_μ can be described as the completion of the range of the mapping $S:E^*\to E$ defined by the Bochner integral

$$Sf=\int_E xf(x)d\mu(x)\qquad f\in E^*,$$

where the completion is in the inner product norm

$$\langle Sf, Sg \rangle_\mu = \int_E f(x)g(x)d\mu(x) \qquad f, g \in E^*.$$

We use $\|\cdot\|_\mu$ to denote the inner product norm induced on H_μ, and for well known properties and various relationships between μ, H_μ, and E, see Lemma 2.1 in Kuelbs [Ku76]. One of the most important facts is that the unit ball $K_\mu = \{x \in H_\mu : \|x\|_\mu \le 1\}$ of H_μ is always compact.

Finally, in order to compare the asymptotic rates, we write $f(x) \preceq g(x)$ as $x \to a$ if $\limsup_{x\to a} f(x)/g(x) < \infty$, and $f(x) \approx g(x)$ as $x \to a$ if $f(x) \preceq g(x)$ and $g(x) \preceq f(x)$.

3.1 Measure of shifted small balls

We first recall Anderson's inequality given in Theorem 2.13 that plays an important role in the estimate of small ball probability. For every convex symmetric subset A of E and every $x \in E$,

$$\mu(A + x) \le \mu(A). \tag{3.1}$$

Note that (3.1) is also an easy consequence of the log-concavity of Gaussian measure given in (2.3) by replacing A with $A + x$ and B with $A - x$ and taking $\lambda = 1/2$.

Next we have the following well known facts about the shift of symmetric convex sets (see, for example, [DHS79] and [dA83]).

Theorem 3.1 *For any $f \in H_\mu$ and $r > 0$,*

$$\exp\{-\|f\|_\mu^2/2\}\mu(x : \|x\| \le r) \le \mu(x : \|x - f\| \le r) \le \mu(x : \|x\| \le r). \tag{3.2}$$

Furthermore,

$$\mu(x : \|x - f\| \le \varepsilon) \sim \exp\Big\{ -\|f\|_\mu^2/2\Big\} \cdot \mu(x : \|x\| \le \varepsilon) \quad \textit{as} \quad \varepsilon \to 0. \tag{3.3}$$

The upper bound follows from Anderson's inequality (3.1). The lower bound follows from the Cameron-Martin formula

$$\mu(A - f) = \int_A \exp\{-\frac{1}{2}\|f\|_\mu^2 - \langle x, f\rangle_\mu\}d\mu(x) \tag{3.4}$$

for Borel subsets A of E, $f \in H_\mu$, together with Hölder's inequality and the symmetry of $\langle x, f\rangle_\mu$ on $A = \{x : \|x\| \le r\}$. Note that $\langle x, f\rangle_\mu$ can be defined as the stochastic inner product for μ almost all x in E. A particularly nice proof of (3.4) is contained in Proposition 2.1 of de Acosta [dA83].

Refinements of (3.2) are the following inequalities which play important roles in the studies of the functional form of Chung's law of the iterated logarithm; see Section 7.4 for more details. They extend the approach of Grill [G91] for Brownian motion, and are given in Kuelbs, Li and Linde [KLL94] and Kuelbs, Li and Talagrand [KLT94].

First we need some additional notations. Let

$$I(x) = \begin{cases} \|x\|_\mu^2/2 & x \in H_\mu \\ +\infty & \text{otherwise,} \end{cases}$$

which is the I-function of large deviations for μ. Furthermore, defining

$$I(f, r) = \inf_{\|f-x\|\le r} I(x),$$

we see $I(f,r) < \infty$ for all $f \in \bar{H}_\mu$, the support of μ in E. It is also the case that all of the properties established for the function $I(x,r)$ in Lemma 1 of [G91], when μ is Wiener measure on $C_0[0,1]$, have analogues for general μ. In particular, if $f \in E$ and $r > 0$, then there is a unique element, call it $h_{f,r}$, such that $\|h_{f,r} - f\| \leq r$ and $I(f,r) = I(h_{f,r})$. The following result is given in [KLL94].

Theorem 3.2 *For all $f \in \bar{H}_\mu$, $r > 0$, and $h = h_{f,\delta r}$*

$$\mu(x : \|x - f\| \leq r) \leq \exp\{-\sup_{\delta>0} \big((\delta-1)\delta^{-1}\langle f, h\rangle_\mu + (2-\delta)\delta^{-1} I(h)\big)\}\mu(x : \|x\| \leq r),$$

and for $0 \leq \delta \leq 1$

$$\mu(x : \|x - f\| \leq r) \geq \exp\{-I(h)\}\mu(x : \|x\| \leq (1-\delta)r).$$

In particular, for all $f \in \bar{H}_\mu$,

$$\mu(x : \|x - f\| \leq r) \leq \exp\{-I(h_{f,r})\}\mu(x : \|x\| \leq r)$$

and for all $f \in H_\mu$,

$$\exp\{-I(f)\} \cdot \mu(x : \|x\| \leq r) \leq \mu(x : \|x - f\| \leq r) \leq \exp\{-I(h_{f,r})\} \cdot \mu(x : \|x\| \leq r).$$

We thus see that the small ball probabilities of shifted balls can be handled by (3.3) if $f \in H_\mu$ and by the above Theorem if $f \notin H_\mu$. Note that the estimates we have in this section can be used to give the convergence rate and constant in the functional form of Chung's LIL; see Section 7.4, which only depends on the shift being in H_μ. So we can also answer the similar problem for points outside H_μ by Theorem 3.2. Other related shift inequalities for Gaussian measures are presented in Kuelbs and Li [KL98].

3.2 Precise links with metric entropy

Let μ denote a centered Gaussian measure on a real separable Banach space E with norm $\|\cdot\|$ and dual E^*. Consider the small ball probability

$$\phi(\varepsilon) = -\log \mu(x : \|x\| \leq \varepsilon) \tag{3.5}$$

as $\varepsilon \to 0$. The complexity of $\phi(\varepsilon)$ is well known, and there are only a few Gaussian measures for which $\phi(\varepsilon)$ has been determined completely as $\varepsilon \to 0$. Kuelbs and Li [KL93a] discovered a precise link between the function $\phi(\varepsilon)$ and the metric entropy of the unit ball K_μ of the Hilbert space H_μ generated by μ.

We recall first that if (E,d) is any metric space and A is a compact subset of (E,d), then the d-metric entropy of A is denoted by $H(A,\varepsilon) = \log N(A,\varepsilon)$ where $N(A,\varepsilon)$ is the minimum covering number defined by

$$N(A,\varepsilon) = \min\left\{n \geq 1 : \exists\, x_1, \ldots, x_n \in A \text{ such that } \cup_{j=1}^n B_\varepsilon(x_j) \supseteq A\right\},$$

where $B_\varepsilon(a) = \{x \in A : d(x,a) < \varepsilon\}$ is the open ball of radius ε centered at a. Since the unit ball $K_\mu = \{x \in H_\mu : \|x\|_\mu \leq 1\}$ of H_μ is always compact, K_μ has finite metric entropy.

Now we can state the precise links between the small ball function $\phi(\varepsilon)$ given in (3.5) and the metric entropy function $H(K_\mu, \varepsilon)$.

Theorem 3.3 *Let $f(x)$ and $g(x)$ be regularly varying functions at 0, and $J(x)$ be a slowly varying function at infinity such that $J(x) \approx J(x^\rho)$ as $x \to \infty$ for each $\rho > 0$.*

(I) *We have* $H(K_\mu, \varepsilon/\sqrt{2\phi(\varepsilon)}) \succeq \phi(2\varepsilon)$. *In particular, if* $\phi(\varepsilon) \preceq \phi(2\varepsilon)$ *and* $\phi(\varepsilon) \succeq \varepsilon^{-\alpha} J(\varepsilon^{-1})$, *where* $\alpha > 0$, *then*

$$H(K_\mu, \varepsilon) \succeq \varepsilon^{-2\alpha/(2+\alpha)} J(1/\varepsilon)^{2/(2+\alpha)}. \tag{3.6}$$

Especially, (3.6) *holds whenever* $\phi(\varepsilon) \approx \varepsilon^{-\alpha} J(\varepsilon^{-1})$.

(II) *If* $\phi(\varepsilon) \preceq f(\varepsilon)$, *then* $H(K_\mu, \varepsilon/\sqrt{f(\varepsilon)}) \preceq f(\varepsilon)$. *In particular, if* $f(\varepsilon) = \varepsilon^{-\alpha} J(\varepsilon^{-1})$ *with* $\alpha > 0$, *then*

$$H(K_\mu, \varepsilon) \preceq \varepsilon^{-2\alpha/(2+\alpha)} J(1/\varepsilon)^{2/(2+\alpha)}.$$

(III) *If* $H(K_\mu, \varepsilon) \succeq g(\varepsilon)$, *then* $\phi(\varepsilon) \succeq g(\varepsilon/\sqrt{\phi(\varepsilon)})$. *In particular, if* $g(\varepsilon) = \varepsilon^{-\alpha} J(1/\varepsilon)$, *where* $0 < \alpha < 2$, *then*

$$\phi(\varepsilon) \succeq \varepsilon^{-2\alpha/(2-\alpha)} (J(1/\varepsilon))^{2/(2-\alpha)}.$$

(IV) *If* $H(K_\mu, \varepsilon) \preceq \varepsilon^{-\alpha} J(1/\varepsilon)$, $0 < \alpha < 2$, *then for* ε *small*

$$\phi(\varepsilon) \preceq \varepsilon^{-2\alpha/(2-\alpha)} (J(1/\varepsilon))^{2/(2-\alpha)}.$$

As a simple consequence, it is easy to see that for $\alpha > 0$ and $\beta \in \mathbb{R}$,

$$\phi(\varepsilon) \approx \varepsilon^{-\alpha} (\log 1/\varepsilon)^{\beta} \quad \text{iff} \quad H(K_\mu, \varepsilon) \approx \varepsilon^{-2\alpha/(2+\alpha)} (\log 1/\varepsilon)^{2\beta/(2+\alpha)}. \tag{3.7}$$

To fully understand this basic result, we would like to make the following remarks. First, since it is known from Goodman [Go90] that $H(K_\mu, \varepsilon) = o(\varepsilon^{-2})$ regardless of the Gaussian measure μ, the restriction on α in part (III) and (IV) of the Theorem is natural.

Second, we see clearly from (I) and (II) of Theorem 3.3 that in almost all cases of interest, small ball probabilities provide sharp estimates on the metric entropy. This approach has been applied successfully to various problems on estimating metric entropy; see Section 7.6 for more details.

Third, the proofs of (I), (II) and (III) given essentially in [KL93a] are based on the relations

$$H(2\varepsilon, \lambda K_\mu) \le \lambda^2/2 - \log \mu(B_\varepsilon(0)) \tag{3.8}$$

and

$$H(\varepsilon, \lambda K_\mu) + \log \mu(B_{2\varepsilon}(0)) \ge \log \Phi(\lambda + \eta_\varepsilon) \tag{3.9}$$

for all $\lambda > 0$ and $\varepsilon > 0$, where $\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} \exp\{-u^2/2\} du$ and $\Phi(\eta_\varepsilon) = \mu(B_\varepsilon(0))$. In fact, (3.8) follows easily from (3.2), and (3.9) is a consequence of the isoperimetric inequality for Gaussian measures which states that $\mu(A+\lambda K_\mu) \ge \Phi(\lambda+\eta)$ for any $\lambda > 0$ and any Borel set A with $\mu(A) \ge \Phi(\eta)$; see Theorem 2.1. The proof of (IV) of Theorem 3.3 given by Li and Linde [LL99] is based on (3.9) with an iteration procedure, and a new connection between small ball probabilities and the l–approximation numbers given in Section 3.5.

Fourth, the recent establishment of (IV) in Theorem 3.3 allows applications of powerful tools and deep results from functional analysis to estimate the small ball probabilities. The following is a very special case of Theorem 5.2 in [LL99], which is a simple consequence of (IV) in Theorem 3.3 for linear transformations of a given Gaussian process. It is worthwhile to stress that the result below, along with many other consequences of (IV) in Theorem 3.3, has no purely probabilistic proof to date.

Theorem 3.4 *Let* $Y = (Y(t))_{t\in[0,1]}$ *be a centered Gaussian process with continuous sample path and assume that*

$$\log \mathbb{P}\left(\sup_{0\le t\le 1} |Y(t)| \le \varepsilon\right) \succeq -\varepsilon^{-\alpha} \left(\log \frac{1}{\varepsilon}\right)^{\beta}$$

for $\alpha > 0$. *If*

$$X(t) = \int_0^1 K(t,s)Y(s)ds \tag{3.10}$$

with the kernel $K(t,s)$ *satisfying the Hölder condition*

$$\int_0^1 \left|K(t,s) - K(t',s)\right| ds \le c\left|t-t'\right|^{\lambda}, \quad t,t' \in [0,1], \tag{3.11}$$

for some $\lambda \in (0,1]$ *and some* $c > 0$, *then*

$$\log \mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \varepsilon\right) \succeq -\varepsilon^{-\alpha/(\alpha\lambda+1)}\left(\log\frac{1}{\varepsilon}\right)^{\beta/(\alpha\lambda+1)}.$$

Some applications of Theorem 3.4 for integrated Gaussian processes are detailed in Section 4.4 and 6.3. Note that the integrated kernel $K(t,s) = 1_{(0,t)}(s)$ satisfies the Hölder condition (3.11) with $\lambda = 1$. So if $Y(t)$ in Theorem 3.4 is a fractional Brownian motion (see Section 4.3) and $X(t)$ in Theorem 3.4 is the integrated fractional Brownian motion (see Section 4.4), then the lower bound given in Theorem 3.4 is sharp by observing (4.14) and (4.15). Other significant applications of (IV) in Theorem 3.3 are mentioned in Section 5.2 on Brownian sheets.

Finally, in the theory of small ball estimates for Gaussian measure, (IV) of Theorem 3.3 together with basic techniques of estimating entropy numbers as demonstrated in [LL99], is one of the most general and powerful among all the existing methods of estimating the small ball lower bound. Another commonly used general lower bound estimate on supremum of Gaussian processes is presented in Sections 3.4 and 4.1.

3.3 Exponential Tauberian theorem

Let V be a positive random variable. Then the following exponential Tauberian theorem connects the small ball type behavior of V near zero with an asymptotic Laplace transform of the random variable V.

Theorem 3.5 *For* $\alpha > 0$ *and* $\beta \in \mathbb{R}$

$$\log \mathbb{P}(V \le \varepsilon) \sim -C_V \varepsilon^{-\alpha} |\log \varepsilon|^{\beta} \quad \textit{as} \quad \varepsilon \to 0^+$$

if and only if

$$\log \mathbb{E}\exp(-\lambda V) \sim -(1+\alpha)\alpha^{-\alpha/(1+\alpha)} C_V^{1/(1+\alpha)} \lambda^{\alpha/(1+\alpha)} (\log\lambda)^{\beta/(1+\alpha)} \quad \textit{as} \quad \lambda \to \infty.$$

A slightly more general formulation of the above result is given in Theorem 4.12.9 of Bingham, Goldie and Teugels [BGT87], and is called de Bruijn's exponential Tauberian theorem. Note that one direction between the two quantities is easy and follows from

$$\mathbb{P}(V \le \varepsilon) = \mathbb{P}(-\lambda V \ge -\lambda\varepsilon) \le \exp(\lambda\varepsilon)\mathbb{E}\exp(-\lambda V),$$

which is just Chebyshev's inequality.

Next we give two typical applications. Let $X_0(t) = W(t)$ and

$$X_m(t) = \int_0^t X_{m-1}(s)ds, \quad t \ge 0, \quad m \ge 1,$$

which is the m'th integrated Brownian motion or the m-fold primitive. Note that using integration by parts we also have the representation

$$X_m(t) = \frac{1}{m!}\int_0^t (t-s)^m dW(s), \quad m \geq 0. \tag{3.12}$$

The exact Laplace transform $\mathbb{E}\exp\left(-\lambda\int_0^1 X_m^2(t)dt\right)$ is computed in Chen and Li [CL99] and one can find from the exact Laplace transform, for each integer $m \geq 0$,

$$\lim_{\lambda\to\infty}\lambda^{-1/(2m+2)}\log\mathbb{E}\exp\left\{-\lambda\int_0^1 X_m^2(t)dt\right\} = -2^{-(2m+1)/(2m+2)}\left(\sin\frac{\pi}{2m+2}\right)^{-1}. \tag{3.13}$$

Then by the Tauberian theorem, (3.13) implies

$$\log\mathbb{P}\left(\int_0^1 X_m^2(t)dt \leq \varepsilon^2\right) \sim 2^{-1}(2m+1)\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-(2m+2)/(2m+1)}\varepsilon^{-2/(2m+1)}. \tag{3.14}$$

For other applications of this type going the other way, see Section 7.10.

Our second application is for sums of independent random variables, and it is an easy consequence of the Tauberian theorem.

Corollary 3.1 *If V_i, $1 \leq i \leq m$, are independent nonnegative random variables such that*

$$\lim_{\varepsilon\to 0}\varepsilon^\gamma\log\mathbb{P}(V_i \leq \varepsilon) = -d_i, \quad 1 \leq i \leq m,$$

for $0 < \gamma < \infty$, then

$$\lim_{\varepsilon\to 0}\varepsilon^\gamma\log\mathbb{P}\left(\sum_{i=1}^m V_i \leq \varepsilon\right) = -\left(\sum_{i=1}^m d_i^{1/(1+\gamma)}\right)^{1+\gamma}.$$

Now we consider the sum of two centered Gaussian random vectors X and Y in a separable Banach space E with norm $\|\cdot\|$.

Theorem 3.6 *If X and Y are independent and*

$$\lim_{\varepsilon\to 0}\varepsilon^\gamma\log\mathbb{P}(\|X\| \leq \varepsilon) = -C_X, \quad \lim_{\varepsilon\to 0}\varepsilon^\gamma\log\mathbb{P}(\|Y\| \leq \varepsilon) = -C_Y \tag{3.15}$$

with $0 < \gamma < \infty$ and $0 \leq C_X, C_Y \leq \infty$, then

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon^\gamma\log\mathbb{P}(\|X+Y\| \leq \varepsilon) &\leq -\max(C_X, C_Y),\\ \liminf_{\varepsilon\to 0}\varepsilon^\gamma\log\mathbb{P}(\|X+Y\| \leq \varepsilon) &\geq -\left(C_X^{1/(1+\gamma)} + C_Y^{1/(1+\gamma)}\right)^{1+\gamma}.\end{aligned}$$

The upper bound follows from

$$\mathbb{P}(\|X+Y\| \leq \varepsilon) \leq \min\left(\mathbb{P}(\|X\| \leq \varepsilon), \mathbb{P}(\|Y\| \leq \varepsilon)\right)$$

by Anderson's inequality and the independence assumption. The lower bound follows from the triangle inequality $\|X+Y\| \leq \|X\| + \|Y\|$ and Corollary 3.1. Note that both upper and lower constants given above are not sharp in the case $X = Y$ in law. It seems a very challenging problem to find the precise constant for $\lim_{\varepsilon\to 0}\varepsilon^\gamma\log\mathbb{P}(\|X+Y\| \leq \varepsilon)$, which we conjecture to exist, in terms of C_X, C_Y, $\|\cdot\|$, γ and possibly the properties of covariance structure of X and Y.

What happens for the sums if X and Y are *not* necessarily independent but with different small ball rates? This is given recently in Li [Li99a] as an application of Theorem 2.14.

Theorem 3.7 *For any joint Gaussian random vectors X and Y such that (3.15) holds with $0 < \gamma < \infty$, $0 < C_X < \infty$ and $C_Y = 0$, we have*

$$\lim_{\varepsilon \to 0} \varepsilon^\gamma \log \mathbb{P}\left(\|X+Y\| \le \varepsilon\right) = -C_X.$$

The proof is so easy now that we have to present it, keeping in mind the very simple argument for (2.12) or (2.13). For the lower bound, we have by the inequality (2.13) with any $0 < \delta < 1$, $0 < \lambda < 1$,

$$\begin{aligned} \mathbb{P}\left(\|X+Y\| \le \varepsilon\right) &\ge \mathbb{P}\left(\|X\| \le (1-\delta)\varepsilon,\ \|Y\| \le \delta\varepsilon\right) \\ &\ge \mathbb{P}\left(\|X\| \le \lambda(1-\delta)\varepsilon\right) \cdot \mathbb{P}\left(\|Y\| \le (1-\lambda^2)^{1/2}\delta\varepsilon\right). \end{aligned}$$

Thus

$$\liminf_{\varepsilon \to 0} \varepsilon^\gamma \log \mathbb{P}\left(\|X+Y\| \le \varepsilon\right) \ge -(\lambda(1-\delta))^{-\gamma} C_X$$

and the lower bound follows by taking $\delta \to 0$ and $\lambda \to 1$. For the upper bound, we have again by the inequality (2.13) with any $0 < \delta < 1$, $0 < \lambda < 1$,

$$\begin{aligned} \mathbb{P}\left(\|X\| \le \frac{\varepsilon}{(1-\delta)\lambda}\right) &\ge \mathbb{P}\left(\|X+Y\| \le \frac{\varepsilon}{\lambda},\ \|Y\| \le \delta \cdot \frac{\varepsilon}{(1-\delta)\lambda}\right) \\ &\ge \mathbb{P}\left(\|X+Y\| \le \varepsilon\right) \cdot \mathbb{P}\left(\|Y\| \le (1-\lambda^2)^{1/2}\delta\frac{\varepsilon}{(1-\delta)\lambda}\right). \end{aligned}$$

Thus

$$\limsup_{\varepsilon \to 0} \varepsilon^\gamma \log \mathbb{P}\left(\|X+Y\| \le \varepsilon\right) \le -(\lambda(1-\delta))^{\gamma} C_X$$

and the upper bound follows by taking $\delta \to 0$ and $\lambda \to 1$.

As a direct consequence of Theorem 3.7, we see easily that under the sup-norm or L_p-norm, Brownian motion and Brownian bridge have exact the same small ball behavior at the log level, and so do Brownian sheets and various tied down Brownian sheets including Kiefer process; see Section 5.2, 6.2 and 7.2.

3.4 Lower bound on supremum under entropy conditions

A Gaussian process $X = (X_t)_{t \in T}$ with index set T is a random process such that each finite linear combination $\sum_i \alpha_i X_{t_i} \in \mathbb{R}$, $t_i \in T$, is a real valued Gaussian variable. We always assume it is separable. For a detailed discussion related to separability, we refer to Section 2.2 of [LT91]. The distribution of the Gaussian process X is therefore completely determined by its covariance sructure $\mathbb{E}\, X_s X_t$, $s, t \in T$. Assume $(X_t)_{t \in T}$ is a centered Gaussian process with entropy number $N(T, d; \varepsilon)$, the minimal number of balls of radius $\varepsilon > 0$, under the Dudley metric

$$d(s,t) = (\mathbb{E}\,|X_s - X_t|^2)^{1/2}, \quad s, t \in T$$

that are necessary to cover T. Then a commonly used general lower bound estimate on the supremum was established in Talagrand [T93] and the following nice formulation was given in [L96], page 257.

Theorem 3.8 *Assume that there is a nonnegative function ψ on $\mathbb{R}_+$ such that $N(T, d; \varepsilon) \le \psi(\varepsilon)$ and such that $c_1\psi(\varepsilon) \le \psi(\varepsilon/2) \le c_2\psi(\varepsilon)$ for some constants $1 < c_1 \le c_2 < \infty$. Then, for some $K > 0$ and every $\varepsilon > 0$ we have*

$$\log \mathbb{P}\left(\sup_{s,t \in T} |X_s - X_t| \le \varepsilon\right) \ge -K\psi(\varepsilon).$$

In particular,

$$\log \mathbb{P}\left(\sup_{t\in T}|X_t| \le \varepsilon\right) \succeq -\psi(\varepsilon).$$

The proof of this theorem is based on the Khatri-Sidak correlation inequality given in (2.8) and standard chaining arguments usual for estimation of *large* ball probabilities via $N(T, d; \cdot)$; see e.g. [L96]. The similar idea of the proof was also used in Shao [S93] and Kuelbs, Li and Shao [KLS95] for some special Gaussian processes. Here is an outline of the method; a similar argument is given at the end of Section 4.1.

Let $(X_t)_{t\in T}$ be a centered Gaussian process. Then, by the Khatri-Sidak inequality

$$\mathbb{P}\left(\sup_{t\in A}|X_t| \le x, |X_{t_0}| \le x\right) \ge \mathbb{P}(|X_{t_0}| \le x)\mathbb{P}\left(\sup_{t\in A}|X_t| \le x\right)$$

for every $A \subset T$, $t_0 \in T$ and $x > 0$. If there are a countable set T_c and a Gaussian process Y on T_c such that

$$\left\{\sup_{t\in T}|X_t| \le x\right\} \supset \left\{\sup_{t\in T_c}|Y_t| \le x\right\},$$

then we have

$$\mathbb{P}\left(\sup_{t\in T}|X_t| \le x\right) \ge \prod_{t\in T_c} \mathbb{P}(|Y_t| \le x).$$

Since Y_t is a normal random variable for each $t \in T_c$, the right hand side above can be easily estimated. So, the key step of estimating the lower bound of $\mathbb{P}\left(\sup_{t\in T}|X_t| \le x\right)$ is to find the countable set T_c and Gaussian process Y.

Although Theorem 3.8 is relatively easy to use, it does *not* always provide sharp lower estimates even when $N(T, d; \varepsilon)$ can be estimated sharply. The simplest example is $X_t = \xi t$ for $t \in T = [0, 1]$, where ξ denotes a standard normal random variable. In this case,

$$\mathbb{P}\left(\sup_{t\in T}|X_t| < \varepsilon\right) = \mathbb{P}\left(|\xi| < \varepsilon\right) \sim (2/\pi)^{1/2} \cdot \varepsilon,$$

but Theorem 3.8 produces an exponential lower bound $\exp(-c/\varepsilon)$ for the above probability. More interesting examples are the integrated fractional Brownian motion given in Section 4.4 and the fractional integrated Brownian motion W_β given in Section 6.3. We know as applications of Theorem 3.4 and a special upper bound estimate,

$$\lim_{\varepsilon\to 0} \varepsilon^{2/\beta} \log \mathbb{P}\left(\sup_{0\le t\le 1}|W_\beta(t)| \le \varepsilon\right) = -k_\beta, \tag{3.16}$$

$0 < k_\beta < \infty$, $\beta > 0$; see [LL98]. But for $\beta > 2$, Theorem 3.8 only implies a lower bound of order ε^{-1} for the log of the probability. When $\beta = 3$,

$$W_3(t) = \int_0^t (t-s)dW_s = \int_0^t W(s)ds, \quad t \ge 0,$$

is the integrated Brownian motion and the sharp lower estimate of order $\varepsilon^{-2/3}$ was first obtained in Khoshnevisan and Shi [KS98a] by using special local time techniques.

The following example in Lifshits [Lif99] suggests that the stationarity plays a big role in the upper estimate in Theorem 4.5, Theorem 4.6 and that L_2-norm entropy $N(T, d; \cdot)$ is not an appropriate tool for the upper bound.

Example. Let $\alpha > 0$, and $\{\xi_i\}$ be i.i.d standard normal random variables. Define $\phi(t) = 1 - |2t-1|$ for $t \in [0,1]$. Let $\{u\}$ denote the fractional part of real number u. Put

$$X_t = \xi_0 t + \sum_{i=1}^{\infty} 2^{-\alpha i/2} \xi_i \phi(\{2^i t\}) \quad \text{for} \quad t \in [0,1].$$

It is easy to see that $\mathbb{E}\,(X_t - X_s)^2 \geq c|t-s|^\alpha$ for all $s, t \in [0,1]$, where $c > 0$ is a constant. However, we have

$$\log(\mathbb{P}\left(\sup_{0\le t\le 1} |X_t| \le \varepsilon\right) \approx -\log^2(1/\varepsilon) \tag{3.17}$$

as $\varepsilon \to 0$. To see the lower bound, we have

$$\begin{aligned}
\mathbb{P}(\sup_{0\le t\le 1} |X_t| \le \varepsilon) &\ge \mathbb{P}\Big(\sum_{i=0}^{\infty} 2^{-\alpha i/2}|\xi_i| \le \varepsilon\Big) \\
&\ge \mathbb{P}\Big(|\xi_i| \le \varepsilon 2^{\alpha i/4}(1-2^{-\alpha/4}), i = 0, 1, \ldots\Big) \\
&= \prod_{i=0}^{\infty} \mathbb{P}\Big(|\xi_1| \le \varepsilon 2^{\alpha i/4}(1-2^{-\alpha/4})\Big) \\
&\ge \exp(-K_2 \log^2(1/\varepsilon))
\end{aligned}$$

for some positive constant K_2. The upper bound can be proven as follows:

$$\begin{aligned}
\mathbb{P}(\sup_{0\le t\le 1} |X_t| \le \varepsilon) &\le \mathbb{P}(\max_{k\ge 2} |X(2^{-k})| \le \varepsilon) \\
&\le \mathbb{P}\Big(\max_{k\ge 2} |\sum_{i=1}^{k-1} 2^{-\alpha i/2}\xi_i 2^{-(k-i-1)}| \le \varepsilon\Big) \\
&\le \mathbb{P}\Big(\max_{k\ge 2} |\xi_k| \le 2\varepsilon 2^{\alpha k/2}\Big) \\
&= \prod_{k\ge 2} \mathbb{P}(|\xi_0| \le 2\varepsilon 2^{\alpha k/2}) \\
&\le \exp(-K_1 \log^2(1/\varepsilon))
\end{aligned}$$

for some positive constant K_1.

For the upper bound estimates, there is no general probabilistic method available in the spirit of Theorem 3.8 at this time. Various special techniques based on Anderson's inequality, Slepian's lemma, exponential Chebychev inequality, iteration procedure, etc, are used in [P78], [S93], [MR95], [KLS95], [St96], [DLL98], [Li99b], [LS99a], [DLL99] and references therein. See Section 4.2 and 5.2 for more information.

3.5 Connections with l-approximation numbers

The small ball behaviour of a Gaussian process is also closely connected with the speed of approximation by "finite rank" processes. For the centered Gaussian random variable X on E, the n^{th} l–approximation number of X is defined by

$$l_n(X) = \inf\left\{ \left(\mathbb{E}\Big\|\sum_{j=n+1}^{\infty} \xi_j x_j\Big\|^2\right)^{1/2} : X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j x_j,\ x_j \in E \right\}. \tag{3.18}$$

where ξ_j are i.i.d. standard normal and the inf is taken over all possible series representations for X. One may consider $l_n(X)$ as a measure of a specific orthogonal approximation of X by random vectors of rank n. Note that $l_n(X) \to 0$ as $n \to \infty$ if X has bounded sample path. Other equivalent definitions and some well known properties of $l_n(X)$ can be found in Li and Linde [LL99] and Pisier [Pi89]. The following results show the connections between the small ball probability of X and its l-approximation numbers $l_n(X)$.

Theorem 3.9 *Let $\alpha > 0$ and $\beta \in \mathbb{R}$.*

(a) *If*

$$l_n(X) \preceq n^{-1/\alpha}(1 + \log n)^{\beta}, \tag{3.19}$$

then

$$-\log \mathbb{P}\left(\|X\| \le \varepsilon\right) \preceq \varepsilon^{-\alpha}\left(\log 1/\varepsilon\right)^{\alpha\beta} . \tag{3.20}$$

(b) *Conversely, if (3.20) holds, then*

$$l_n(X) \preceq n^{-1/\alpha}(1 + \log n)^{\beta+1} . \tag{3.21}$$

Moreover, if E is K–convex (e.g. L_p, $1 < p < \infty$), i.e. E does not contain l_1^n's uniformly (see Thm. 2.4 in [Pi89]*), then (3.19) holds and thus (3.19) and (3.20) are equivalent in this case.*

(c) *If*

$$-\log \mathbb{P}\left(\|X\| \le 2\varepsilon\right) \succeq -\log \mathbb{P}\left(\|X\| \le \varepsilon\right) \succeq \varepsilon^{-\alpha}\left(\log 1/\varepsilon\right)^{\alpha\beta},$$

then

$$l_n(X) \succeq n^{-1/\alpha}(1 + \log n)^{\beta-1/\alpha}.$$

(d) *If E is K-convex and*

$$l_n(X) \approx n^{-1/\alpha}(1 + \log n)^{\beta},$$

then

$$-\log \mathbb{P}\left(\|X\| \le \varepsilon\right) \approx \varepsilon^{-\alpha}\left(\log 1/\varepsilon\right)^{\alpha\beta} .$$

Parts (a), (b) and (c) of Theorem 3.9 are given by Li and Linde [LL99] and part (d) is a very nice observation of Ingo Steinwart. There are several natural and important open questions. Does $l_n(X) \succeq n^{-1/\alpha}$ imply a lower estimate for $-\log \mathbb{P}\left(\|X\| \le \varepsilon\right)$? What is the optimal power for the log–term in (3.21) ? Recently it is shown in [D-99] that under the sup-norm

$$l_n(B_{d,\alpha}) \approx n^{-\alpha/2}(1 + \log n)^{d(\alpha+1)/2-\alpha/2} \tag{3.22}$$

for the fractional Brownian sheets $B_{d,\alpha}(t)$, $d \ge 1$ and $0 < \alpha < 2$; see Section 5.2 for definition. Hence the best known lower bound (5.8) for $B_{d,\alpha}$ under the sup-norm over $[0,1]^d$ follows from (3.22) and part (a) of Theorem 3.9. On the other hand, the correct rates of small ball probabilities for Brownian sheets $B_{d,1}$, $d \ge 3$, are still unknown under the sup-norm. See Section 5.3. This suggests that $l_n(X)$ may be easier to work with. In fact, finding the upper bound for $l_n(X)$ is relatively easy since we only need to find one good series expansion for X. But it may not be sharp even for the standard Brownian motion $W(t) = B_{1,1}(t)$, $0 \le t \le 1$, under the sup-norm since $l_n(W) \approx n^{-1/2}(1 + \log n)^{1/2}$ and $\log \mathbb{P}\left(\sup_{0 \le t \le 1} |W(t)| \le \varepsilon\right) \sim -(\pi^2/8)\varepsilon^{-2}$. Consequently, we also see that (3.20) does not imply (3.21) with log–power β in general. At least $\beta + 1/2$ is needed.

Finally, we mention that the links to approximation theory are not restricted only to metric entropy and the l-approximation numbers. The small ball probability for X is also related to many other different approximation quantities such as Gelfand numbers, Kolmogorov numbers and volume numbers of the compact linear operator from H_μ to E associated with X, although the links are not as precise as those with the entropy numbers; see [LL99] and [Pi89].

3.6 A connection between small ball probabilities

Let X and Y be any two centered Gaussian random vectors in a separable Banach space E with norm $\|\cdot\|$. We use $|\cdot|_{\mu(X)}$ to denote the inner product norm induced on H_μ by $\mu = \mathcal{L}(X)$. The following relation discovered recently in Chen and Li [CL99] can be used to estimate small ball probabilities under any norm via a relatively easier L_2-norm estimate.

Theorem 3.10 *For any $\lambda > 0$ and $\varepsilon > 0$,*

$$\mathbb{P}\left(\|Y\| \le \varepsilon\right) \ge \mathbb{P}\left(\|X\| \le \lambda\varepsilon\right) \cdot \mathbb{E}\exp\{-2^{-1}\lambda^2\, |Y|^2_{\mu(X)}\}. \tag{3.23}$$

In particular, for any $\lambda > 0$, $\varepsilon > 0$ and $\delta > 0$,

$$\mathbb{P}\left(\|Y\| \le \varepsilon\right) \cdot \exp\{-\lambda^2\delta^2/2\} \ge \mathbb{P}\left(\|X\| \le \lambda\varepsilon\right) \mathbb{P}\left(|Y|_{\mu(X)} \le \delta\right).$$

Note that we need $Y \in H_{\mu(X)} \subset E$ almost surely. Otherwise for $f \notin H_{\mu(X)}$, $|f|_{\mu(X)} = \infty$ and the result is trivial. Thus the result can also be stated as follows. Let $(H, |\cdot|_H)$ be a Hilbert space and Y be a Gaussian vector in H. Then for any linear operator $L : H \to E$ and the Gaussian vector X in E with covariance operator LL^*

$$\mathbb{P}\left(\|LY\| \le \varepsilon\right) \ge \mathbb{P}\left(\|X\| \le \lambda\varepsilon\right) \cdot \mathbb{E}\exp\{-2^{-1}\lambda^2 |Y|^2_H\}$$

for any $\lambda > 0$ and $\varepsilon > 0$.

The proof of Theorem 3.10 is very simple and based on both directions of the well known shift inequalities (3.2). Without loss of generality, assume X and Y are independent. Then

$$\mathbb{P}\left(\|Y\| \le \varepsilon\right) \ge \mathbb{P}\left(\|X - \lambda Y\| \le \lambda\varepsilon\right) \ge \mathbb{P}\left(\|X\| \le \lambda\varepsilon\right) \cdot \mathbb{E}\exp\{-2^{-1}\lambda^2\, |Y|^2_{\mu(X)}\}.$$

To see the power of Theorem 3.10, we state and prove the following special case of a result for $X_m(t)$ given in Chen and Li [CL99], where $X_m(t)$ is defined in (3.12). The particular case of $m = 1$, or so called integrated Brownian motion, was studied in Khoshnevisan and Shi [KS98a] using local time techniques.

Theorem 3.11 *We have*

$$\lim_{\varepsilon \to 0} \varepsilon^{2/3} \log \mathbb{P}\left(\sup_{0 \le t \le 1} \left|\int_0^t W(s)ds\right| \le \varepsilon\right) = -\kappa \tag{3.24}$$

with

$$3/8 \le \kappa \le (2\pi)^{2/3} \cdot 3/8. \tag{3.25}$$

The existence of the limit is by subadditivity; see Section 6.3. The lower bound for κ in (3.25) follows from $\mathbb{P}\left(\sup_{0\le t\le 1}\left|\int_0^t W(s)ds\right| \le \varepsilon\right) \le \mathbb{P}\left(\int_0^1 \left|\int_0^t W(s)ds\right|^2 dt \le \varepsilon^2\right)$ and the L_2 estimate given in (3.14). The upper bound for κ in (3.25) follows from Theorem 3.10, the L_2 estimate given in (3.14) and the well known estimate $\log \mathbb{P}\left(\sup_{0\le t\le 1}|W(t)| \le \varepsilon\right) \sim -(\pi^2/8)\varepsilon^{-2}$. To be more precise, take $Y(t) = \int_0^t W(s)ds$ and $X = W(t)$; then

$$\mathbb{P}\left(\sup_{0\le t\le 1}\left|\int_0^t W(s)ds\right| \le \varepsilon\right) \ge \mathbb{P}\left(\sup_{0\le t\le 1}|W(t)| \le \lambda\varepsilon\right) \cdot \mathbb{E}\exp\{-2^{-1}\lambda^2 \int_0^1 W^2(s)ds\}.$$

Taking $\lambda = \lambda_\varepsilon = (\pi^2/2)^{1/3}\varepsilon^{-2/3}$, the bound follows. It is of interest to note that both bounds rely on easier L_2 estimates and the constant bounds for κ are the sharpest known.

4 Gaussian processes with index set $T \subset \mathbb{R}$

This section focuses on small ball probabilities of Gaussian processes with index set $T \subset \mathbb{R}$. For the sake of easy presentation, we assume $T = [0, 1]$. Some of the results are covered by general approaches in the last section. But we still state them in different forms for comparison and historical purposes.

4.1 Lower bounds

We first present a general result on the lower bound.

Theorem 4.1 *Let $\{X_t, t \in [0,1]\}$ be a centered Gaussian process with $X(0) = 0$. Assume that there is a function $\sigma^2(h)$ such that*

$$\forall\ 0 \le s, t \le 1,\ \mathbb{E}\,(X_s - X_t)^2 \le \sigma^2(|t-s|) \tag{4.1}$$

and that there are $0 < c_1 \le c_2 < 1$ such that $c_1\sigma(2h \wedge 1) \le \sigma(h) \le c_2\sigma(2h \wedge 1)$ for $0 \le h \le 1$. Then, there is a positive and finite constant K_1 depending only on c_1 and c_2 such that

$$\mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \sigma(\varepsilon)\right) \ge \exp(-K_1/\varepsilon). \tag{4.2}$$

The above result was given in Csörgő and Shao [CS94], Kuelbs, Li and Shao [KLS95]. It can also be derived from the Talagrand lower bound in Section 3.4. Its detailed proof is similar to the outline we give after the Theorem 4.3. The next result is intuitively appealing. It says that the small ball probability $\mathbb{P}(\sup_{0\le t\le 1}|X_t| \le \sigma(\varepsilon))$ is determined by $\mathbb{P}(\max_{1\le i\le 1/\delta\varepsilon}|X(i\delta\varepsilon)| \le \sigma(\varepsilon))$ as long as δ is sufficiently small. We refer to Shao [S99] for a proof.

Theorem 4.2 *Under the condition of Theorem 4.1, there are positive constants K_1 and θ depending only on c_1, c_2 such that $\forall\ 0 < \delta < 1, 0 < \varepsilon < 1$*

$$\mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le (1+\delta^\theta)\sigma(\varepsilon)\right) \ge \exp\Big(-K_1\,\delta/\varepsilon\Big)\,\mathbb{P}\Big(\max_{0\le i\le 1/(\delta\varepsilon)} |X(i\delta\varepsilon)| \le \sigma(\varepsilon)\Big). \tag{4.3}$$

Our next result is a generalization of Theorem 4.1, which may be useful particularly for a differentiable Gaussian process. An application is given in Section 4.4 for integrated fractional Brownian motion.

Theorem 4.3 *Let $\{X_t, t \in [0,1]\}$ be a centered Gaussian process with $X(0) = 0$. Assume that there is a function $\sigma^2(h)$ such that*

$$\forall\ 0 \le h < 1/2, h < t \le 1-h,\ \mathbb{E}\,(X(t+h) + X(t-h) - 2X_t)^2 \le \sigma^2(h). \tag{4.4}$$

Assume that there are $0 < c_1 \le c_2 < 1$ such that $c_1\sigma(2h \wedge 1) \le \sigma(h) \le c_2\sigma(2h \wedge 1)$ for $0 \le h \le 1$. Then, there is a positive and finite constant K_1 depending only on c_1 and c_2 such that

$$\mathbb{P}\left(\sup_{0\le t\le 1} |X(t)| \le \sigma(\varepsilon)\right) \ge \exp(-K_1/\varepsilon). \tag{4.5}$$

The idea of the proof has been explained in Section 3.4. To illustrate the idea more precisely, it is worthwhile to outline the proof as follows.

The assumptions already imply that X is almost surely continuous. Therefore,

$$\sup_{0\le t\le 1}|X(t)|\le |X(1)|+\sum_{k=1}^{\infty}\max_{1\le i\le 2^k}|X((i+1)2^{-k})+X((i-1)2^{-k})-2X(i2^{-k})| \quad a.s.$$

Without loss of generality, assume $0<\varepsilon<1$. Let n_0 be an integer such that

$$2^{-n_0}\le\varepsilon\le 2^{-n_0+1}$$

and define

$$\varepsilon_k=\sigma(\varepsilon 1.5^{-|n_0-k|})/K,\ \ k=1,2,\dots,$$

where K is a constant. It is easy to see that

$$\sum_{k=1}^{\infty}\varepsilon_k\le\sigma(\varepsilon)/2$$

provided that K is sufficiently large. Hence by the Khatri-Sidak inequality

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le 1}|X(t)|\le\sigma(\varepsilon)\right)\\
\ge\ &\mathbb{P}\left(|X(1)|\le\sigma(\varepsilon)/2,\ \max_{1\le i\le 2^k}|X((i+1)2^{-k})+X((i-1)2^{-k})-2X(i2^{-k})|\le\varepsilon_k, k=1,2,\dots\right)\\
\ge\ &\mathbb{P}(|X(1)|\le\sigma(\varepsilon)/2)\prod_{k=1}^{\infty}\prod_{1\le i\le 2^k}\mathbb{P}\left(|X((i+1)2^{-k})+X((i-1)2^{-k})-2X(i2^{-k})|\le\varepsilon_k\right).
\end{aligned}$$

A direct argument then gives (4.5).

4.2 Upper bounds

The upper bound of small ball probabilities is much more challenging than the lower bound. This can be seen easily from the precise links with the metric entropy given in Section 3.2. The upper bound of small ball probabilities gives the lower estimate of the metric entropy and vice versa. The lower estimates for metric entropy are frequently obtained by a volume comparison, i.e. for suitable finite dimensional projections, the total volume of the covering balls is less than the volume of the set being covered. As a result, when the volumes of finite dimensional projections of K_μ do not compare well with the volumes of the same finite dimensional projection of the unit ball of E, sharp lower estimates for metric entropy (upper bounds for small ball probabilities) are much harder to obtain. Some examples are given in Section 7.6.

We start with the following general result. Although it is not as general as Theorem 4.1, it does cover many special cases known so far. Note further that the example given in Section 3.4 shows that L_2-norm entropy is not an appropriate tool for the upper bound.

Theorem 4.4 *Let $\{X_t, t\in[0,1]\}$ be a centered Gaussian process. Then $\forall\ 0<a\le 1/2, \varepsilon>0$*

$$\mathbb{P}\Big(\sup_{0\le t\le 1}|X_t|\le\varepsilon\Big)\le\exp\Big(-\frac{\varepsilon^4}{16a^2\sum_{2\le i,j\le 1/a}(\mathbb{E}\,(\xi_i\xi_j))^2}\Big) \tag{4.6}$$

provided that

$$a \sum_{2\le i\le 1/a} \mathbb{E}\,\xi_i^2 \ge 32\varepsilon^2, \tag{4.7}$$

where $\xi_i = X(ia) - X((i-1)a)$ *or* $\xi_i = X(ia) + X((i-2)a) - 2X((i-1)a)$.

As a consequence of the above result, we have

Theorem 4.5 *Let* $\{X_t, t \in [0,1]\}$ *be a centered Gaussian process with stationary increments and* $X_0 = 0$. *Put*

$$\sigma^2(|t-s|) = \mathbb{E}\,|X_t - X_s|^2, \quad s,t \in [0,1].$$

Assume that there are $1 < c_1 \le c_2 < 2$ *such that*

$$c_1\sigma(h) \le \sigma(2h) \le c_2\sigma(h) \quad \text{for } 0 \le h \le 1/2. \tag{4.8}$$

Then there exists a positive and finite constant K_2 *such that*

$$\forall\ 0 < \varepsilon \le 1,\ \mathbb{P}\left(\sup_{0\le t\le 1} |X_t| \le \sigma(\varepsilon)\right) \le \exp(-K_2/\varepsilon) \tag{4.9}$$

if one of the following conditions is satisfied.

(*i*) σ^2 *is concave on* $(0,1)$;

(*ii*) *There is* $c_0 > 0$ *such that* $(\sigma^2(a))''' \le c_0 a^{-3}\sigma^2(a)$ *for* $0 < a < 1/2$.

When (i) is satisfied, the result is due to Shao [S93]. The original proof is lengthy; a short proof based on Slepian's inequality was given in Kuelbs, Li and Shao [KLS95]. Here we use Theorem 4.4. Let $a = \varepsilon A$, where $A \ge 2$ will be specified later. Without loss of generality, assume $0 < a < 1/4$. Define $\xi_i = X(ia) - X((i-1)a)$. It is easy to see that

$$a \sum_{2\le i\le 1/a} \mathbb{E}\,\xi_i^2 \ge (1-2a)\sigma^2(a) \ge 32\sigma^2(\varepsilon).$$

Noting that $\mathbb{E}\,(\xi_i\xi_j) \le 0$ for $i < j$, we have

$$\sum_{2\le i,j\le 1/a} (\mathbb{E}\,(\xi_i\xi_j))^2 \le \sigma^2(a) \sum_{2\le i,j\le 1/a} |\mathbb{E}\,(\xi_i\xi_j)| \le \sigma^4(a)/a. \tag{4.10}$$

Now (4.9) follows from Theorem 4.4.

When (ii) is satisfied, let $a = \varepsilon A$, where $A \ge 2$, and let $\eta_i = X(ia) + X((i-2)a) - 2X((i-1)a)$. Noting that

$$\mathbb{E}\,(\eta_3\eta_i) = 4\sigma^2((i-2)a) + 4\sigma^2((i-3)a) - 6\sigma^2((i-3)a) - \sigma^2((i-1)a) - \sigma^2((i-5)a)$$

for $i \ge 6$, we have by the Taylor expansion

$$\begin{aligned}\sum_{6\le i\le 1/a} |\mathbb{E}\,(\eta_3\eta_i)|^2 &\le K \sum_{6\le i\le 1/a} (a^3\sigma^2(ia)/(ia)^3)^2 \\ &\le \sum_{6\le i\le 1/a} (i^2\sigma^2(a)/(ia)^3)^2 \\ &\le K\sigma^4(a).\end{aligned}$$

Similarly, one can see that (4.7) is satisfied as long as A is large enough. Hence (4.9) holds, by Theorem 4.4.

We now turn to prove Theorem 4.4 which indeed is a consequence of the following lemma.

Lemma 4.1 *For any centered Gaussian sequence* $\{\xi_i\}$ *and for any* $0 < x < \sum_{i\le n} \mathbb{E}\,\xi_i^2$*, we have*

$$\mathbb{P}\Big(\sum_{i\le n} \xi_i^2 \le x\Big) \le \exp\Big(-\frac{(\sum_{i\le n}\mathbb{E}\,\xi_i^2 - x)^2}{4\sum_{1\le i,j\le n}(\mathbb{E}\,\xi_i\xi_j)^2}\Big). \tag{4.11}$$

The proof of Lemma 4.1 given here is of independent interest. It is easy to see that there exists a sequence of independent mean zero normal random variables η_i such that

$$\sum_{i=1}^n \xi_i^2 = \sum_{i=1}^n \eta_i^2. \tag{4.12}$$

Let

$$\lambda = \frac{\sum_{i\le n}\mathbb{E}\,\xi_i^2 - x}{2\sum_{i\le n}(\mathbb{E}\,\eta_i^2)^2}.$$

Then for any $0 < x < \sum_{i=1}^n \mathbb{E}\,\xi_i^2$

$$\begin{aligned}
\mathbb{P}(\sum_{i=1}^n \xi_i^2 \le x) &= \mathbb{P}(\sum_{i=1}^n \eta_i^2 \le x) \\
&\le e^{\lambda x}\prod_{i\le n}\mathbb{E}\,e^{-\lambda\eta_i^2} \\
&= \exp\Big(\lambda x - \frac{1}{2}\sum_{i\le n}\log(1+2\lambda\mathbb{E}\,\eta_i^2)\Big) \\
&\le \exp\Big(-(\sum_{i\le n}\mathbb{E}\,\xi_i^2 - x)\lambda + \lambda^2\sum_{i\le n}(\mathbb{E}\,\eta_i^2)^2\Big) \\
&= \exp\Big(-\frac{(\sum_{i\le n}\mathbb{E}\,\xi_i^2 - x)^2}{4\sum_{i\le n}(\mathbb{E}\,\eta_i^2)^2}\Big).
\end{aligned}$$

Note further that

$$\sum_{i\le n}(\mathbb{E}\,\eta_i^2)^2 = \frac{1}{2}\mathrm{Var}(\sum_{i=1}^n \eta_i^2) = \frac{1}{2}\mathrm{Var}(\sum_{i=1}^n \xi_i^2) = \sum_{i,j}(\mathbb{E}\,(\xi_i\xi_j))^2$$

for $\mathbb{E}\,\xi_i^2\xi_j^2 = (\mathbb{E}\,\xi_i^2)(\mathbb{E}\,\xi_j^2) + 2(\mathbb{E}\,\xi_i\xi_j)^2$. The lemma follows from the above inequalities.

4.3 Fractional Brownian motions

A centered Gaussian process $X = \{X_t, t \in [0,1]\}$ is called a fractional Brownian motion of order $\alpha \in (0,2)$, denoted by $X \in \mathrm{fBm}_\alpha$, if $X_0 = 0$ and

$$\forall\ 0 \le s,t \le 1,\quad \mathbb{E}\,|X_t - X_s|^2 = |t-s|^\alpha. \tag{4.13}$$

When $\alpha = 1$, it is the ordinary Brownian motion. The name of the fractional Brownian motion was first introduced in Mandelbrot and Van Ness [MN68], but their sample path properties were already studied by Kolmogorov in the 1940's. The study of the fractional Brownian motion was motivated by natural time series in economics, fluctuations in solids, hydrology and more recently by new problems in mathematical finance and telecommunication networks.

It is easy to see that the assumption in Theorem 4.1 and condition (ii) in Theorem 4.5 are satisfied for fractional Brownian motions. Hence we have the following sharp bound of small ball probabilities due to Monrad and Rootzen [MR95] and Shao [S93].

Theorem 4.6 *Let $X \in fBm_\alpha$, $\alpha \in (0,2)$. Then there exist $0 < K_1 \le K_2 < \infty$ depending only on α such that* $\forall\ 0 < \varepsilon \le 1$

$$-K_2\varepsilon^{-2/\alpha} \le \log \mathbb{P}\Big(\sup_{0\le t\le 1} |X_t| \le \varepsilon \Big) \le -K_1 \varepsilon^{-2/\alpha}. \tag{4.14}$$

Other small ball probabilities for fractional Brownian motions under non-uniform norms, such as Hölder norm and Sobolev norm, are discussed in Baldi and Roynette [BR92], Kuelbs and Li [KL93b]. The first result below is given in Kuelbs, Li and Shao [KLS95] and the second in Li and Shao [LS99a] which include the L_p-norm, $p \ge 1$. One may refer to Stolz [St93] for a universal approach to different norms based on Schauder decomposition and a detailed discussion on the approach is given in [L96]. The third result below is given in Stolz [St96]. Related results for increments can be found in Zhang [Zh96b]. In Section 6.3, the existence of the small ball constants for results below is indicated.

Theorem 4.7 *Let $X \in fBm_\alpha$, $\alpha \in (0,2)$ and let $0 \le \beta < \alpha/2$. Then there exist $0 < K_1 \le K_2 < \infty$ depending only on α and β such that* $\forall\ 0 < \varepsilon \le 1$

$$-K_2\varepsilon^{-2/(\alpha-2\beta)} \le \log \mathbb{P}\Big(\sup_{s,t\in[0,1]} \frac{|X_t - X_s|}{|t-s|^\beta} \le \varepsilon \Big) \le -K_1 \varepsilon^{-2/(\alpha-2\beta)}.$$

Theorem 4.8 *Let $X \in fBm_\alpha$, $\alpha \in (0,2)$, and let $p > 0$, $0 \le q < 1 + p\alpha/2$, $q \ne 1$. Then there are $0 < K_1 \le K_2 < \infty$ depending only on α, p and q such that* $\forall\ 0 < \varepsilon \le 1$

$$-K_2\varepsilon^{-\theta} \le \log \mathbb{P}\Big(\int_0^1 \int_0^1 \frac{|X(t) - X(s)|^p}{|t-s|^q}\, dtds \le \varepsilon \Big) \le -K_1 \varepsilon^{-\theta},$$

where $\theta = 1/(\alpha/2 - \max(0, q-1))$.

Theorem 4.9 *Let $X \in fBm_\alpha$, $\alpha \in (0,2)$. For $0 < 1/p < \beta < 1/2$ and $1 < q \le \infty$,*

$$\log \mathbb{P}\left(\|X_t\|_{\beta,p,q} \le \varepsilon \right) \approx -\varepsilon^{-2/(\alpha-2\beta)}$$

where the Besov norm

$$\|f\|_{\beta,p,q} = \|f\|_p + \left(\int_0^1 \left(\frac{\omega_p(t,f)}{t^\beta} \right)^q \frac{dt}{t} \right)^{1/q}$$

with

$$\omega_p(t,f) = \sup_{|h|\le t} \left(\int_{I_h} |f(x-h) - f(x)|^p dx \right)^{1/p}$$

and $I_h = \{x \in [0,1] \,:\, x - h \in [0,1]\}$.

4.4 Integrated fractional Brownian motions

Consider the integrated fractional Brownian motion

$$Y(t) = \int_0^t X_u du,$$

where X is the fractional Brownian motion of order $\alpha \in (0,2)$. The following result is a very special case of what is given in [LL99]; see the remarks at the end of Section 3.2. In fact, the so called small ball constant exists; see Section 6.3 for details. But the proofs given in [LL99] are not probabilistic, in particular the lower bound. Here we give a direct pure probabilistic proof based on Theorems 4.3 and 4.4.

Theorem 4.10 *There exist* $0 < K_1 \le K_2 < \infty$ *depending only on* α *such that* $\forall\ 0 < \varepsilon \le 1$

$$-K_2\varepsilon^{-2/(2+\alpha)} \le \log \mathbb{P}\Big(\sup_{t\in[0,1]} |Y(t)| \le \varepsilon\Big) \le -K_1\varepsilon^{-2/(2+\alpha)}. \tag{4.15}$$

We only provide an outline of the proof. For $0 < h < 1/4$ and $h < s \le t \le 1-h$, we can write

$$Y(t+h) + Y(t-h) - 2Y(t) = \int_0^h (X(t+u) - X(t-u))du$$

and

$$Y(s+h) + Y(s-h) - 2Y(s) = \int_0^h (X(s+v) - X(s-v))dv.$$

Note that

$$\begin{aligned}
&\Big|\mathbb{E}\Big((Y(t+h) + Y(t-h) - 2Y(t))(Y(s+h) + Y(s-h) - 2Y(s))\Big)\Big| \\
=\ & \Big|\int_0^h \int_0^h \mathbb{E}\Big(X(t+u) - X(t-u))(X(s+v) - X(s-v))\Big)dudv\Big| \\
=\ & \frac{1}{2}\Big|\int_0^h \int_0^h \Big(|t-s+u-v|^\alpha + |t-s-u+v|^\alpha - |t-s-u-v|^\alpha - |t-s+u+v|^\alpha\Big)dudv\Big| \\
\approx\ & (|t-s|+h)^{\alpha-2}h^4.
\end{aligned}$$

Hence, applying Theorems 4.3 and 4.4 yields the result.

5 Gaussian processes with index set $T \subset \mathbb{R}^d$, $d \ge 2$

There are two versions of extension of the fractional Brownian motion in d-dimensional space. One is the so called Lévy fractional Brownian motion of order $\alpha \in (0,2)$ defined by

$$X_0 = 0,\ \mathbb{E}\,X_t = 0,\ \mathbb{E}\,(X_t - X_s)^2 = |t-s|^\alpha \ \text{ for }\ s,t \in [0,1]^d.$$

The other is the so called fractional Brownian sheet of order $\alpha \in (0,2)$ if the covariance satisfies

$$\forall\ \ s,t \in [0,1]^d,\ \ \mathbb{E}\,(X_tX_s) = \prod_{j=1}^d \frac{1}{2}(s_j^\alpha + t_j^\alpha - |s_j - t_j|^\alpha).$$

The classical Brownian sheet corresponds to $\alpha = 1$.

5.1 Lévy's fractional Brownian motions

Following Shao and Wang [SWa95], and Talagrand [T93, T95] we have sharp bounds of small ball probability for Lévy fractional Brownian motions.

Theorem 5.1 *Let* $\{X_t, t \in [0,1]^d\}$ *be a Lévy fractional Brownian motion of order* $\alpha \in (0,2)$*. Then there exist* $0 < K_1 \le K_2$ *depending only on* α *and* d *such that* $\forall\ 0 < \varepsilon \le 1$

$$-K_2\varepsilon^{-2d/\alpha} \le \log \mathbb{P}\Big(\sup_{t\in[0,1]^d} |X_t| \le \varepsilon\Big) \le -K_1\varepsilon^{-2d/\alpha}. \tag{5.1}$$

Here is a proof for the upper bound. For $i = (i_1, \ldots, i_d)$, write $t_i = i\varepsilon^{2/\alpha}$. Clearly,

$$\mathbb{P}\left(\sup_{t\in[0,1]^d} |X(t)| \le \varepsilon\right) \le \mathbb{P}\left(\max_{1\le i\le \varepsilon^{-2/\alpha}} |X(t_i)| \le \varepsilon\right)$$

and for $1 \le j \le \varepsilon^{-2/\alpha}$

$$\begin{aligned} &\mathbb{P}\left(\max_{1\le i\le \varepsilon^{-2/\alpha}} |X(t_i)| \le \varepsilon\right) \\ &= \mathbb{E}\left(I\{\max_{1\le i\le \varepsilon^{-2/\alpha}, i\ne j} |X(t_i)| \le \varepsilon\}\mathbb{P}\Big(|X(t_j)| \le \varepsilon \mid X(t_i), 1 \le i \le \varepsilon^{-2/\alpha}, i \ne j\Big)\right). \end{aligned}$$

In terms of Lemma 7.1 of Pitt [P78], there is a positive constant $C = C(\alpha, d)$ such that

$$\mathrm{Var}(X(t_j) \mid X(t_i), 1 \le i \le \varepsilon^{-2/\alpha}, i \ne j) \ge \mathrm{Var}(X(t_j) \mid X(s) : |s - t_i| \ge \varepsilon^{2/\alpha}) = C\,\varepsilon^2.$$

Hence,

$$\mathbb{P}\left(|X(t_j)| \le \varepsilon \mid X(t_i), 1 \le i \le \varepsilon^{-2/\gamma}, i \ne j\right) \le \mathbb{P}\left(|\xi| \le 1/\sqrt{C}\right) < 1,$$

where ξ is the standard normal random variable, and

$$\mathbb{P}\left(\max_{1\le i\le \varepsilon^{-2/\alpha}} |X(t_i)| \le \varepsilon\right) \le \mathbb{P}\left(|\xi| \le 1/\sqrt{C}\right) \mathbb{P}\left(\max_{1\le i\le \varepsilon^{-2/\alpha}, i\ne j} |X(t_i)| \le \varepsilon\right).$$

We thus have the upper bound by recurrence.

The lower bound is a consequence of the following general result which can be derived from Theorem 3.8 (see also [SWa95]).

Theorem 5.2 *Let $\{X_t, t \in [0,1]^d\}$ be a centered Gaussian process with $X_0 = 0$. Assume that*

$$\forall\ s, t \in [0,1]^d, \quad \mathbb{E}\,|X_t - X_s|^2 \le \sigma^2(|t - s|) \tag{5.2}$$

and that there are $0 < c_1 \le c_2 < 1$ such that $c_1\sigma(2h \wedge 1) \le \sigma(h) \le c_2\sigma(2h \wedge 1)$ for $0 \le h \le 1$. Then, there is a positive and finite constant K_1 depending only on c_1 and c_2 such that for all $\varepsilon > 0$

$$\log \mathbb{P}\Big(\sup_{t\in[0,1]^d} |X(t)| \le \sigma(\varepsilon)\Big) \ge -K_1/\varepsilon^d\ . \tag{5.3}$$

An upper bound for a general class of stationary Gaussian processes was given by Tsyrelson (see Lifshits and Tsyrelson [LTs86]) in terms of spectral density. In particular, if X is a homogeneous process on $\mathbb{R}^d$ with spectral density f satisfying $f(u) \succeq |u|^{-d-\alpha}$ as $u \to \infty$, then

$$\log \mathbb{P}\Big(\sup_{[0,1]^d} |X_t| \le \varepsilon\Big) \preceq -K\varepsilon^{-2d/\alpha}\ .$$

5.2 Brownian sheets

We first state a precise result of Csáki [C82] on the small ball probability of Brownian sheet under L_2 norm.

Theorem 5.3 *Let $\{X_t, t \in [0,1]^d\}$ be the Brownian sheet. Then*

$$\log \mathbb{P}\Big((\int_{t\in[0,1]^d} |X_t|^2 dt)^{1/2} \le \varepsilon \Big) \sim -c_d \varepsilon^{-2} |\log(\varepsilon)|^{2d-2}, \tag{5.4}$$

where $c_d = 2^{d-2}/(\sqrt{2}\pi^{d-1}(d-1)!)$.

Various non–Brownian multiparameter generalizations of the above result are given in Li [Li92a]. Next, we consider the case $d = 2$.

Theorem 5.4 *Let $\{X_t,\ t \in [0,1]^2\}$ be a Brownian sheet. Then there exist $0 < K_1 \le K_2 < \infty$ such that $\forall\ 0 < \varepsilon \le 1$*

$$-K_2 \varepsilon^{-2} \log^3(1/\varepsilon) \le \log \mathbb{P}\Big(\sup_{t\in[0,1]^2} |X_t| \le \varepsilon \Big) \le -K_1 \varepsilon^{-2} \log^3(1/\varepsilon). \tag{5.5}$$

The lower bound is due to Lifshits (see [LTs86]) and Bass [Ba88], and the upper bound to Talagrand [T94]. A simplified upper bound proof can be found in Dunker [D98]. The following small ball probabilities under the mixed sup-L_2 norm and L_2-sup norm may be of some interest.

Theorem 5.5 *Let $\{X_t,\ t \in [0,1]^2\}$ be a Brownian sheet. Then there exist $0 < K_1 \le K_2 < \infty$ such that $\forall\ 0 < \varepsilon \le 1$*

$$-K_2 \varepsilon^{-1} \log^2(1/\varepsilon) \le \log \mathbb{P}\Big(\sup_{t_1\in[0,1]} \int_0^1 |X(t_1,t_2)|^2 dt_2 \le \varepsilon \Big) \le -K_1 \varepsilon^{-1} \log^2(1/\varepsilon) \tag{5.6}$$

and

$$-K_2 \varepsilon^{-1} \log^3(1/\varepsilon) \le \log \mathbb{P}\Big(\int_0^1 \sup_{t_1\in[0,1]} |X(t_1,t_2)|^2 dt_2 \le \varepsilon \Big) \le -K_1 \varepsilon^{-1} \log^3(1/\varepsilon). \tag{5.7}$$

The upper bound of (5.6) follows from (5.4), and the lower bound is given in Horváth and Shao [HS99]. The lower bound of (5.7) is from (5.5) and the upper bound can be shown with modification of the arguments used in the proof of the upper bound of (5.5) given in [T94].

For $d \ge 3$, the situation becomes much more difficult as the combinatorial arguments used for $d = 2$ fail and there is still a gap between the upper and lower bounds.

Theorem 5.6 *Let $d \ge 3$, and $\{X_t, t \in [0,1]^d\}$ be the Brownian sheet. Then there exist $0 < K_1, K_2 < \infty$ such that $\forall\ 0 < \varepsilon \le 1$*

$$-K_2 \varepsilon^{-2} \log^{2d-1}(1/\varepsilon) \le \log \mathbb{P}\Big(\sup_{t\in[0,1]^d} |X_t| \le \varepsilon \Big) \le -K_1 \varepsilon^{-2} \log^{(2d-2)}(1/\varepsilon). \tag{5.8}$$

The upper bound above follows from (5.4), and the lower bound was recently proved by Dunker, Kuhn, Lifshits, and Linde [D-98] (a slightly weaker lower bound is given in Belinskii [Bel98]). It should be pointed out that the proofs of the lower bound in [D-98] and [Bel98] are based on approximation theory and part (IV) of Theorem 3.3, and hence are not probabilistic. Another way to obtain the lower bound in (5.4) is to use the estimates on the l-approximation numbers $l_n(X)$ (a probabilistic concept) given in (3.22) and part (a) of Theorem 3.9. But both proofs of (3.22) on $l_n(X)$ and part (a) of Theorem 3.9 use various approximation concepts and hence are also not probabilistic. It would be interesting to find a pure probabilistic proof of the lower bound in (5.4). The only known probabilistic proof for the lower bound with $d \ge 2$ is presented in Bass [Ba88] which gives $3d-3$ for the power of the log-term.

Similar to Theorem 5.6, Dunker [D99] obtained the following upper and lower bounds for the fractional Brownian sheet using methods detailed above.

Theorem 5.7 *Let $\{X_t, t \in [0,1]^d\}$ be the fractional Brownian sheet of order $\alpha \in (0,2)$. Then there exist $0 < K_1, K_2 < \infty$ such that $\forall\ 0 < \varepsilon \le 1$*

$$-K_2\varepsilon^{-2/\alpha}\log^{(1+\alpha)d/\alpha-1}(1/\varepsilon) \le \log \mathbb{P}\Big(\sup_{t\in[0,1]^d}|X_t| \le \varepsilon\Big) \le -K_1\varepsilon^{-2/\alpha}\log^{(1+\alpha)d/\alpha-2}(1/\varepsilon). \tag{5.9}$$

6 The small ball constants

So far we have been mainly interested in the asymptotic order (up to a constant factor) of the small ball rate function $\phi(\varepsilon)$ given in (3.5). In this section, we will present results in which the exact constants are known or known to exist, and we call them small ball constants. Keep in mind that results of this type (even just the existence) play a more important role in applications of small ball estimates as can be seen in Section 7. In the Hilbert space l_2, the full asymptotic formula is known. And with the help of a comparison result, most small ball probabilities under the L_2-norm can thus be treated at least in principle, and in particular when the Karhunen-Loeve expansion for a given Gaussian process can be found in some reasonable form. This is the case for Brownian motion and Brownian sheets, etc. Other exact values of small ball constants are known only with a pure analytic representation. It is no surprise that most of them are related to Brownian motion in one way or another. The most elusive small ball constants are those shown to exist and they may even connect with each other. It is challenging to show the existence and to find those unknown but existing ones at least in terms of a pure analytic representation. We only present some basic tools and results here.

Throughout this section, we use

$$\|f\|_p = \begin{cases} (\int_0^1 |f(t)|^p dt)^{1/p} & \text{for} \quad 1 \le p < \infty \\ \\ \sup_{0\le t\le 1}|f(t)| & \text{for} \quad p = \infty \end{cases}$$

to denote the L_p-norm on $C[0,1]$, $1 \le p \le \infty$.

6.1 Exact estimates in Hilbert space

Consider a continuous Gaussian process $\{X(t) : a \le t \le b\}$ with mean zero and covariance function $\sigma(s,t) = \mathbb{E}\, X(s)X(t)$ for $s,t \in [a,b]$. We are interested in the exact asymptotic behaviour of

$$\mathbb{P}\left(\int_a^b X^2(t)dt \le \varepsilon^2\right)$$

as $\varepsilon \to 0$. By the well known Karhunen-Loeve expansion, we have in distribution

$$\int_a^b X^2(t)dt = \sum_{n\geq 1} \lambda_n \xi_n^2,$$

where $\lambda_n > 0$ for $n \geq 1$, $\sum_{n\geq 1} \lambda_n < \infty$, are the eigenvalues of the equation

$$\lambda f(t) = \int_a^b \sigma(s,t) f(s) ds, \quad a \leq t \leq b.$$

Thus the problem reduces to finding the asymptotic behavior of

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2\right)$$

as $\varepsilon \to 0$, where $\{\xi_n\}$ are i.i.d. $N(0,1)$ random variables. Theoretically, the problem has been solved by Sytaya [Sy74]. Namely

Theorem 6.1 *If $\lambda_n > 0$ and $\sum_{n=1}^{\infty} \lambda_n < +\infty$, then as $\varepsilon \to 0$*

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2\right) \sim \left(4\pi \sum_{n=1}^{\infty} \left(\frac{\lambda_n \gamma_\lambda}{1 + 2\lambda_n \gamma_\lambda}\right)^2\right)^{-1/2} \cdot \exp\left(\varepsilon^2 \gamma_\lambda - \frac{1}{2} \sum_{n=1}^{\infty} \log(1 + 2\lambda_n \gamma_\lambda)\right),$$

where $\gamma_\lambda = \gamma_\lambda(\varepsilon)$ is uniquely determined, for $\varepsilon > 0$ small enough, by the equation

$$\varepsilon^2 = \sum_{n=1}^{\infty} \frac{\lambda_n}{1 + 2\lambda_n \gamma_\lambda}.$$

Note that the given asymptotic behaviour is still an implicit expression that is highly inconvenient for concrete computations and applications. This is primarily due to the series form for the asymptotic and the implicit relation between ε and γ_λ in Theorem 6.1. A number of papers, Dudley, Hoffmann–Jørgensen and Shepp [DHS79], Ibragimov [I82], Zolotarev [Z86], Dembo, Mayer-Wolf and Zeitouni [DMZ95], Dunker, Lifshits, and Linde [DLL98], have been devoted to finding the asymptotic behaviour of $\mathbb{P}\left(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2\right)$ as $\varepsilon \to 0$, or sharp estimates at the log level for some particular λ_n, after the work of Sytaya because of the difficulties in applying Theorem 6.1. Most of the results of these papers involve difficult calculations that most often depend very much on special properties of the sequence λ_n. Nevertheless, the problem is considered solved completely when eigenvalues λ_n can be found explicitly.

When eigenvalues λ_n can not be found explicitly, the following comparison principle given by Li [Li92a] provides a very useful computational tool.

Theorem 6.2 *If $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$, then as $\varepsilon \to 0$*

$$\mathbb{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leq \varepsilon^2\right) \sim \left(\prod_{n=1}^{\infty} b_n/a_n\right)^{1/2} \mathbb{P}\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \leq \varepsilon^2\right),$$

where a_n, b_n are positive and $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} b_n < \infty$. Furthermore, if $a_n \geq b_n$ for n large, then $\mathbb{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leq \varepsilon^2\right)$ and $\mathbb{P}\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \leq \varepsilon^2\right)$ have the same order of magnitude as $\varepsilon \to 0$ if and only if $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$.

The following simple example demonstrates a way of using the comparison theorem, and more examples can be found in Li [Li92a, Li92b]. Let $\{B(t) \ : \ 0 \le t \le 1\}$ be the Brownian bridge and consider weighted L_2-norms for $B(t)$.

Proposition 6.1 *For $\alpha > 0$ and $\beta = 1 - \alpha^{-1} < 1$,*

$$\begin{aligned} & \mathbb{P}\Big(\int_0^1 B^2(t^\alpha)dt \le \varepsilon^2\Big) \quad \Big(= \mathbb{P}\Big(\int_0^1 \frac{1}{t^\beta} B^2(t)dt \le \alpha\varepsilon^2\Big)\Big) \\ \sim \quad & c_\alpha \varepsilon^{-\frac{\alpha-1}{2(\alpha+1)}} \exp\Big(-\frac{\alpha}{2(\alpha+1)^2}\cdot\frac{1}{\varepsilon^2}\Big) \quad as \quad \varepsilon \to 0, \end{aligned}$$

where c_α is a positive constant.

To see this, note that by using the Karhunen-Loeve expansion, the eigenvalues are solutions of

$$J_{\alpha/\alpha+1}\Big(2(\alpha+1)^{-1}\sqrt{\alpha/\lambda}\Big) = 0$$

where $J_\nu(x)$ is the Bessel function. Hence by the asymptotic formula for zeros of the Bessel function, we have

$$\frac{2}{\alpha+1}\sqrt{\frac{\alpha}{\lambda_n}} = \Big(n + \frac{\alpha-1}{4(\alpha+1)}\Big)\pi + O\Big(\frac{1}{n}\Big)$$

which shows that

$$\sum_{n=1}^\infty \left| \frac{4\alpha}{(\alpha+1)^2\pi^2}\Big(n + \frac{\alpha-1}{4(\alpha+1)}\Big)^{-2} \cdot \frac{1}{\lambda_n} - 1 \right| < \infty.$$

Thus by Theorem 6.2 and Theorem 6.1, we obtain as $\varepsilon \to 0$

$$\begin{aligned} \mathbb{P}\Big(\int_0^1 B^2(t^\alpha)dt \le \varepsilon^2\Big) &= \mathbb{P}\Big(\sum_{n=1}^\infty \lambda_n \xi_n^2 \le \varepsilon^2\Big) \\ &\sim D_\alpha \mathbb{P}\left(\sum_{n=1}^\infty \frac{4\alpha}{(\alpha+1)^2\pi^2}\cdot\Big(n + \frac{\alpha-1}{4(\alpha+1)}\Big)^{-2} \xi_n^2 \le \varepsilon^2\right) \\ &\sim c_\alpha \varepsilon^{-\frac{\alpha-1}{2(\alpha+1)}} \exp\Big(-\frac{\alpha}{2(\alpha+1)^2}\cdot\frac{1}{\varepsilon^2}\Big). \end{aligned}$$

Next we mention that (see [Li92a]) for any positive integer N,

$$\log \mathbb{P}\left(\sum_{n=1}^\infty a_n\xi_n^2 \le \varepsilon^2\right) \sim \log \mathbb{P}\left(\sum_{n \ge N} a_n \xi_n^2 \le \varepsilon^2\right) \quad \text{as} \quad \varepsilon \to 0,$$

which shows that the small ball rate function will not change at the logarithmic level if we delete a finite number of the terms.

Finally, we mention that for any (even moving) shifts and any (going to zero or infinity) radius in this l_2 setting, the exact asymptotic behaviours similar to Theorem 6.1 are studied in Li and Linde [LL93] and Kuelbs, Li and Linde [KLL94].

6.2 Exact value of small ball constants

Let $\{W(t); 0 \le t \le 1\}$ be the standard Brownian motion and $\{B(t); 0 \le t \le 1\}$ be a standard Brownian bridge, which can be realized as $\{W(t) - tW(1); 0 \le t \le 1\}$. First we present the exact value of small ball constants for $W(t)$ and $B(t)$ under the L_p-norm, $1 \le p \le \infty$. Its generalization and extension to other related processes are given in Theorem 6.4.

Theorem 6.3 *For any* $1 \le p \le \infty$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\|W(t)\|_p \le \varepsilon \right) = \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\|B(t)\|_p \le \varepsilon \right) = -\kappa_p, \tag{6.1}$$

where

$$\kappa_p = 2^{2/p} p \left(\lambda_1(p)/(2+p)\right)^{(2+p)/p} \tag{6.2}$$

and

$$\lambda_1(p) = \inf \left\{ \int_{-\infty}^{\infty} |x|^p \phi^2(x) dx + \frac{1}{2} \int_{-\infty}^{\infty} \left(\phi'(x)\right)^2 dx \right\} > 0, \tag{6.3}$$

the infimum takes over all $\phi \in L_2(-\infty, \infty)$ *such that* $\int_{-\infty}^{\infty} \phi^2(x) dx = 1$.

The cases $p = 2$ and $p = \infty$ with $\kappa_2 = 1/8$ and $\kappa_\infty = \pi^2/8$ are also well known, and the exact distributions in terms of infinite series are known; see Smirnov [Sm37], Chung [Ch48] and Doob [Do49]. The only other case, for which the exact distribution is given in terms of Laplace transform, is in Kac [K46] for $p = 1$. Namely, for $\lambda \ge 0$

$$\mathbb{E} \exp\left\{ -\lambda \int_0^1 |W(s)| ds \right\} = \sum_{j=1}^{\infty} \theta_j \exp\{-\delta_j \lambda^{2/3}\} \tag{6.4}$$

where $\delta_1, \delta_2, \cdots$ are the positive roots of the derivative of

$$P(y) = 3^{-1}(2y)^{1/2} \left(J_{-1/3}(3^{-1}(2y)^{3/2}) + J_{1/3}(3^{-1}(2y)^{3/2}) \right),$$

$J_\alpha(x)$ are the Bessel functions of parameter α, and $\theta_j = (3\delta_j)^{-1}(1 + 3\int_0^{\delta_j} P(y)dy)$. The extension of (6.4) to values of $\lambda < 0$ remains open as far as we know. By using the exponential Tauberian theorem given as Theorem 3.5, we have from (6.4), $\kappa_1 = (4/27)\delta_1^3$ where δ_1 is the smallest positive root of the derivative of $P(y)$.

Now from asymptotic point of view for the Laplace transform, it was shown in Kac [K51] using the Feynman-Kac formula and the eigenfunction expansion that

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ -\int_0^t |W(s)|^p ds \right\} = -\lambda_1(p) \tag{6.5}$$

and $\lambda_1(p)$ is the smallest eigenvalue of the operator

$$Af = -\frac{1}{2} f''(x) + |x|^p f(x) \tag{6.6}$$

on $L_2(-\infty, \infty)$. Thus from (6.6) and the classical variation expression for eigenvalues, we obtain (6.3). A different and extremely powerful approach was given in Donsker and Varadhan [DV75a] so that the direct relation between (6.5) and (6.3)

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ -\int_0^t |W(s)|^p ds \right\} = -\inf \left\{ \int_{-\infty}^{\infty} |x|^p \phi^2(x) dx + \frac{1}{2} \int_{-\infty}^{\infty} \left(\phi'(x)\right)^2 dx \right\} \tag{6.7}$$

holds as a very special case of their general theory on occupation measures for Markov processes. Both approaches work for more general functions $V(x)$ than the ones we used here with $V(x) = |x|^p$, $1 \le p < \infty$, and thus the statement for W in Theorem 6.3 also holds for $0 < p < 1$.

On the other hand, from the small ball probability or small deviation point of view, Borovkov and Mogulskii [BM91] obtained

$$\mathbb{P}\left(\|W\|_p \le \varepsilon\right) \sim c_1(p)\varepsilon \exp\{-\lambda_1(p)\varepsilon^{-2}\}$$

by using a similar method to that of Kac [K51], but more detailed analysis for the polynomial term. Unfortunately, they did not realize that the variation expression (6.3) for $\lambda_1(p)$ and the polynomial factor ε is missing in their original statement due to an algebraic error.

Theorem 6.3 is first formulated explicitly this way as a lemma in Li [Li99c]. And for the Brownian motion part, it follows from (6.5) or (6.7), which is by Brownian scaling

$$\lim_{\lambda\to\infty} \lambda^{-2/(2+p)} \log \mathbb{E} \exp\left\{-\lambda \int_0^1 |W(s)|^p ds\right\} = -\lambda_1(p),$$

and the exponential Tauberian theorem given as Theorem 3.5 with $\alpha = 2/p$. The result for the Brownian Bridge follows from Theorem 3.7; see [Li99c] for a traditional argument.

Next we mention the following far reaching generalization of the basic Theorem 6.3.

Theorem 6.4 *Let $\rho : [0,\infty) \to [0,\infty]$ be a Lebesgue measurable function satisfying the following conditions:*

(i) $\rho(t)$ is bounded or non-increasing on $[0,a]$ for some $a > 0$;

(ii) $\rho(t)\cdot t^{(2+p)/p}$ is bounded or non-decreasing on $[T,\infty)$ for some T with $a < T < \infty$;

(iii) $\rho(t)$ is bounded on $[a,T]$ and $\rho(t)^{2p/(2+p)}$ is Riemann integrable on $[0,\infty)$.

Then for $1 \le p \le \infty$

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\left(\int_0^\infty |\rho(t)W(t)|^p dt\right)^{1/p} \le \varepsilon\right) = -\kappa_p \left(\int_0^\infty \rho(t)^{2p/(2+p)} dt\right)^{(2+p)/p} \tag{6.8}$$

where κ_p is given in (6.2).

Before we give some interesting examples, some brief history and remarks are needed. In the case of the sup-norm ($p = \infty$) over a finite interval $[0,T]$, similar results were given in Mogulskii [M74] under the condition $\rho(t)$ is bounded, in Berthet and Shi [BS98] under the condition that $\rho(t)$ is nonincreasing, and in Li [Li99b] under the critical case that $\int_0^T \rho^2(t)dt = \infty$. In the case of the sup-norm ($p = \infty$) over an infinite interval $[0,\infty)$, the results were treated in Li [Li99a] as an application of Theorem 2.14. The proof of Theorem 6.4 is given in Li [Li99c] together with connections to Gaussian Markov processes. For related results and associations to Volterra operators, see Lifshits and Linde [LifL99]. Below we present some interesting examples of Theorem 6.4.

Example 1. Consider $X_1(t) = t^{-\alpha}W(t)$ on the interval $[0,1]$ for $\alpha < (2+p)/2p$, $p \ge 1$. Then Theorem 6.4 together with a simple calculation implies

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^1 |t^{-\alpha}W(t)|^p dt \le \varepsilon^p\right) = -\kappa_p \left(\frac{2+p}{2+p-2\alpha p}\right)^{(2+p)/p}. \tag{6.9}$$

In Section 7.10 we will see the implication of this result in terms of the asymptotic Laplace transform.
Example 2. Let $U(t)$ be the stationary Gaussian Markov process or the Ornstein–Uhlenbeck process with $\mathbb{E}\,U(s)U(t) = \sigma^2 e^{-\theta|t-s|}$ for $\theta > 0$ and any $s, t \in [a, b]$, $-\infty < a < b < \infty$. Then we have for $1 \le p \le \infty$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\|U(t)\|_p \le \varepsilon \right) = -2\sigma^2 \theta (b-a)^{(2+p)/p} \kappa_p.$$

In the case $p = 2$, the above result and its refinement are given in Li [Li92a] by using the Karhunen-Loeve expansion and the comparison Theorem 6.2. Other interesting examples that are a consequence of Theorem 6.4 can be found in [Li99c].

Next we mention the corresponding results in higher dimensions under the sup norm. Let $\{W^d(t);\ t \ge 0\}$ be a standard d–dimensional Brownian motion and $\{B^d(t);\ 0 \le t \le 1\}$ be a standard d–dimensional Brownian bridge, $d \ge 1$. We use the convention $1/\infty = 0$ and denote by "$\|\cdot\|_{(d)}$" the usual Euclidean norms in $\mathbb{R}^d$.

Theorem 6.5 *Let $g : (0, \infty) \mapsto (0, \infty]$ satisfy the conditions:*

(i) $\inf_{0<t<\infty} g(t) > 0$ *or $g(t)$ is nondecreasing in a neighborhood of* 0.

(ii) $\inf_{0<t<\infty} t^{-1} g(t) > 0$ *or $t^{-1}g(t)$ is nonincreasing for t sufficiently large;*

Then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t<\infty} \frac{\|W^d(t)\|_{(d)}}{g(t)} \le \varepsilon \right) = -\frac{j^2_{(d-2)/2}}{2} \int_0^\infty g^{-2}(t)\,dt,$$

where $j_{(d-2)/2}$ is the smallest positive root of the Bessel function $J_{(d-2)/2}$ and $j_{-1/2} = \pi/2$.

Theorem 6.6 *Assume that* $\inf_{a \le t \le b} g(t) > 0$ *for all* $0 < a \le b < 1$. *If* $\inf_{0 \le t \le 1} g(t) > 0$ *or both $(1-t)^{-1}g(t)$ and $(1-t)^{-1}g(1-t)$ are nondecreasing in a neighborhood of* 0, *then*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t<1} \frac{\|B^d(t)\|_{(d)}}{g(t)} \le \varepsilon \right) = -\frac{j^2_{(d-2)/2}}{2} \int_0^\infty g^{-2}(t)\,dt,$$

where $j_{(d-2)/2}$ is defined in Theorem 6.5.

Both results above, which extend earlier work of Mogulskii [M74] and Berthet and Shi (1998) for the sup-norm over a finite interval, are given in Li [Li99a] as applications of Theorem 2.14. For some applications of Theorem 6.6 to weighted empirical processes, we refer to Csáki [C94].

6.3 Existence of small ball constants

As we have seen in the previous section, there are relatively few cases where the exact small ball constants can be given explicitly or represented analytically. A natural question, the next best thing we can hope for, is to show the existence of the small ball constants. As we all know, proving the existence or finding the exact value of various constants plays an important part in the history of mathematics. The most fruitful benefits are the methods developed along the way to obtain the existence or the exact value of an interesting constant, the heart of the matter in many problems. Another benefit, as we can see from (6.10) below, is that related constants can be represented in terms of a few basic unknown, but proven to exist, constants. For the small ball constants, they play important roles in problems such as the integral test for lower limits (see Talagrand [T96]), and various functional LIL results; see de Acosta [dA83], Kuelbs, Li and Talagrand [KLT94], Kuelbs and Li [KL00].

We start with the work of de Acosta [dA83] on the existence of the small ball constants for finite dimensional vector-valued Brownian motion under the sup-type norm. To be more precise, let E be a finite dimensional Banach space with norm $Q(\cdot)$ and let μ be a centered Gaussian measure on E. Let $\{W^E(t) : t \geq 0\}$ be an E-valued μ-Brownian motion; that is, $\{W^E(t) : t \geq 0\}$ is an E-valued stochastic process with stationary independent increments, $W^E(0) = 0$, W^E has continuous paths and $\mathcal{L}(W^E(1)) = \mu$.

Theorem 6.7 *The limit*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0\leq t\leq 1} Q(W^E) \leq \varepsilon\right) = -c_{\mu,Q}$$

exists and $0 < c_{\mu,Q} < \infty$.

Note that in the case μ is the canonical Gaussian measure on $E = \mathbb{R}^d$, $d \geq 1$, $c_{\mu,Q} = j^2_{(d-2)/2}/2$ in Theorem 6.5 when $Q(\cdot) = \|\cdot\|_{(d)}$ is the usual Euclidean norm, and $c_{\mu,Q} = d \cdot j^2_{-1/2}/2 = d(\pi^2/8)$ when $Q(x) = \max_{1\leq i\leq d} |x_i|$ for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. The method of proof is formulated as a scaling argument (and same as the well known subadditive argument in this problem) with upper bound on the probability.

Next we mention the result for the standard Brownian motion on $\mathbb{R}$ under the Hölder norms given in Kuelbs and Li [KL93b].

Theorem 6.8 *For any* $0 < \beta < 1/2$ *the limit*

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(1-2\beta)} \log \mathbb{P}\left(\sup_{0\leq s,t\leq 1} \frac{|W(t) - W(s)|}{|t-s|^\beta} \leq \varepsilon\right) = -c_\beta$$

exists and $0 < c_\beta < \infty$.

Note that no value of c_β is known, though reasonable upper and lower bounds are given in Kuelbs and Li [KL93b]. The existence part of the proof is similar to the one used in de Acosta [dA83].

In Khoshnevisan and Shi [KS98a], the existence of integrated Brownian motion is shown by using the subadditive argument with upper bound on the probability. Related existence results for fractional integrated fractional Brownian motion are given in Li and Linde [LL98, LL99] under the sup-norm.

For the remainder of this section, we focus on the existence of the small ball constants for the fractional Brownian motion $B_\alpha(t)$ under the sup-norm, due independently to Li and Linde [LL98] and Shao [S99]. The definition of fractional Brownian motion can be found in Section 4.4. The following statement from Li and Linde [LL98] also provides the relation with the constant for the self-similar Gaussian process $W_\beta(t)$, $\beta > 0$ given in (6.12).

Theorem 6.9 *We have*

$$\lim_{\varepsilon\to 0} \varepsilon^{2/\alpha} \log \mathbb{P}\left(\sup_{0\leq t\leq 1} |B_\alpha(t)| \leq \varepsilon\right) = \lim_{\varepsilon\to 0} \varepsilon^{2/\alpha} \log \mathbb{P}\left(\sup_{0\leq t\leq 1} |W_\alpha(t)| \leq \sqrt{a_\alpha} \cdot \varepsilon\right) = -C_\alpha, \tag{6.10}$$

where $0 < C_\alpha < \infty$,

$$a_\alpha = \alpha^{-1} + \int_{-\infty}^0 ((1-s)^{(\alpha-1)/2} - (-s)^{(\alpha-1)/2})^2 ds \tag{6.11}$$

and

$$W_\alpha(t) = \int_0^t (t-s)^{(\alpha-1)/2} dW(s). \tag{6.12}$$

It was proved in Shao [S96] that

$$\Big(\frac{.08}{\sqrt{\alpha}}\Big)^{2/\alpha} < C_\alpha < \Big(\frac{10}{\sqrt{\alpha}}\Big)^{2/\alpha} \quad \text{for } 0 < \alpha < 1.$$

The existence of the constant in (6.10) was explicitly asked in Talagrand [T96] in connection with the integral test established in that paper. The constants C_α in (6.10) also play the role of the principle eigenvalues of certain operators in the proper domain. Note that (6.10) can be rewritten as

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}\,(\tau > t) = -C_\alpha, \tag{6.13}$$

where the exit time $\tau = \inf\{t : B_\alpha(t) \notin [-1,1]\}$. In the Brownian motion case, $\alpha = 1$, the constant in (6.13) is principle eigenvalue of the Laplacian on the domain $[-1,1]$. For other results related to (6.10), see Theorem 6.10 and (7.36).

Next we discuss three and a half different proofs available for the existence of the small ball constants for $B_\alpha(t)$ under the sup-norm.

The proof given in Li and Linde [LL98] is based on the following useful representation when $\alpha \neq 1$ (see [MN68]),

$$B_\alpha(t) = \sqrt{a_\alpha}\,(W_\alpha(t) + Z_\alpha(t))\,, \quad 0 \le t \le 1, \tag{6.14}$$

where a_α is given in (6.11), $W_\alpha(t)$ is given in (6.12) and

$$Z_\alpha(t) = \int_{-\infty}^{0} \{(t-s)^{(\alpha-1)/2} - (-s)^{(\alpha-1)/2}\} dW(s).$$

Furthermore, $W_\alpha(t)$ is independent of $Z_\alpha(t)$. Observe that the centered Gaussian process $W_\beta(t)$ is defined for all $\beta > 0$ as a fractional Wiener integral and $W_3(t)$ is the integrated Brownian motion mentioned in Section 3.6. The existence of constants for $W_\beta(t)$, $\beta > 0$, under the sup-norm follows by using the subadditive argument with upper bound on the probability.

The proof given in Shao [S99] is based on the following correlation inequality: There exists $d_\alpha > 0$ such that

$$\begin{aligned} &\mathbb{P}\Big(\sup_{0\le t\le a} |B_\alpha(t)| \le x,\ \sup_{a\le t\le b} |B_\alpha(t) - B_\alpha(a)| \le y\Big) \\ \ge\ & d_\alpha \mathbb{P}\Big(\sup_{0\le t\le a} |B_\alpha(t)| \le x\Big)\, \mathbb{P}\Big(\sup_{a\le t\le b} |B_\alpha(t) - B_\alpha(a)| \le y\Big) \end{aligned}$$

for any $0 < a < b$, $x > 0$ and $y > 0$. The existence part follows from a modified scaling argument with lower bound on the probability. This approach was first used in an early version of Li and Shao (1999); see the half proof part below for more details.

The third proof is in Li [Li00] and is based on the representation (6.14) and the Gaussian correlation inequality given in Theorem 2.14. The existence part follows from a refined scaling lemma that allows error terms. Various modifications of this approach are most fruitful since the techniques can be used to show the existence of the limit

$$\lim_{\varepsilon\to 0} \varepsilon^{\gamma} \log \mathbb{P}\,(||X|| \le \varepsilon) = -\kappa(||\cdot||, X)$$

for various self-similar Gaussian processes X such as $B_\alpha(t)$ and W_β under norms $||\cdot||$ such as sup-norm, L_p-norm and Hölder norm, with $0 < \kappa(||\cdot||, X) < \infty$ and suitable $0 < \gamma < \infty$. It seems that the method also works for the Sobolev type norm given in Theorem 4.8 and the Besov norm given in Theorem 4.9, but the details still need to be checked. It should be emphasized that the refined scaling lemma formulated in Li [Li00] for the existence of a constant is weaker than all the competing subadditive type results we examined. Furthermore, the estimates used are for the lower bound on the probability, rather than the upper bound when using the subadditive argument mentioned earlier.

Now we turn to the "half" proof given in Li and Shao [LS99b]. It asserts the existence of the constants under the following weaker Gaussian correlation conjecture:

$$\mu(A_1 \cap A_2) \geq \alpha^2$$

for any $\mu(A_i) = \alpha$, $i = 1, 2$. Our early version of the paper in 1997 used the Gaussian correlation conjecture (2.6). we hope the "half" proof sheds light on the conjecture and points out new directions for useful partial results.

Finally, we mention the following result given in Kuelbs and Li [KL00] as a consequence of (7.36). Note in particular that the constant C_α plays an important role here.

Theorem 6.10 *Let $\rho : [0,1] \to [0,\infty)$ be a bounded function such that $\rho(t)^{2/\alpha}$ is Riemann integrable on $[0,1]$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\Big(\sup_{0 \leq t \leq 1} |\rho(t) B_\alpha(t)| \leq \varepsilon\Big) = -C_\alpha \int_0^1 \rho(t)^{2/\alpha} dt$$

where C_α is the small ball constant given in (6.10).

The small ball estimates for weighted L_p-norm of $B_\alpha(t)$ similar to Theorem 6.3 can also be obtained using the techniques in Li [Li99c, Li00].

7 Applications of small ball probabilities

We have presented some of the direct consequences or implications of the small ball probability in earlier Sections. In this section we want to point out more applications of the small ball probability to demonstrate the usefulness of this wide class of probability estimates. Many of the tools and inspiration for small ball probability come from these applications. For example, the study of the rate of convergence of Strassen's functional LIL (Section 7.3) leads to the discovery of the precise links with metric entropy; the need to complete the link forces the use of the l-approximation numbers and other approximation quantities. Further, the tools and ideas developed for these applications have also been used in studying other related problems. Finally, we must stress that we limited ourselves to Gaussian and related processes in all the applications mentioned in this section, and there are much more applications for other processes as indicated in the Introduction. In particular, one can use strong approximation theorems (see Csörgő and Révész [CR81]) to extend the results to partial sum processes.

7.1 Chung's laws of the iterated logarithm

Let $\{W(t),\ 0 \leq t \leq 1\}$ be a Brownian motion. It is well-known that by the Lévy [Le37] law of the iterated logarithm

$$\limsup_{h \to 0} \sup_{0 \leq s \leq h} \frac{|W(t+s) - W(t)|}{(2h \log\log(1/h))^{1/2}} = 1 \quad a.s. \tag{7.1}$$

for every $0 \le t < 1$, and furthermore by the modulus of continuity

$$\lim_{h\to 0} \sup_{0\le s\le h} \sup_{0\le t\le 1-h} \frac{|W(t+s)-W(t)|}{(2h\,\log(1/h))^{1/2}} = 1 \quad a.s. \tag{7.2}$$

On the other hand, Chung's LIL [Ch48] gives the lower limit of the convergence rate

$$\liminf_{h\to 0} \sup_{0\le s\le h} \frac{|W(t+s)-W(t)|}{(h/\log\log(1/h))^{1/2}} = \frac{\pi}{\sqrt{8}} \quad a.s. \tag{7.3}$$

for every $0 \le t < 1$, while Csörgő and Révész [CR78] obtain the modulus of non-differentiability

$$\lim_{h\to 0} \inf_{0\le t\le 1-h} \sup_{0\le s\le h} \frac{|W(t+s)-W(t)|}{(h/\log(1/h))^{1/2}} = \frac{\pi}{\sqrt{8}} \quad a.s. \tag{7.4}$$

We refer to [CR81] for more general results on the increments of the Brownian motion. For a general centered Gaussian process $\{X(t), t \in [0,1]\}$ with stationary increments, the law of the iterated logarithm remains true under certain regularity conditions. One can refer to the insightful review of Bingham [Bi86], to Nisio [N67], Marcus [Ma68] and Lai [Lai73] for real valued Gaussian processes, and to Csörgő and Csáki [CC92], Csörgő and Shao [CS93] for ℓ^p-valued Gaussian processes. A typical result is

$$\limsup_{h\to 0} \sup_{0<s\le h} \frac{|X(t+s)-X(t)|}{\sigma(h)(2\,\log\log(1/h))^{1/2}} = 1 \quad a.s. \tag{7.5}$$

and

$$\lim_{h\to 0} \sup_{0<s\le h} \sup_{0<t\le 1-h} \frac{|X(t+s)-X(t)|}{\sigma(h)(2\,\log(1/h))^{1/2}} = 1 \quad a.s. \tag{7.6}$$

where $\sigma(h) = (\mathbb{E}\,(X(t+h)-X(t))^2)^{1/2}$. In particular, (7.5) and (7.6) hold for the fractional Brownian motions of order $\alpha \in (0,2)$.

On the other hand, we are interested in Chung type LIL such as (7.3) and (7.4). It is well-known that a key step in establishing a Chung type law of the iterated logarithm is the small ball probability. The following general result gives a precise implication of the small ball probability to Chung's LIL.

Theorem 7.1 *Let $X = \{X(s), s \in [0,1]^d\}$ be a centered Gaussian process with stationary increments, that is*

$$\sigma^2(|t-s|) = \mathbb{E}\,|X(t)-X(s)|^2. \tag{7.7}$$

Assume that $X(0)=0$, $\sigma(x)/x^\alpha$ is non-decreasing on $[0,1]$ for some $\alpha > 0$ and that there is $0 < \theta < 2$ such that

$$\forall \;\; 0 < h < 1/2, \quad \sigma^2(2\,h) \le \theta\,\sigma(h). \tag{7.8}$$

If there exist $0 < c_1 \le c_2 < \infty$ such that

$$\exp\left\{-c_2(h/x)^d\right\} \le \mathbb{P}\left(\sup_{s\in[0,h]^d} |X(s)| \le \sigma(x)\right) \le \exp\left\{-c_1(h/x)^d\right\} \tag{7.9}$$

for some $0 < h_0 < 1$ and for any $0 < x \le h_0\, h \le h_0^2$, then

$$1 \le \liminf_{h\to 0} \frac{\sup_{s\in[0,h]^d} |X(s)|}{\sigma\left(h\,(c_1/\log\log(1/h))^{1/d}\right)} \qquad a.s. \tag{7.10}$$

$$\liminf_{h\to 0} \frac{\sup_{s\in[0,h]^d} |X(s)|}{\sigma(h\,(c_2/\log(1/h))^{1/d})} \le 1 \quad a.s. \tag{7.11}$$

Note that (7.10) follows from the right hand side of (7.9) and the subsequence method. To prove (7.11), let

$$M(h) = \sup_{s\in[0,h]^d} |X(s)|.$$

For arbitrary $0 < \varepsilon < 1$, put

$$s_k = \exp(-k^{1+\varepsilon}), \quad d_k = \exp(k^{1+\varepsilon} + k^{\varepsilon}), \quad \text{and} \quad \sigma_k = \sigma\left(s_k \left(c_2/\log\log(1/s_k)\right)^{1/d}\right). \tag{7.12}$$

It suffices to show that

$$\liminf_{k\to\infty} M(s_k)/\sigma_k \le 1 + \varepsilon^{\alpha} \quad a.s. \tag{7.13}$$

To prove (7.13), we use the spectral representation of X, as Monrad and Rootzen (1995) did. In what follows $\langle s, \lambda\rangle$ denotes $\sum_{i=1}^d s_i\,\lambda_i$. It is known from Yaglom [Y57] that $\mathbb{E}\,X(s)X(t)$ has a unique Fourier representation of the form

$$\mathbb{E}\left\{X(s)X(t)\right\} = \int_{\mathbb{R}^d} \left(e^{i\langle s,\lambda\rangle} - 1\right)\left(e^{-i\langle t,\lambda\rangle} - 1\right) \Delta(d\lambda) + \langle s, Bt\rangle. \tag{7.14}$$

Here $B = (b_{ij})$ is a positive semidefinite matrix and $\Delta(d\lambda)$ is a nonnegative measure on $\mathbb{R}^d - \{\mathbf{0}\}$ satisfying

$$\int_{\mathbb{R}^d} \frac{||\lambda||^2}{1+||\lambda||^2}\,\Delta(d\lambda) < \infty.$$

Moreover, there exist a centered, complex-valued Gaussian random measure $W(d\lambda)$ and a Gaussian random vector Y which is independent of W such that

$$X(s) = \int_{\mathbb{R}^d} \left(e^{i\langle s,\lambda\rangle} - 1\right) W(d\lambda) + \langle\, \mathrm{Y}, s\rangle. \tag{7.15}$$

The measures W and Δ are related by the identity $\mathbb{E}\{W(A)\overline{W(B)}\} = \Delta(A\cap B)$ for all Borel sets A and B in $\mathbb{R}^d$. Furthermore, $W(-A) = \overline{W(A)}$. It follows from (7.14) and (7.7) that

$$\sigma^2(|t-s|) = 2\int_{\mathbb{R}^d} (1-\cos(\langle t-s,\lambda\rangle))\,\Delta(d\lambda) + \langle t-s, B(t-st)\rangle.$$

In particular, for $0 < h < 1$ and for every $i = 1, 2, \ldots, d$

$$\sigma^2(h) = 2\int_{\mathbb{R}^d} (1-\cos(h\lambda_i))\,\Delta(d\lambda) + h^2\, b_{ii} \ge 2\int_{\mathbb{R}^d} (1-\cos(h\lambda_i))\,\Delta(d\lambda). \tag{7.16}$$

For $0 < h < 1$ and $1 \le i \le d$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d,\,|\lambda_i|\ge 1/h} \Delta(d\lambda) &\le \frac{1}{1-\sin 1}\int_{\mathbb{R}^d,\,|\lambda_i|\ge 1/h} \left(1 - \frac{\sin(h\lambda_i)}{h\lambda_i}\right)\Delta(d\lambda)\\
&= \frac{1}{(1-\sin 1)h}\int_{\mathbb{R}^d,\,|\lambda_i|\ge 1/h}\int_0^h (1-\cos(u\lambda_i))\,du\,\Delta(d\lambda)\\
&= \frac{1}{(1-\sin 1)h}\int_0^h\int_{\mathbb{R}^d,\,|\lambda_i|\ge 1/h} (1-\cos(u\lambda_i))\,\Delta(d\lambda)\,du\\
&\le 4\,\sigma^2(h)
\end{aligned}$$

and hence

$$\int_{|\lambda|\ge 1/h} \Delta(d\lambda) \le 4\, d\, \sigma^2(d\, h) \le 4\, d^3\, \sigma^2(h). \tag{7.17}$$

Similarly, by (7.16)

$$\begin{aligned}\int_{|\lambda|\le 1/h} |\lambda|^2\, \Delta(d\lambda) &\le d\, h^{-2} \sum_{i=1}^{d} \int_{\mathbb{R}^d, |\lambda_i|\le 1/h} (h\lambda_i)^2\, \Delta(d\lambda) \\ &\le 4\, d\, h^{-2} \sum_{i=1}^{d} \int_{\mathbb{R}^d, |\lambda_i|\le 1/h} (1-\cos(h\lambda_i))\; \Delta(d\lambda) \\ &\le 4\, d^2\, h^{-2} \sigma^2(h). \end{aligned} \tag{7.18}$$

Define for $k = 1, 2 \ldots$ and $0 \le s \le 1$,

$$X_k(s) = \int_{|\lambda|\in(d_{k-1}, d_k]} \left(e^{i\langle s,\lambda\rangle} - 1\right) W(d\lambda), \qquad \tilde{X}_k(s) = \int_{|\lambda|\notin(d_{k-1}, d_k]} \left(e^{i\langle s,\lambda\rangle} - 1\right) W(d\lambda).$$

Clearly,

$$\liminf_{k\to\infty} \frac{M(s_k)}{\sigma_k} \le \liminf_{k\to\infty} \frac{\sup_{t\in[0,s_k]^d} |X_k(t)|}{\sigma_k} + \limsup_{k\to\infty} \frac{\sup_{t\in[0,s_k]^d} |\tilde{X}_k(t)|}{\sigma_k} + \limsup_{k\to\infty} \frac{s_k \,||\, \mathrm{Y}||}{\sigma_k}. \tag{7.19}$$

It is easy to see that

$$\limsup_{k\to\infty} s_k \,||\, \mathrm{Y}||/\sigma_k \le ||\, \mathrm{Y}||\, \limsup_{k\to\infty} s_k \left((\log\log(1/s_k))^{1/d} / (s_k\, c_2^{1/d}) \right)^{1-\delta} = 0 \quad a.s. \tag{7.20}$$

By the Anderson inequality [A55],

$$\mathbb{P}\Big(\sup_{t\in[0,s_k]^d} |X_k(t)| \le (1+\varepsilon^\alpha)\sigma_k \Big) \ge \mathbb{P}(M(s_k) \le (1+\varepsilon^\alpha)\sigma_k).$$

Therefore, by (7.9), and (7.12)

$$\begin{aligned} &\sum_{k=1}^{\infty} \mathbb{P}\left(\sup_{t\in[0,s_k]^d} |X_k(t)| \le (1+\varepsilon^\alpha)\sigma_k \right) \qquad (7.21) \\ &\ge \sum_{k=1}^{\infty} \mathbb{P}\left(M(s_k) \le \sigma\left(s_k(1+\varepsilon)(c_2/\log\log(1/s_k))^{1/d} \right) \right) \\ &\ge \sum_{k=1}^{\infty} \exp\left\{ -(1+\varepsilon)^{-d} \log\log(1/s_k) \right\} = \infty. \end{aligned}$$

Since $\{\sup_{t\in[0,s_k]^d} |X_k(t)|,\, k \ge 1\}$ are independent, by the Borel-Cantelli lemma, it follows from (7.21) that

$$\liminf_{k\to\infty} \sup_{t\in[0,s_k]^d} |X_k(t)|/\sigma_k \le 1 + \varepsilon^\alpha \quad a.s. \tag{7.22}$$

We next estimate the second term on the right hand side of (7.19). From (7.17), (7.18), and (7.12), we obtain, for $0 \le s \le s_k$

$$\begin{aligned}
\mathrm{Var}(\tilde{X}_k(s)) &= 2\int_{|\lambda|\notin(d_{k-1},d_k]} (1-\cos(\langle s,\lambda\rangle))\ \Delta(d\lambda) \\
&\le \int_{|\lambda|\le d_{k-1}} |s|^2|\lambda|^2\,\Delta(d\lambda) + 4\int_{|\lambda|\ge d_k} \Delta(d\lambda) \\
&\le 4\,d^3\,s_k^2\,d_{k-1}^2\sigma^2(s_k\,/(s_k\,d_{k-1})) + 4\,d^3\,\sigma^2(s_k/(s_k\,d_k)) \\
&\le 4\,d^3\,(s_k\,d_{k-1})^{2\delta}\sigma^2(s_k) + 4\,d^3\,(s_k\,d_k)^{-2\alpha}\sigma^2(s_k) \\
&\le 8\,d^4\,\exp(^{-\varepsilon\,k^{\varepsilon}})\sigma^2(s_k).
\end{aligned}$$

Therefore

$$\mathrm{Var}(\tilde{X}_k(t) - \tilde{X}_k(s)) \le \tilde{\sigma}_k^2(h) \tag{7.23}$$

for every $0 < s,t \le s_k,\ |s-t| \le h \le s_k$, where $\tilde{\sigma}_k^2(h) = \min(\sigma^2(h), 16\,d^4\,\exp(^{-\varepsilon\,k^{\varepsilon}})\sigma^2(s_k))$. Applying an inequality of Fernique [F75] and the Borel-Cantelli lemma yields

$$\limsup_{k\to\infty}\ \sup_{0\le s\le s_k} |\tilde{X}_k(s)|/\sigma_k = 0 \quad a.s. \tag{7.24}$$

This proves (7.13), by (7.19), (7.20), (7.22) and (7.24), as desired.

For Lévy fractional Brownian motion we have

Theorem 7.2 *Let $\{X(s), s \in [0,1]^d\}$ be a fractional Lévy Brownian motion of order $\alpha \in (0,2)$. Then we have*

$$\liminf_{h\to 0} \frac{(\log\log(1/h))^{\alpha/(2d)}}{h^{\alpha/2}} \sup_{0\le s\le h} |X(s)| = c \quad a.s. \tag{7.25}$$

for some $0 < c < \infty$.

The result (7.25) follows immediately from Theorems 5.1 and 7.1 and the zero-one law of Pitt and Tran [PT79]. In general, it is not too difficult to derive a Chung type law of the iterated logarithm when the small ball probabilities are available. Below is a direct consequence of Theorem 5.5 on Brownian sheet. Other liminf types of LIL for Brownian sheet can be found in Zhang [Zh96a].

Theorem 7.3 *Let $\{X_t, t \in [0,1]^2\}$ be a Brownian sheet. Then*

$$\liminf_{h\to 0} \frac{(\log\log(1/h))^2}{h(\log\log\log(1/h))^{3/2}} \sup_{t\in[0,h]^2} |X_t| = c \quad a.s.$$

for some constant $0 < c < \infty$.

We end this subsection with the following integral test of Talagrand [T96].

Theorem 7.4 *Let $\{X(t), t \in [0,1]\}$ be a fractional Brownian motion of order $\alpha \in (0,2)$, and $a(t)$ be a nondecreasing function with $a(t) \ge 1$ and $a(t)/t^{\alpha/2}$ bounded. Then*

$$\mathbb{P}\Big(\sup_{s\in[0,t]} |X_s| < a(t), i.o.\Big)$$

is equal to zero or one according to whether the integral

$$\int_0^\infty a(t)^{-2/\alpha}\psi(a(t)t^{-\alpha/2})dt$$

is convergent or divergent, where $\psi(h) = \mathbb{P}\Big(\sup_{s\in[0,1]}|X_s| \le h\Big)$.

The above result for the Brownian motion case, $\alpha = 1$, dates back to Chung [Ch48].

7.2 Lower limits for empirical processes

Let $u_1, u_2, \ldots$ be independent uniform $(0,1)$ random variables. We define the empirical process U_n by

$$U_n(s) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1_{\{u_i\le s\}} - s) \quad \text{for } 0 \le s \le 1,$$

and the partial sum process $\mathcal{K}_n$ by

$$\mathcal{K}_n(s) = \sqrt{n}\,U_n(s) = \sum_{i=1}^{n}(1_{\{u_i\le s\}} - s)$$

for $0 \le s \le 1$.

We let $\mathcal{K} = \{\mathcal{K}(s,t), s \ge 0, t \ge 0\}$ denote a Kiefer process, that is, $\mathcal{K}$ is a Gaussian process with mean zero and

$$E\{\mathcal{K}(s_1,t_1)\mathcal{K}(s_2,t_2)\} = (s_1 \wedge s_2)\Big((t_1 \wedge t_2) - t_1 t_2\Big)$$

for $s_1, s_2, t_1, t_2 \ge 0$. It is easy to see that

$$\{\mathcal{K}(s,t), s \ge 0, t \ge 0\} \overset{d.}{=} \{B(s,t) - tB(s,1),\ s,t \ge 0\}$$

where B is the Brownian sheet.

By the celebrated strong approximation theorem of Komlós-Major-Tusnády [KMT75], there exists a Kiefer process $\mathcal{K}$ (on a possibly expanded probability space) such that

$$\max_{1\le i\le n}\sup_{0\le s\le 1}|\mathcal{K}_i(s) - \mathcal{K}(i,s)| = O\Big(n^{-1/2}(\log n)^2\Big) \quad a.s. \tag{7.26}$$

Hence, many strong limit theorems for the empirical process follow from the corresponding result for the Kiefer process. We give below several Chung type LIL results for the related empirical processes, based on the small ball probabilities presented in previous sections. We refer to Shorack and Wellner [SW86] and Csörgő and Horváth [CH93] for the general theory of empirical processes. The first result below is given in Mogulskii [M80] and the second is given in Horváth and Shao [HS99]. Other related work can be found in Shi [Shi91] and Csáki [C94] and the references therein.

Theorem 7.5 *Let* $U_n(t)$ *be the empirical process defined above. Then*

$$\liminf_{n\to\infty}(\log\log n)^{1/2}\sup_{0\le t\le 1}|U_n(t)| = \frac{\pi}{\sqrt{8}} \quad a.s.$$

and

$$\liminf_{n\to\infty}(\log\log n)\int_0^1 U_n^2(t)dt = \frac{1}{8} \quad a.s.$$

Theorem 7.6 *Let $\mathcal{K}_n(t)$ be the partial sum process defined above. Then*

$$\liminf_{n\to\infty} \frac{(\log\log n)^{1/2}}{\sqrt{n}(\log\log\log n)^{3/2}} \sup_{0\le s\le 1} \max_{1\le i\le n} |\mathcal{K}_i(s)| = c_1 \quad a.s.$$

and

$$\liminf_{n\to\infty} \frac{\log\log n}{n(\log\log\log n)^2} \max_{1\le i\le n} \int_0^1 |\mathcal{K}_i(s)|^2 ds = c_2 \quad a.s.$$

where c_1 and c_2 are positive finite constants.

7.3 Rates of convergence of Strassen's functional LIL

Recall that $\{W(t) : t \ge 0\}$ is the standard Brownian motion. Then $H_W \subseteq C[0,1]$, the reproducing Hilbert space generated by the Wiener measure $\mu = \mathcal{L}(W)$, is the Hilbert space of absolutely continuous functions on $C[0,1]$ whose unit ball is the set

$$K_W = \left\{ f(t) = \int_0^t f'(s)\,ds \ , \ 0 \le t \le 1 \quad : \quad \int_0^1 |f'(s)|^2 ds \le 1 \right\}. \tag{7.27}$$

Here the inner product norm of H_W is given by

$$|f|_W = \left(\int_0^1 |f'(s)|^2 ds \right)^{1/2}, \qquad f \in H_W, \tag{7.28}$$

Let $\|f\|_\infty$ denote the usual sup-norm on $C[0,1]$. If

$$\eta_n(t) = W(nt)/(2n\log\log n)^{1/2}, \qquad 0 \le t \le 1,$$

then Strassen's functional LIL can be considered to consist of two parts:

$$\lim_{n\to\infty} \inf_{f\in K_W} \|\eta_n - f\|_\infty = 0 \tag{7.29}$$

and

$$\lim_{n\to\infty} \|\eta_n - f\|_\infty = 0 \quad \text{for all} \quad f \in K_W. \tag{7.30}$$

It is natural to ask for rates of convergence in (7.29) and (7.30).

Bolthausen [Bol78] was the first to examine the rate of convergence in (7.29) and the problem was further investigated by Grill [G87], Goodman and Kuelbs [GK91]. Finally Grill [G92] and Talagrand [T92] have shown that

$$0 < \limsup_{n\to\infty} (\log\log n)^{2/3} \inf_{f\in K_W} \|\eta_n - f\|_\infty < \infty. \tag{7.31}$$

where the lower bound also follows from a result of Goodman and Kuelbs [GK91].

When examining a more general formulation of the problem for any Gaussian measure μ, the rate is in fact determined by the small ball probability in (3.5). One motivation for proving results about Gaussian random vectors is that once this is accomplished, then one can translate, in some fashion or other, the Gaussian result to a variety of non-Gaussian situations. For example, this is the approach of Strassen [Str64], which produced the functional law of the iterated logarithm for Brownian motion and then, via Skorohod embedding, for polygonal processes. Goodman and Kuelbs [GK89] also contains polygonal process results obtained by a suitable strong approximation of their analogous Gaussian results.

Next we formulate the results in terms of Gaussian random samples. Let X, X_1, $X_2, \ldots$ denote an i.i.d sequence of centered Gaussian random vectors in a real separable Banach space E with norm $\|\cdot\|$. Let the small ball function $\phi(\varepsilon)$ be given as in (3.5). Since the exact computation of ϕ is in practice impossible, we state the following result due to Talagrand [T93], requiring only either an upper or a lower bound for ϕ.

Consider two continuous and non-increasing functions $\phi_1 \geq \phi \geq \phi_2$, and assume that for some constants $c_i > 1$, $\phi_i(\varepsilon/c_i) \geq 2\phi_i(\varepsilon)$ for all $\varepsilon > 0$ small, $i = 1, 2$.

Theorem 7.7 *Let $\varepsilon_{n,i}$ be the unique root of the equation*

$$\phi_i(\varepsilon)/\varepsilon = \sqrt{2\log n}/\sigma, \quad i = 1, 2$$

for n large enough, where $\sigma^2 = \sup_{\|h\|_{E^*}\leq 1} \mathbb{E}\,(h^2(X))$. Then for some constants C_1 and C_2,

$$\limsup_{n\to\infty} \frac{\sqrt{\log n}}{\varepsilon_{n,1}} \inf_{f\in K_\mu} \left\| X_n/\sqrt{2\log n} - f \right\| \;\leq\; C_1 < \infty \quad a.s.$$
$$\limsup_{n\to\infty} \frac{\sqrt{\log n}}{\varepsilon_{n,2}} \inf_{f\in K_\mu} \left\| X_n/\sqrt{2\log n} - f \right\| \;\geq\; C_2 > 0 \quad a.s.$$

Note that the probability estimates needed for the proof of Theorem 7.7 can be viewed as a type of refined large deviation estimates (enlarged balls) for which the small ball rate function ϕ plays a crucial role in the second order term, see Chapter 7 in [L96]. The result (7.31) follows from Theorem 7.7 by a scaling argument along a suitable exponential subsequence; see [GK91] for details.

The rate of convergence in (7.30) for general Gaussian measure μ is also determined by the small ball probability in (3.5) and can be viewed as a Chung type functional law of the iterated logarithm; see next section for details.

7.4 Rates of convergence of Chung type functional LIL

The rate of convergence in (7.30) can also be seen as a functional form of Chung's law of the iterated logarithm given in Csáki [C80], and in more refined form in de Acosta [dA83], which implies for each f in $C[0,1]$ that with probability one

$$\liminf_{n\to\infty} \log\log n\, \|\eta_n - f\|_\infty = \begin{cases} \pi/4 \cdot (1 - \|f\|_W^2)^{-1/2} & \text{if } |f|_W < 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{7.32}$$

Note that if $f \equiv 0$ then (7.32) is just Chung's law of the iterated logarithm given in (7.1) with a time inversion. The proof of (7.32) is based on the small ball probability

$$\log \mathbb{P}\left(\sup_{0\leq t\leq 1} |W(t)| \leq \varepsilon \right) \sim -(\pi^2/8)\varepsilon^{-2},$$

the shifting results given in Section 3.1 and variations of well-known techniques in iterated logarithm proofs. The precise rates for various classes of functions f with $|f|_W = 1$ can be found in Csáki [C80, C89], Grill [G91], Kuelbs, Li and Talagrand [KLT94], Gorn and Lifshits [GL99].

When examining more general formulations for Gaussian samples, the rate clearly is determined by the small ball probability in (3.5) and the function $f \in K_\mu$. The following result is given in Kuelbs, Li and Talagrand [KLT94]. We use the same notations as in the previous section.

Theorem 7.8 *Assume that for $\varepsilon > 0$ sufficiently small the function ϕ satisfies both of the following:*

$$\varepsilon^{-q} \le \phi(\varepsilon) \le \varepsilon^{-p} \quad \textit{for some} \quad p \ge q > 0$$

and for each $\alpha \in (0,1)$ there is a $\beta > 0$ such that

$$\phi(\alpha\varepsilon) - \phi(\varepsilon) \ge \phi(\beta\varepsilon).$$

If $f \in K_\mu$ and d_n is the unique solution of the equation

$$\log n(1 - I(f, d_n)) = \phi((2\log n)^{1/2} d_n)$$

where $I(f, \delta_n)$ is defined in Section 3.1, then with probability one

$$1 \le \liminf d_n^{-1} \left\| X_n/(2\log\log n)^{1/2} - f \right\| \le 2.$$

It is still an open question to find the exact constant in the case of $\|f\|_\mu = 1$.

7.5 A Wichura type functional law of the iterated logarithm

There are various motivations for extending results classical for Brownian motion to the fractional Brownian motion $B_\alpha(t)$, $0 \le \alpha < 2$. It is not only the importance of these processes, but also the need to find proofs that rely upon general principles at a more fundamental level by moving away from crucial properties (such as the Markov property) of Brownian motion. Below we mention an extension of a classical result for Brownian motion to fractional Brownian motion by using pure Gaussian techniques. Let

$$M_\alpha(t) = \sup_{0 \le s \le t} |B_\alpha(s)|, \quad t \ge 0 \tag{7.33}$$

and

$$H_n(t) = M_\alpha(nt)/(C_\alpha n^\alpha / \log\log n)^{1/2}, \quad t \ge 0 \tag{7.34}$$

where C_α is given in (6.10). In the Brownian motion case, i.e. $\alpha = 1$, it is well known that $C_1 = \pi^2/8$. Denote by $\mathcal{M}$ the non-decreasing functions $f : [0,\infty) \to [0,\infty]$ with $f(0) = 0$ and which are right continuous except possibly at zero. Let

$$K_\alpha = \left\{ f \in \mathcal{M} : \int_0^\infty (f(t))^{-2/\alpha} dt \le 1 \right\} \tag{7.35}$$

for $\alpha \in (0,2)$, and topologize $\mathcal{M}$ with the topology of weak convergence, i.e. pointwise convergence at all continuity points of the limit.

When $\alpha = 1$, Wichura [W73] proved, in an unpublished paper, that the sequence $\{H_n(t)\}$ has a deterministic limit set K_1 with probability one, yielding a functional analogue of Chung's LIL for the maximum of the absolute value of Brownian motion. Wichura obtained his result for Brownian motion as a special case of a result for the radial part of a Bessel diffusion. The key probability estimate in Wichura's work follows from a computation of the Laplace transform of the first passage time of the radial part of the Bessel diffusion. The first passage time process in his setting has independent increments, and hence is relatively easy to study.

Very recently, the following is proved in [KL00] based on the Gaussian nature of these processes.

Theorem 7.9 *The sequence $\{H_n(t)\}$ is relatively compact and all possible subsequential limits of $\{H_n(t)\}$ in the weak topology is K_α given in (7.35).*

One of the key steps in the proof is the small ball estimate

$$\lim_{\varepsilon\to 0}\varepsilon^{2/\alpha}\log\mathbb{P}\left(a_i\varepsilon \le \sup_{0\le s\le t_i}|B_\alpha(s)| \le b_i\varepsilon, \quad 1\le i\le m\right) = -C_\alpha\sum_{i=1}^{m}(t_i-t_{i-1})/b_i^{2/\alpha} \tag{7.36}$$

for any fixed sequences $\{t_i\}_{i=0}^m$, $\{a_i\}_{i=0}^m$ and $\{b_i\}_{i=0}^m$ such that $0=t_0<t_1<\cdots<t_m$ and

$$0\le a_1<b_1\le a_2<b_2\le\cdots a_i<b_i\le a_{i+1}\cdots\le a_m<b_m,$$

where C_α is given in (6.10).

7.6 Fractal geometry for Gaussian random fields

There has been much effort devoted to the study of fractal properties of random sets generated by various stochastic processes. The development of the techniques used in early papers were surveyed in Taylor [Ta86] and more recent results associated to Gaussian random fields were surveyed in Xiao [X97]. The crucial ingredient in most of the early works is the strong Markov property. For the (N,d,α)-Gaussian random fields $Y(t;N,d,\alpha)$ from R^N to R^d with

$$\mathbb{E}\left(|Y(t;N,d,\alpha)-Y(s;N,d,\alpha)|^2\right)=d|t-s|^\alpha,$$

also called Lévy's multiparameter fractional Brownian motion, the small ball probability

$$-\varepsilon^{-2N/\alpha}\preceq\log\mathbb{P}\left(\sup_{|t|\le 1}|Y(t;N,1,\alpha)|\le\varepsilon\right)\preceq-\varepsilon^{-2N/\alpha} \tag{7.37}$$

plays an important role, where $0<\alpha<2$ and $|\cdot|$ denotes the Euclidean norm. The upper bound is proved essentially in Pitt [P78] and the lower bound follows from Theorem 3.8.

The exact Hausdorff measure of the image set of $Y(t;N,d,\alpha)$ in the transient case ($2N<d\alpha$) was given in Talagrand [T95], with a significantly shorter proof of previous known special cases, by using the small ball estimates (7.37) and other related techniques. Later in Talagrand [T98], the multiple points of the trajectories $Y(t;N,d,\alpha)$ were studied. The key to the success is the use of a direct "global" approach to obtain a lower bound for a certain sojourn time which is a small ball type estimate. Furthermore, the detailed arguments rely heavily on the chaining argument and the Khatri-Sidak lemma as in the proof of Theorem 3.8 mentioned in Section 3.4. Recently, Xiao [X96, X98] has studied various other aspects of the fractal structure of fractional Brownian motions using various small ball type estimates.

7.7 Metric entropy estimates

The precise links between the small ball probability and the metric entropy given in Section 3.2 allow one to establish new results about the metric entropy of the various sets K_μ in instances when one knows the behaviour of $\phi(\varepsilon)$. Two examples from [KL93a] are given below.

The first one relates to the unit ball K_W, given in (7.27), which is generated by the Wiener measure. In this case, the small ball function $\phi(\varepsilon)$ is known very precisely for the L_2-norm $\|\cdot\|_2$ and the sup-norm $\|\cdot\|_\infty$ on $C[0,1]$; see Theorem 6.3. Thus the key relations (3.8) and (3.9) can yield the following correspondingly precise estimates of $H(\varepsilon,K_W)$ which are much sharper than those known before.

Proposition 7.1 *If K_W is as in (7.27), then for each $\delta > 0$ as $\varepsilon \to 0$*

$$(1-\delta)(2-\sqrt{3})/4 \le \varepsilon \cdot H(\varepsilon, K_W, \|\cdot\|_2) \le 1+\delta \tag{7.38}$$

and

$$(1-\delta)(2-\sqrt{3})\pi/4 \le \varepsilon \cdot H(\varepsilon, K, \|\cdot\|_\infty) \le \pi(1+\delta). \tag{7.39}$$

Note that (7.38) is more precise than what is given in Theorem XVI of Kolmogorov and Tihomirov [KT61], and for (7.39) there are no constant bounds in Birman and Solomjak [BiS67].

The second is related to the small ball probability for the 2-dimensional Brownian sheet (see Section 5.2), and it was first solved in Talagrand [T93]. The unit ball of the generating Hilbert space for the 2-dimensional Brownian sheet is

$$K = \left\{ f \in C([0,1]^2) \,:\, f(s,t) = \int_0^s \int_0^t g(u,v)dudv, \ \int_0^1 \int_0^1 g^2(u,v)dudv \le 1 \right\}.$$

Thus combining Theorem 5.4 and Theorem 3.3 implies

$$H(\varepsilon, K, \|\cdot\|_\infty) \approx \varepsilon^{-1}(\log 1/\varepsilon)^{3/2},$$

which solves an interesting problem left open in approximation theory. Later, a direct proof inspired by this line of argument was given in Temlyakov [Te95]. Further research in this direction can be found in Dunker [D98] and for dimensions bigger than two, the analogous problem remains open, as well as the corresponding Brownian sheet problem discussed in detail in Section 5.2.

7.8 Capacity in Wiener space

Let $\{W^E(t) : t \ge 0\}$ be an E-valued μ-Brownian motion as given before Theorem 6.7. Define the E-valued μ-Ornstein-Uhlenbeck process O by

$$O(t) = e^{-t/2} W^E(e^t), \qquad t \ge 0.$$

Note that O is also an E–valued stationary diffusion whose stationary measure is μ. Let $Q(\cdot)$ be a continuous semi-norm on E and define the λ-capacity of the "ball" $\{x \in E : Q(x) \le \varepsilon\}$ by

$$\mathrm{Cap}_\lambda(\varepsilon) = \int_0^\infty e^{-\lambda T} \mathbb{P}\left(\inf_{0 \le t \le T} Q(O_t^E) \le \varepsilon \right) dT, \quad \varepsilon > 0.$$

We refer the readers to Üstünel [U95] and Fukushima, Oshima and Takeda [FOT94] for details. Then the following result is given in Khoshnevisan and Shi [KS98b].

Theorem 7.10 *Suppose Q is a nondegenerate, transient semi–norm on E. Then, for all $\lambda > 0$ and all $\kappa > 1$, there exists a constant $c \in (1,\infty)$ such that for all $\varepsilon \in (0, 1/c)$,*

$$\frac{\mu(x : Q(x) \le \varepsilon)}{c\,\varepsilon^2} \le Cap_\lambda(\varepsilon) \le \frac{c\,\mu(x : Q(x) \le \varepsilon)}{(\lambda_Q(\varepsilon;\kappa) - \varepsilon)^2}$$

where

$$\lambda_Q(\varepsilon;\kappa) = \sup\{a > 0 : \mu(x : Q(x) \le a) \le \kappa \mu(x : Q(x) \le \varepsilon)\}, \qquad \varepsilon > 0$$

is the approximate inverse to $\mu(x : Q(x) \le \varepsilon)$.

It turns out that, under very general conditions, $\lambda_Q(\varepsilon;\kappa) - \varepsilon$ has polynomial decay rate. Thus, Theorem 7.10 and its relatives given in [KS98b] provide exact (and essentially equivalent) asymptotics between the λ–capacity of a small ball and the small ball probability $\mu(x : Q(x) \le \varepsilon)$. But since $\lambda_Q(\varepsilon;\kappa)$ is much harder to find directly, the results are in fact applications of small ball probabilities to the λ-capacity.

7.9 Natural rates of escape for infinite dimensional Brownian motions

For any stochastic process $\{X(t), t \geq 0\}$ taking values in a real Banach space with norm $\|\cdot\|$, a nondecreasing function $\gamma : [0,\infty) \to (0,\infty)$ is called a natural rate of escape for X if $\liminf \|X(t)\| / \gamma(t) = 1$ as $t \to \infty$ with probability one. The following result and related definitions of a vector valued Brownian motion and an admissible function are given in Erickson [Er80].

Theorem 7.11 *Let X be a genuinely d-dimensional Brownian motion on a Banach space $(E, \|\cdot\|)$ with $3 \leq d \leq \infty$, and let h be an admissible function. Fix $b > 1$ and put $\gamma(t) = t^{1/2}h(t)$. Then*

$$\liminf_{t\to\infty} \|X(t)\| / \gamma(t) \lesseqqgtr 1 \qquad a.s.$$

depending on whether

$$\sum_{k\geq 1} h(b^k)^{-2}\mathbb{P}\left(\|X(1)\| \leq h(b^k)\right) \quad \begin{cases} \textit{converges} \\ \textit{diverges} \end{cases} . \tag{7.40}$$

Since h is nonincreasing and slowly varying at infinity, we see that (7.40) completely depends on the small ball probability $\mathbb{P}(\|X(1)\| \leq \varepsilon)$ as $\varepsilon \to 0$. A natural conjecture mentioned in [Er80] is that if X is genuinely infinite dimensional Brownian motion and if $\mathbb{P}(\|X(t)\| \leq \varepsilon) > 0$ for all $\varepsilon > 0$, $t > 0$, then

$$0 < \liminf_{t\to\infty} \|X(t)\| / \gamma(t) < \infty \tag{7.41}$$

for some $\gamma(t)$; see also [Co82] for some related works. Note that (7.41) does not hold in the finite dimensional setting by the Dvoretzky-Erdos test given in [DE51]. The difference between the infinite and finite dimensional cases is that $\mathbb{P}(\|X(1)\| \leq \varepsilon)$ is $o(\varepsilon^n)$ as $\varepsilon \to 0$ for all n when X is genuinely infinite dimensional, whereas $\mathbb{P}(\|X(1)\| \leq \varepsilon) \neq o(\varepsilon^n)$ for $n \geq d$ when X is d-dimensional with $d < \infty$.

7.10 Asymptotic evaluation of Laplace transform for large time

When proper scaling properties hold for certain functionals of a given random process, small ball estimates are equivalent to asymptotic evaluation of Laplace transform for large time via the exponential Tauberian theorem given in Theorem 3.5. Here we present two more examples.

The first one is the consequence of (6.9) from which

$$\lim_{\lambda\to\infty} \lambda^{-2/(2+p)} \log \mathbb{E} \exp\left\{-\lambda \int_0^1 \frac{|W(s)|^p}{s^{\alpha p}} ds\right\} = \frac{(2+p)^2}{2+p-2\alpha p} \cdot \left(\kappa_p/2^{2/p}p\right)^{p/(2+p)}$$

by the exponential Tauberian theorem given in Theorem 3.5. Now by using the scaling property of Brownian motion and (6.2)

$$\lim_{t\to\infty} t^{-(2+p-2\beta)/(2+p)} \log \mathbb{E} \exp\left\{-\lambda \int_0^t \frac{|W(s)|^p}{s^{\beta}} ds\right\} = -\frac{2+p}{2+p-2\beta}\lambda_1(p)\lambda^{2/(2+p)}$$

for $\beta < (2+p)/2$ and $\lambda > 0$.

The second one is the consequence of (6.10) from which

$$\log \mathbb{E} \exp\left\{-\lambda \sup_{0\leq t\leq 1} |B_\alpha(t)|\right\} \sim -(2+\alpha)(C_\alpha/\alpha)^{\alpha/(2+\alpha)}(\lambda/2)^{2/(2+\alpha)}$$

as $\lambda \to \infty$ by the exponential Tauberian theorem given in Theorem 3.5. Now by the scaling property of fractional Brownian motion $B_\alpha(t)$, i.e. $\{B_\alpha(at), t \ge 0\} = \{a^{\alpha/2} B_\alpha(t), t \ge 0\}$ in law,

$$\lim_{t\to\infty} t^{-\alpha/(2+\alpha)} \log \mathbb{E} \exp\left\{-\lambda \sup_{0\le s\le t} |B_\alpha(s)|\right\} = -(2+\alpha)(C_\alpha/\alpha)^{\alpha/(2+\alpha)}(\lambda/2)^{2/(2+\alpha)}$$

for $0 < \alpha < 2$ and any $\lambda > 0$. Note that C_α is the small ball constant given in (6.10).

7.11 Onsager-Machlup functionals

For any measure ν on a metric space E with metric $d(\cdot,\cdot)$, the Onsager-Machlup function is defined as

$$F(a,b) = \log\left(\lim_{\varepsilon\to 0} \frac{\nu(x\colon d(x,a) \le \varepsilon)}{\nu(x\colon d(x,b) \le \varepsilon)}\right) \tag{7.42}$$

if the above limit exists.

For the Gaussian measure, the existence of (7.42) and related conditional exponential moments are studied in Shepp and Zeitouni [SZ92], Bogachev [Bog95] and Ledoux [L96]. Both correlation type inequalities and small ball probabilities play an important role in the study.

In Capitaine [Ca95], a general result in the Cameron-Martin space for diffusions is proved for rotational invariant norms with known small ball behavior, including in particular Hölder norms and Sobolev type norms. Other related work can be found in Carmona and Nualart [CN92] and Chaleyat-Maurel and Nualart [CmN95].

7.12 Random fractal laws of the iterated logarithm

Let $\{W(t) : t \ge 0\}$ denote a standard Wiener process, and for any $\eta \in [0,1]$, set

$$E_\eta = \left\{t \in [0,1) : \quad \limsup_{h\to 0} (2h\log(1/h))^{-1/2} \left(W(t+h) - W(t)\right) \ge \eta\right\}. \tag{7.43}$$

Orey and Taylor [OT74] proved that E_η is a random fractal and established that with probability one the Hausdorff dimension of this set is given by $\dim(E_\eta) = 1 - \eta^2$. Recently, Deheuvels and Mason [DM98] show that one can derive the following functional refinement of (7.43). We use notations given in Section 7.3 and 7.4, and in particular, the Strassen set K_W and the inner product norm $|\cdot|_W$ are given in (7.27) and (7.28).

Theorem 7.12 *For each $f \in K_W$ and $c > 1$, let $E(f,c)$ denote the set of all $t \in [0,1]$ such that*

$$\liminf_{h\to 0} |\log h| \times \|(2h|\log h|)^{-1/2}\xi(h,t,\cdot) - f\|_\infty \le c 2^{-1/2} \frac{\pi}{8^{1/2}}(1 - |f|_W^2)^{-1/2}$$

where $\xi(h,t,s) = W(t+hs) - W(t)$. Then for $|f|_W < 1$, $\dim E(f,c) = (1 - |f|_W^2)(1 - c^{-2})$ with probability one.

A key in the proof is the small ball estimate given in de Acosta [dA83] discussed in Section 7.4. The case $|f|_W = 1$ is also related to Section 7.4. Very recently, Khoshnevisan, Peres and Xiao [KPX99] present a general approach to many random fractals defined by limsup operations. In particular, their result yields extensions of Theorem 7.12 by applying appropriate small ball estimates.

References

[A55] Anderson, T.W. (1955). The integral of a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6** , 170–176 .

[BR92] Baldi, P. and Roynette, B. (1992). Some exact equivalents for the Brownian motion in Hölder norm. *Probab. Thy. Rel. Fields.* **93**, 457-484.

[Ba88] Bass, R.F. (1988). Probability estimates for multiparameter Brownian processes. *Ann. Prob.* **16**, 251-264.

[Bel98] Belinsky, E. (1998). Estimates of entropy numbers and Gaussian measures for classes of functions with bounded mixed derivative. *J. Approx. Th.* **93**, 114-127.

[Be85] Berman, S.M. (1985). An asymptotic bound for the tail of the distribution of the maximum of a Gaussian process with stationary increments. *Ann. Inst. H. Poincaré* **21**, 47-57.

[BS98] Berthet, P. and Shi, Z. (1998). Small ball estimates for Brownian motion under a weighted sup-norm. (Preprint)

[Bi86] Bingham, N.H. (1986). Variants on the law of the iterated logarithm. *Bull. London Math. Soc.* **18**, 433-467.

[BGT87] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation.* Cambridge Univ. Press. Cambridge.

[BiS67] Birman, M.S. and Solomjak, M.Z. (1967). Piecewise polynomial approximation of the functions of the classes W_p^α. *Math. USSR-Sbornik* **2**, 295-317.

[Bog95] Bogachev, V.I. (1995). Onsager-Machlup functions for Gaussian measures. *Dokl. Akad. Nauk*, **344**, 439–441.

[Bog98] Bogachev, V.I. (1998). *Gaussian Measures.* Amer. Math. Soc., Mathematical Surveys and Monographs, Vol 62.

[Bol78] Bolthausen, E. (1978), On the speed of convergence in Strassen's law of the Iterated logarithm. *Ann. Probab.* **6**, 668–672.

[B75] Borell, C. (1975). The Brunn-Minkowski inequality in Gauss space. *Invent. Math.* **30**, 207-216.

[B76] Borell, C. (1976). Gaussian Radon measures on locally convex spaces. *Math. Scand.* **38**, 265-285.

[BM91] Borovkov, A. and Mogulskii, A. (1991). On probabilities of small deviations for stochastic processes, *Siberian Advances in Mathematics* **1**, 39-63.

[Ca95] Capitaine, M. (1995). Onsager-Machlup functional for some smooth norms on Wiener space. *Probab. Theory Related Fields*, **102**, 189–201.

[CN92] Carmona, R. and Nualart, D. (1992). Traces of random variables on Wiener space and the Onsager-Machlup functional. *J. Funct. Anal.* **107**, 402–438.

[CmN95] Chaleyat-Maurel, M. and Nualart, D. (1995). Onsager-Machlup functionals for solutions of stochastic boundary value problems. *Lecture Notes in Math.*, **1613**, 44–55, Springer, Berlin.

[CL99] Chen, X. and Li, W.V. (1999). Small ball probabilities for primitives of Brownian motion. *Preprint.*

[Ch48] Chung, K.L. (1948). On the maximum partial sums of sequence of independent random variables. *Amer. Math. Soc.* **64**, 205-233.

[Co82] Cox, D.D. (1982). On the existence of natural rate of escape functions for infinite dimensional Brownian motions. *Ann. Prob.* **10**, 623-638.

[C80] Csáki, E. (1980). A relation between Chung's and Strassen's law of the iterated logarithm, *Z. Wahrsch. verw. Gebiete* **54**, 287-301.

[C82] Csáki, E. (1982). On small values of the square integral of a multiparameter Wiener process. *Statistics and Probability. Proc. of the 3rd Pannonian Symp. on Math. Stat.* D.Reidel Publishing Company, pp. 19-26.

[C89] Csáki, E. (1989). A lim inf result in Strassen's law of the iterated logarithm. *Colloquia Math. Soc. J. Bolyai* **57**, Limit theorems in probability and statistics. 83–93

[C94] Csáki, E. (1994). Some limit theorems for empirical processes. In: *Recent Advances in Statistics and Probability* (Proc. 4th IMSIBAC, eds.: J.P. Vilaplana and M.L. Puri) pp. 247–254. VSP, Utrecht.

[CC92] Csáki, E. and Csörgő, M. (1992). Inequalities for increments of stochastic processes and moduli of continuity. *Ann. Probab.* **20**, 1031–1052.

[CH93] Cśorgő, M. and Horváth, L. (1993). *Weighted Approximations in Probability and Statistics.* John Wiley & Sons, New York.

[CR78] Csörgő, M. and Révész, P. (1978). How small are the increments of a Wiener process? *Stoch. process and their appl.* **8** , 119–129 .

[CR81] Csörgő, M. and Révész, P. (1979). *Strong Approximations in Probability and Statistics.* Academic Press, New York.

[CS93] Csörgő, M. and Shao, Q.M. (1993). Strong limit theorems for large and small increments of ℓ^p-valued Gaussian processes. *Ann. Probab.* **21**, 1958–1990.

[CS94] Csörgő, M. and Shao, Q.M. (1994). On almost sure limit inferior for B-valued stochastic processes and applications. *Probab. Th. Rel. Fields* **99**, 29-54.

[dA83] de Acosta, A. (1983). Small deviations in the functional central limit theorem with applications to functional laws of the iterated logarithm. *Ann. Probab.* **11**, 78–101.

[DM98] Deheuvels, P. and Mason, D. (1998). Random fractal functional laws of the iterated logarithm. *Studia Sci. Math. Hungar.* **34**, 89-106.

[DMZ95] Dembo, A., Mayer-Wolf, E. and Zeitouni, O. (1995). Exact behavior of Gaussian seminorms. *Stat. Prob. Lett.* **23**, 275-280.

[DZ98] Dembo, A. and Zeitouni, O. (1998). *Large deviations techniques and applications.* 2nd Ed. Springer, New York.

[DS89] Deuschel, J. and Stroock, D. (1989). *Large Deviations*. Academic Press, Boston.

[DV75a] Donsker, M.D. and Varadhan, S.R.S. (1975a). Asymptotic evaluation of certain Wiener integrals for large time. *Functional integration and its applications (Proc. Interna. Conf.)*, 15-33.

[DV75b] Donsker, M.D. and Varadhan, S.R.S. (1975b). Asymptotics for the Wiener Sausage. *Comm. Pure Appl. Math.*, **28**, 525–565.

[Do49] Doob, J.L. (1949). Heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Stat.* **20**, 393-403.

[Du67] Dudley, R. (1967). The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *J. Funct. Anal.* **1**, 290-330.

[DHS79] Dudley, R.M., Hoffmann–Jørgensen, J. and Shepp, L.A. (1979). On the lower tail of Gaussian seminorms. *Ann. Probab.* **9**, 319-342.

[D98] Dunker T. (1998) Small ball estimates for the fractional Brownian sheet. PhD Thesis, F.Schiller Universität, Jena.

[D99] Dunker, T. (1999). Estimates for the small ball probabilities of the fractional Brownian sheet. *J. Theor. Proba.* to appear

[D-98] Dunker, T., Kuhn, T., Lifshits, M. A. and Linde, W. (1998). Metric entropy of integration operator and small ball probabilities of the Brownian sheet. *C. R. Acad. Sci. Paris* **326**, 347-352.

[D-99] Dunker, T., Kuhn, T., Lifshits, M. A. and Linde, W. (1999). Approximation numbers and l-norm. Preprint.

[DLL99] Dunker, T., Li, W.V. and Linde, W. (1999). Small ball probabilities of integrals of weighted Brownian motion. *Stat. Prob. Letter*, to appear.

[DLL98] Dunker, T., Lifshits, M.A., and Linde W. (1998) Small deviations of sums of independent variables. *Progress in Probability.* **43**, 59-74, Birkhauser.

[DE51] Dvoretzky, A. and Erdos, P. (1951). Some problems on random walk in space. *Proc. Second Berkely Symp. Math. Statist. Probability*, 353-367.

[E83] Ehrhard, A. (1983). Symétrisation dans l'espace de Gauss. *Math. Scand.* **53**, 281–301.

[E84] Ehrhard, A. (1984). Inéqalités Isopérimétriques et intégrales de Dirichlet Gaussiens. *Ann. Scient Éc Norm. Sup.* **17** , 317-332.

[E86] Ehrhard, A. (1986). Eléments extrémaux pour les inégalités de Brunn-Minkowski gaussiennes. *Ann. H. Poincaré*, **22**, 149-168.

[Er80] Erickson, K. (1980). Rates of escape of infinite dimensional Brownian motion, *Ann. Probab.* **8**, 325–338.

[F75] Fernique, X. (1975). Régularité des trajectoires des fonctions aléatoires gaussiennes. École Été de Probabilités de St.–Flour. *Lecture Notes in Math.*, **480**, 1–96.

[FOT94] Fukushima, M., Y. Oshima and M. Takeda, (1994). *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, New York.

[Go90] Goodman, V. (1990). Some probability and entropy estimates for Gaussian measures. *Progress in Probability*, **20**, 150-156, Birkhauser.

[GK89] Goodman, V. and Kuelbs, J. (1989). Rates of convergence for the functional LIL. *Ann. Prob.* **17** , 301–316 .

[GK91] Goodman, V. and Kuelbs, J. (1991). Rates of clustering in Strassen's LIL for Brownian motion. *J. Theory Prob.* **4**, 285–309.

[Gor85] Gordon, Y. (1985). Some inequalities for Gaussian processes and applications. *Israel J. Math.* **50**, 265-289.

[GL99] Gorn N.L., Lifshits M.A. (1999). Chung's law and Csáki function. *J. Theor. Probab.* **12**, 399-420.

[G87] Grill, K. (1987). On the rate of convergence in Strassen's law of the iterated logarithm. *Probab. Rel. Fields.* **74** , 583–589 .

[G91] Grill, K. (1991). A lim inf result in Strassen's law of the iterated logarithm. *Probab. Th. Rel. Fields,* **89**, 149-157.

[G92] Grill, K. (1992). Exact rate of convergence in Strassen's law of the iterated logarithm. *J. Theoret. Probab.* **5**, 197–204.

[Gr70] Gross, L. (1970). *Lectures in modern analysis and applications II.* Lecture Notes in Math. Vol 140, Springer, Berlin.

[G-72] Das Gupta, S., Eaton, M., Olkin, I., Perlman, M., Savage, L. and Sobel, M. (1972). Inequalities on the probability content of convex regions for elliptically contoured distributions. *Proc. Sixth Berkeley Symp. Math. Statist. Prob. 3*, 241–264.

[Ha98] Hargé, G. (1998). Une inégalité de décorrélation pour la mesure gaussienne. *C. R. Acad. Sci. Paris Sér.* **326**, 1325–1328.

[H-98] Hitczenko, P., Kwapień, S., Li, W.V., Schechtman, G., Schlumprecht, T. and Zinn, J. (1998). Hypercontractivity and comparison of moments of iterated maxima and minima of independent random variables. *Electronic Journal of Probability* Vol. **3**, Paper no. 2, 1-26.

[HS99] Horváth, L. and Shao, Q.M. (1999). Liminf for weighted empirical processes. In preparation.

[Hu97] Hu, Y. (1997). Ito-Wiener chaos expansion with exact residual and correlation, variance inequalities. *J. Theoret. Probab.* **10**, 835–848.

[I82] Ibragimov, I.A. (1982). On the probability that a Gaussian vector with values in a Hilbert space hits a sphere of small radius. *J. Sov. Math.* **20**, 2164-2174.

[J70] Jogdeo, K. (1970). A simple proof of an inequality for multivariate normal probabilities of rectangles. *Ann Math. Statist.* **41**, 1357-1359.

[K46] Kac, M. (1946). On the average of a certain Wiener functional and a related limit theorem in calculus of probability. *Trans. Amer. Math. Soc.* **59**, 401–414 .

[K51] Kac, M. (1951). On some connections between probability theory and differential and integral equations, *Proc. Second Berkeley Symposium on Math. Stat. and Probability*, 189–215, University of California Press, Berkeley.

[Kh67] Khatri, C.G. (1967). On certain inequalities for normal distributions and their applications to simultaneous confidence bounds. *Ann. Math. Stat.* **38**, 1853–1867.

[KPX99] Khoshnevisan, D., Peres, Y. and Xiao, Y. (1999). Limsup random fractals, preprint.

[KS98a] Khoshnevisan, D. and Shi, Z. (1998a). Chung's law for integrated Brownian motion. *Trans. Amer. Math. Soc.* **350**, 4253-4264.

[KS98b] Khoshnevisan, D. and Shi, Z. (1998b). Gaussian measure of a small ball and capacity in Wiener space, *Asymptotic methods in Probability and statistics.* 453-465, North-Holland, Amsterdam.

[KT61] Kolmogorov, A.N. and Tihomirov, V.M. (1961). ε-entropy and ε-capacity of sets in function spaces. *Amer. Math. Soc. Transl.* **17**, 277-364.

[KMT75] Komlós, J. Major, P., and Tusnády, G. (1975). An approximation of partial sums of independent rv's and the sample distribution function, I. *Z. Wahrsch. verw. Gebiete* **32**, 111-131.

[Ku76] Kuelbs, J. (1976). A strong convergence theorem for Banach space valued random variables. *Ann. Probab.* **4**, 744–771.

[KL93a] Kuelbs, J. and Li, W.V. (1993a). Metric entropy and the small ball problem for Gaussian measures. *J. Funct. Anal.* **116**, 133-157.

[KL93b] Kuelbs, J. and Li, W.V. (1993b). Small ball estimates for Brownian motion and the Brownian sheet. J. Theoret. Probab. **6**, 547-577.

[KL95] Kuelbs, J. and Li, W.V. (1995). An extension of Ehrhard's Theorem. *Interaction Between Functional Analysis, Harmonic Analysis, and Probability.* Lecture notes in pure and applied mathematics. Vol **175**, 291-300.

[KL98] Kuelbs, J. and Li, W.V. (1998). Some shift inequalities for Gaussian measures. *Progress in Probability* **43**, 233-243, Birkhauser.

[KL00] Kuelbs, J. and Li, W.V. (2000). Wichura type functional law of the iterated logarithm for fractional Brownian. In preparation.

[KLL94] Kuelbs, J., Li, W.V. and Linde, W. (1994). The Gaussian measure of shifted balls. *Proba. Theor. and Rel. Fields*, Vol. **98**, 143-162. (1994).

[KLS95] Kuelbs, J., Li, W.V. and Shao, Q. (1995). Small ball estimates for fractional Brownian motion under Hölder norm and Chung's functional LIL. *J. Theor. Probab.* **8**, 361-386.

[KLT94] Kuelbs, J., Li, W.V. and Talagrand, M. (1994). Lim inf results for Gaussian samples and Chung's functional LIL. *Ann. Probab* **22**, 1879-1903.

[Kw94] Kwapień, S. (1994). A remark on the median and the expectation of convex functions of Gaussian vectors. *Progress in Probability* **35**, 271-272, Birkhauser, Boston.

[KwS93] Kwapień, S. and Sawa, J. (1993). On some conjecture concerning Gaussian measures of dilatations of convex symmetric sets. *Studia Math.* **105**, 173–187.

[Lai73] Lai, T.L. (1973). Gaussian processes, moving averages and quick detection problems. *Ann. Probab.* **1**, 825-837.

[LaS70] Landau, H.J. and Shepp, L.A. (1970). On the supremum of a Gaussian process. *Sankhya* **32**, 369–378.

[La96] Latala, R. (1996). A note on the Ehrhard's inequality. *Studia Math.* **118**, 169-174.

[LO99] Latała, R. and Oleszkiewicz, K. (1999). Gaussian measures of dilatations of convex symmetric sets. *Preprint.*

[L96] Ledoux, M. (1996). Isoperimetry and Gaussian Analysis, *Lectures on Probability Theory and Statistics, Lecture Notes in Math.* **1648** 165–294, Springer-Verlag.

[LT91] Ledoux, M. and Talagrand, M. (1991). *Probability on Banach Spaces*, Springer, Berlin.

[Le37] Lévy, P. (1937). Theorie de l'Addition des Variables Aleatoires. Gauthier-Villars, Paris.

[LP99] Lewis, T. and Pritchard, G. (1999). Correlation measures. *Elect. Comm. in Probab.* **4**, 77–85.

[Li92a] Li, W.V. (1992a). Comparison results for the lower tail of Gaussian seminorms, *J. Theoret. Probab*, **5**, 1-31. (1992).

[Li92b] Li, W.V. (1992b). Lim inf results for the Wiener process and its increments under the L_2-norm. *Proba. Theory and Rel. Fields*, **92**, 69-90.

[Li92c] Li, W.V. (1992c). On the lower tail of Gaussian measures on l_p, *Progress in Probability,* **30**, 106-117, Birkhauser.

[Li98a] Li, W.V. (1998a). A lim inf result for the Brownian motion, *Asymptotic methods in Probability and statistics*, 281-292, (ICAMPS'97), edited by B. Szyszkowicz, North-Holland, Amsterdam.

[Li99a] Li, W.V. (1999a). A Gaussian correlation inequality and its applications to small ball probabilities. *Elect. Comm. in Probab.*, **4**, 111-118.

[Li99b] Li, W.V. (1999b). Small deviations for Gaussian Markov processes under the sup–norm. *J. Theoret. Probab.*, **12**, 971-984.

[Li99c] Li, W.V. (1999c). Small ball estimates for Gaussian Markov processes under the L_p-norm, Preprint.

[Li00] Li, W.V. (2000). The existence of small ball constants for Gaussian processes under various norms, in preparation.

[LL93] Li, W.V. and Linde, W. (1993). Small ball problems for non-centered Gaussian measures. *Prob. Math. Stat.* **14**, 231-251.

[LL98] Li, W.V. and Linde, W. (1998). Existence of small ball constants for fractional Brownian motions. *C.R. Acad. Sci. Paris*, **326** , 1329-1334.

[LL99] Li, W.V. and Linde, W. (1999). Approximation, metric entropy and small ball estimates for Gaussian measures. *Ann. Probab.* **27**, 1556-1578.

[LS99a] Li, W.V. and Shao, Q.M. (1999a). Small ball estimates for Gaussian processes under the Sobolov norm. *J. Theor. Prob.*, **12**, 699-720.

[LS99b] Li, W.V. and Shao, Q.M. (1999b). A note on the Gaussian correlation conjecture, to appear.

[Lif95] Lifshits, M.A. (1995). *Gaussian Random Functions.* Kluwer Academic Publishers, Boston.

[Lif99] Lifshits, M. (1999). Asymptotic behavior of small ball probabilities. In: Probability theory and Mathematical Statistics. Proceedings of Seventh Vilnius Conference, p.453-468. TEV and VSP editions.

[LifL99] Lifshits, M.A. and Linde, W. (1999). Entropy numbers of Volterra operators with application to Brownian motion, *preprint.*

[LTs86] Lifshits, M.A., and Tsyrelson, B.S. (1986). Small ball deviations of Gaussian fields. *Theor. Probab. Appl.* **31**, 557-558.

[MN68] Mandelbrot, B. and Van Ness, J. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Review* **10**, 422-437.

[Ma68] Marcus, M. (1968). Hölder conditions for continuous Gaussian processes. *Trans. Amer. Math. Soc.* **134**, 29-52.

[Ma70] Marcus, M. (1970) A bound for the distribution of the maximum of continuous Gaussian processes. *Ann. Math. Statist.* **41** 305-309.

[MS72] Marcus, M. and Shepp, L. (1972). Sample behavior of Gaussian processes. *Proc. Sixth Berkeley Symp. Math. Stat. Prob.* **2**, 423-441.

[M74] Mogulskii, A.A. (1974). Small deviations in space of trajectories. *Th. Probab. Appl.* **19**, 726–736.

[M80] Mogulskii, A.A. (1980). On the law of the iterated logarithm in Chung's form for functional spaces. *Theor. Probab. Appl.* **24**, 405-413.

[MR95] Monrad, D. and Rootzen, H. (1995). Small values of Gaussian processes and functional laws of the iterated logarithm. *Probab. Th. Rel. Fields* **101**, 173-192.

[N67] Nisio, M. (1967). On the extreme values of Gaussian processes. *Osaka J. Math.* **4**, 313-326.

[OT74] Orey, S. and Taylor, S. (1974). How often on a Brownian motion path does the law of iterated logarithm fail? *Pro. Lon. Math. Soc.* **28**, 174–192 .

[Pi89] Pisier, G. (1989). *The Volume of Convex Bodies and Banach Space Geometry.* Cambridge Univ. Press. Cambridge.

[Pit82] Piterbarg, V.I. (1982). Gaussian random processes. [Progress in Science and Technology] *Teor. Veroyatnost. Mat. Statist. Teor Kibernet.* **9**, 155-198.

[P77] Pitt, L. (1977). A Gaussian correlation inequality for symmetric convex sets. *Ann. Probab.* **5**, 470–474.

[P78] Pitt, L. (1978). Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27**, 309-330.

[PT79] Pitt, L. and Tran, L.T. (1979). Local sample path properties of Gaussian fields. *Ann Probab.* **7**, 477-493.

[SSZ98] Schechtman, G., Schlumprecht, T. and Zinn, J. (1998). On the Gaussian measure of the intersection. *Ann. Probab.* **26**, 346-357.

[S93] Shao, Q.M. (1993). A note on small ball probability of Gaussian processes with stationary increments. *J. Theoret. Probab.* **6**, 595-602.

[S96] Shao, Q.M. (1996). Bounds and estimators of a basic constant in extreme value theory of Gaussian processes. *Statist. Sinica* **7**, 245-257.

[S99] Shao, Q.M. (1999). A Gaussian correlation inequality and its applications to the existence of small ball constant, preprint.

[SWa95] Shao, Q.M. and Wang, D. (1995). Small ball probabilities of Gaussian fields. *Probab. Th. Rel. Fields* **102**, 511-517.

[SZ92] Shepp, L. and Zeitouni, O. (1992). A note on conditional exponential moments and Onsager-Machlup functionals. *Ann. Probab.* **20**, 652–654.

[Shi91] Shi, Z. (1991). A generalization of the Chung-Mogulskii law of the iterated logarithm, *J. Multi. Analy.* **37**, 269-278.

[SW86] Shorack, G. R. and Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics.* John Wiley & Sons, New York.

[Si67] Sidak, Z. (1967). Rectangular confidence regions for the means of multivariate normal distributions. *J. Amer. Statist. Assoc.* **62**, 626-633.

[Si68] Sidak, Z. (1968). On multivariate normal probabilities of rectangles: their dependence on correlations. *Ann. Math. Stat.* **39**, 1425–1434.

[Sl62] Slepian, D. (1962). The one-sided barrier problem for Gaussian noise. *Bell System Tech. J* **41**, 463–501.

[Sm37] Smirnov, N.V. (1937). On the distribution of ω^2-criterion of Mises. *Math. Sbornik* **2**, 973-993.

[St93] Stolz, W. (1993). Une méthods élémentaire pour l'évaluation de petites boules Browniennes. *C.R. Acad. Sci. Paris* **315**, 1217–1220.

[St96] Stolz, W. (1996). Some small ball probabilities for Gaussian processes under nonuniform norms. *J. Theor. Probab.* **9**, 613-630.

[Str64] Strassen, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrs. verw. Gebiete* **3** , 211-226.

[Sy74] Sytaya, G. N. (1974). On some asymptotic representation of the Gaussian measure in a Hilbert space. *Theory of Stochastic Processes.* Ukrainian Academy of Sciences, **2**, 93-104.

[Su69] Sudakov, V. N. (1969). Gaussian measures, Cauchy measures and ε–entropy. *Soviet Math. Dokl.* **10**, 310–313.

[ST74] Sudakov, V.N. and Tsirelson, B.S. (1974). Extremal properties of half-spaces for spherically invariant measures. *J. Sov. Math.* **9**, 9-18, (1978); translated from Zap. Nauch. Sem. L.O.M.I. **41**, 14-24.

[SzW99] Szarek, S. and Werner, E. (1999). A nonsymmetric correlation inequality for Gaussian measure. *J. Multi. Anal.* **68**, 193-211.

[T92] Talagrand, M. (1992). On the rate of convergence in Strassen's LIL. *Progress in Probability*, Birkhauser, Boston, 339–351.

[T93] Talagrand, M. (1993). New Gaussian estimates for enlarged balls. *Geometric and Funct. Anal.*, **3**, 502-526.

[T94] Talagrand, M. (1994). The small ball problem for the Brownian sheet. *Ann. Probab.* **22**, 1331-1354.

[T95] Talagrand, M. (1995). Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **23**, 767–775.

[T96] Talagrand, M. (1996). Lower classes for fractional Brownian motion. *J. Th. Probab.* **9**, 191-213.

[T98] Talagrand, M. (1998). Multiple points of trajectories of multiparameter fractional Brownian motion. *Proba. Th. Rel. Fields* **112**, 545-563.

[Ta86] Taylor, S.J. (1986). The measure theory of random fractals. *Math. Proc. Cambridge Philos. Soc.* **100**, 383–406.

[Te95] Temlyakov, V. (1995). An inequality for trigonometric polynomials and its application for estimating the entropy numbers. *Journal of Complexity* **11**, 293-307.

[To80] Tong, Y. L. (1980). *Probability Inequalities in Multi-variate Distributions.* Academic Press, New York.

[U95] Üstünel, A.S. (1995). *An Introduction to Analysis on Wiener Space.* Springer, Berlin.

[V84] Varadhan, S.R.S. (1984). *Large Deviations and Applications.* S.I.A.M. Philadelphia.

[Vi96] Vitale, R. (1996). The Wills Functional and Gaussian Processes. *Ann. Probab.* **24**, 2172-2178.

[Vi99a] Vitale, R. (1999). Majorization and Gaussian Correlation. *Stat. Probab. Letters*, **45**, 247–251.

[Vi99b] Vitale, R. (1999). A log–concavity proof for a Gaussian exponential bound. *Contemporary Math.: Advances in Stochastic Inequalities* **234**, 209–212.

[Vi99c] Vitale, R. (1999). Some Comparisons for Gaussian Processes. *Proc. Amer. Math. Soc.* to appear.

[W73] Wichura, M. (1973). A functional form of Chung's Law of the iterated logarithm for maximun absolute partial sums, unpublished.

[X96] Xiao, Y. (1996). Hausdorff measure of the sample paths of Gaussian random fields. *Osaka J. Math.* **33**, 895–913.

[X97] Xiao, Y. (1997). Fractal measures of the sets associated to Gaussian random fields, *preprint*.

[X98] Xiao, Y. (1998). Hausdorff-type measures of the sample paths of fractional Brownian motion. *Stoch. Process. Appl.* **74**, 251–272.

[Y57] Yaglom, A.M. (1957). Some classes of random fields in n-dimensional space, related to stationary random processes. *Theory Probab. Appl.* **2**, 273-320.

[Zh96a] Zhang, L. X. (1996a). Two different kinds of liminfs on the LIL for two-parameter Wiener processes. *Stoch. Proc. Their Appl.*, **63**, 175-188.

[Zh96b] Zhang, L. X. (1996b). Some liminf results on increments of fractional Brownian motions, *Acta Math. Hungar.* **71**, 215-240.

[Z86] Zolotarev, V. M. (1986). Asymptotic behavior of the Gaussian measure in l_2. *J. Sov. Math.* **24**, 2330-2334.