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Small ball probabilities for Gaussian Markov processes under the L_p-norm ☆

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Abstract

Let $\{X(t);\ 0 \leqslant t \leqslant 1\}$ be a real-valued continuous Gaussian Markov process with mean zero and covariance $\sigma(s,t) = EX(s)X(t) \neq 0$ for $0 < s,\ t < 1$. It is known that we can write $\sigma(s,t) = G(\min(s,t))H(\max(s,t))$ with $G > 0$, $H > 0$ and G/H nondecreasing on the interval $(0,1)$. We show that for the L_p-norm on $C[0,1]$, $1 \leqslant p \leqslant \infty$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\|X(t)\|_p < \varepsilon) = -\kappa_p \left(\int_0^1 (G'H - H'G)^{p/(2+p)}\, \mathrm{d}t \right)^{(2+p)/p}$$

and its various extensions.

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1. Introduction

The small ball probability studies the behavior of

$$\log \mu(x\colon \|x\| \leqslant \varepsilon) = -\phi(\varepsilon) \quad \text{as } \varepsilon \to 0 \tag{1.1}$$

for a given measure μ and a norm $\|\cdot\|$. In the literature, small ball probabilities of various types are studied and applied to many problems of interest under different names such as small deviation, lower tail behaviors, two-sided boundary crossing probability and exit time.

For a Gaussian measure and any norm on a separable Banach space, there is a precise link, discovered in Kuelbs and Li (1993) and completed in Li and Linde (1999), between the function $\phi(\varepsilon)$ and the metric entropy of the unit ball of the reproducing kernel Hilbert space generated by μ. This powerful connection allows the use of tools and results from functional analysis and approximation theory to estimate small ball probabilities. The survey paper of Li and Shao (2000) on small ball probabilities for

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Gaussian processes, together with its extended references, covers much of the recent progress in this area. In particular, various applications and connections with other areas of probability and analysis are discussed.

For Markov processes, there is no general result available unless the process and the norm have the correct scaling (or self-similar) property. In that case (1.1) can be rewritten in terms of the first exit time of certain region and much more general results are known, and the problems are related to the large deviation for occupation measures developed by Donsker and Varadhan (1977).

In this paper, we study small ball probabilities for Gaussian Markov processes under the L_p-norms, which can also be viewed as for Brownian motion under weighted L_p-norms. It is somewhat surprising that we are able to find the exact small ball constants here since the main results in many works in this area determine only the asymptotic behavior in (1.1) up to some constant factor in front of the rate. Even for Brownian motion and Brownian bridge under various norms, these constants are known under L_p-norms and the sup-norm, but not under the Hölder norms.

Now we need some notations. Let $X(t)$ be a real-valued continuous Gaussian Markov processes on the interval $[0,1]$ with mean zero. It is known (cf. Feller, 1967; Borisov, 1982) that the covariance function $\sigma(s,t)=EX(s)X(t)<\infty,\ 0\leqslant s,\ t\leqslant 1$, satisfies the relation

$$\sigma(s,t)\sigma(t,u)=\sigma(t,t)\sigma(s,u), \quad 0\leqslant s<t<u\leqslant 1,$$

and this relation actually implies the Markov property of $X(t)$. Hence it is easy to obtain and *characterize* the Gaussian Markov process $X(t)$ with $\sigma(s,t)\neq 0,\ 0<s\leqslant t<1$, by

$$\sigma(s,t)=G(\min(s,t))H(\max(s,t)), \tag{1.2}$$

with $G>0,\ H>0$ and G/H nondecreasing on the interval $(0,1)$. Moreover, the functions G and H are unique up to a constant multiple. Throughout this paper, we use

$$\|f\|_p=\begin{cases}\left(\displaystyle\int_0^1 |f(t)|^p\,\mathrm{d}t\right)^{1/p} & \text{for } 1\leqslant p<\infty\\ \displaystyle\sup_{0\leqslant t\leqslant 1}|f(t)| & \text{for } p=\infty\end{cases}$$

to denote the L_p-norm on $C[0,1],\ 1\leqslant p\leqslant\infty$.

Theorem 1.1. *Let the Gaussian Markov process $X(t)$ be defined as above. Assume H and G are absolutely continuous and G/H is strictly increasing on the interval* $[0,1]$. *If*

$$\sup_{0<t\leqslant 1} H(t)<\infty, \text{ or}$$

$$H(t) \text{ is nonincreasing in a neighborhood of } 0,$$

then

$$\lim_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}(\|X(t)\|_p<\varepsilon)=-\kappa_p\left(\int_0^1 (G'H-H'G)^{p/(2+p)}\,\mathrm{d}t\right)^{(2+p)/p}$$

where

$$\kappa_p = 2^{2/p} p(\lambda_1(p)/(2+p))^{(2+p)/p} \tag{1.3}$$

and

$$\lambda_1(p) = \inf\left\{\int_{-\infty}^{\infty} \|x\|^p \phi^2(x)\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} (\phi'(x))^2\,\mathrm{d}x\right\} > 0, \tag{1.4}$$

the infimum is taken over all $\phi \in L_2(-\infty,\infty)$ *such that* $\int_{-\infty}^{\infty} \phi^2(x)\,\mathrm{d}x = 1$.

Several remarks are in order here. First note that $0 < \kappa_p < \infty$ are the constants associated with Brownian motion and it is well known that $\kappa_2 = 1/8$ and $\kappa_\infty = \pi^2/8$. Other related facts and history are given after Lemma 2.3. The sup-norm case ($p = \infty$) of the above result was presented in Li (1999a). Second we observe that there is nothing special about the interval $[0,1]$ and it can be replaced by any finite interval $[a,b]$ as long as $\sigma(s,t) \neq 0$ for s,t in (a,b) and the analogous regularity condition on $H(t)$ holds. If $\sigma(s,t) = 0$ for some s and t inside the finite interval of interest, then the process $X(t)$ can break up into uncorrelated Gaussian Markov processes on disjoint open intervals such that $\sigma(s,t) \neq 0$ in each of the sub-intervals. The details are given in Borisov (1982) and hence we can apply our result to each of the sub-interval and then put independent pieces together by Lemma 2.2 to cover this case. Finally, we can also handle the L_p-norm over the whole positive real line, based on a Gaussian correlation inequality given in Li (1999b), see Lemma 2.2 in Section 2. This is conveniently given in the following for Brownian motion under the weighted L_p-norms. The connection between Theorem 1.1 and Theorem 1.2 is the following representation for Gaussian Markov processes

$$X(t) = h(t)W(g(t)) \tag{1.5}$$

with $g(t) > 0$ nondecreasing on the interval $(0,1)$ and $h(t) > 0$ on the interval $(0,1)$. It is easy to see the connection between (1.2) and (1.5), $h(t) = H(t)$ and $g(t) = G(t)/H(t)$. Thus our Theorem 1.1 follows easily from the following general result.

Theorem 1.2. *Let* $\rho : [0,\infty) \to [0,\infty]$ *be a Lebesgue measurable function satisfying the following conditions for* $1 \leqslant p \leqslant \infty$.

(i) $\rho(t)$ *is bounded or nonincreasing on* $[0,a]$ *for some* $a > 0$;
(ii) $\rho(t) \cdot t^{(2+p)/p}$ *is bounded or nondecreasing on* $[T,\infty)$ *for some* $T < \infty$;
(iii) $\rho(t)$ *is bounded on* $[a,T]$ *and* $\rho(t)^{2p/(2+p)}$ *is Riemann integrable on* $[0,\infty)$.

Then

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\left(\int_0^{\infty} |\rho(t)W(t)|^p\,\mathrm{d}t\right)^{1/p} \leqslant \varepsilon\right) = -\kappa_p \left(\int_0^{\infty} \rho(t)^{2p/(2+p)}\,\mathrm{d}t\right)^{(2+p)/p} \tag{1.6}$$

where κ_p *is given in* (1.3).

Some brief history and remarks are needed. In the case of sup-norm ($p = \infty$) over finite interval $[0,T]$, similar results were given in Mogulskii (1974) under the condition

$\rho(t)$ is bounded, in Berthet and Shi (1998) under the condition $\rho(t)$ is nonincreasing, and in Li (1999a) under the critical case that $\int_0^T \rho^2(t)\,\mathrm{d}t=\infty$. In the case of sup-norm ($p=\infty$) over infinite interval $[0,\infty)$, the results were treated in Li (1999b) as an application of a Gaussian correlation inequality, see Lemma 2.4 in Section 2. As we can see from examples in Section 2, the integration over $(0,\infty)$ is very useful. By the precise link with small ball probabilities, we can obtain the entropy numbers for the generating integral operator

$$(Tf)(t)=H_t\int_0^t H_s^{-1}(G_s'H_s-H_s'G_s)^{1/2}f(s)\,\mathrm{d}s,\quad t\geqslant 0, \tag{1.7}$$

associated to our Gaussian Markov process $X(t)$ for $f\in L_2(\mathbb{R}^+)$. For its associations to Volterra operators, we refer to Lifshits and Linde (1999). Furthermore, Theorem 1.2 is also proved there under a slightly weaker regularity assumptions on $\rho(t)$ at zero and infinity. Their proof is based on our Lemmas 2.2 and 2.4, and a more flexible procedure of approximation using step functions. In addition, they show that the regularity condition on $\rho(t)$ cannot be weakened to the most natural and general one that $\rho(t)^{2p/(2+p)}$ is Lebesgue integrable. Finally we mention that Theorem 1.2 can easily be extend to the $\mathbb{R}^d$ valued Brownian motion with independent component, see Shi (1996) and Berthet and Shi (1998).

The remaining of the paper is arranged as follows. We present some interesting examples and basic lemmas in Section 2. Some applications are also indicated. The proof of Theorem 1.2 is given in Section 3 in three steps. [1]

2. Examples and lemmas

Next we apply Theorems 1.1 and 1.2 to some well-known Gaussian Markov processes or weighted Brownian motion over finite or infinite intervals. Let $\{W(t);\ 0\leqslant t\leqslant 1\}$ be the standard Brownian motion and $\{B(t);\ 0\leqslant t\leqslant 1\}$ be a standard Brownian bridge, which can be realized as $\{W(t)-tW(1);\ 0\leqslant t\leqslant 1\}$.

Example 1. Consider $X_1(t)=t^{-\alpha}W(t)$ on the interval $[0,1]$ for $\alpha<(2+p)/2p,\ p\geqslant 1$. Then our main result together with simple calculation implies

$$\lim_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\left(\int_0^1|t^{-\alpha}W(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)=-\kappa_p\left(\frac{2+p}{2+p-2\alpha p}\right)^{(2+p)/p}. \tag{2.1}$$

Thus by the exponential Tauberian theorem given as Lemma 2.1 below,

$$\lim_{\lambda\to\infty}\lambda^{-2/(2+p)}\log\mathbb{E}\exp\left\{-\lambda\int_0^1\frac{|W(s)|^p}{s^{\alpha p}}\,\mathrm{d}s\right\}=\frac{(2+p)^2}{2+p-2\alpha p}(\kappa_p/2^{2/p}p)^{p/(2+p)}.$$

[1] During the preparation of this paper, Professor Zhan Shi kindly informed me the work Shi (1999) on the L_2-norm of α-symmetric stable processes over finite interval. The over-lapping part is the $p=\alpha=2$ case on finite interval. Even in this case, our conditions are slightly weaker for non-increasing functions near zero.

Now by using the scaling property of Brownian motion and (1.3)

$$\lim_{t\to\infty} t^{-(2+p-2\beta)/(2+p)} \log \mathbb{E} \exp\left\{-\lambda \int_0^t \frac{|W(s)|^p}{s^\beta}\,\mathrm{d}s\right\} = -\frac{2+p}{2+p-2\beta}\lambda_1(p)\lambda^{2/(2+p)}$$

for $\beta < (2+p)/2$ and $\lambda > 0$, where $\lambda_1(p)$ is given in (1.4).

As the second application of (2.1), we have the following Chung type law of the iterated logarithm.

$$\liminf_{T\to\infty} \frac{(\log\log T)^{1/2}}{T^{(2+p-2\alpha p)/(2p)}} \left(\int_0^T |t^{-\alpha} W(t)|^p\,\mathrm{d}t\right)^{1/p} = \kappa_p^{1/2} \left(\frac{2+p}{2+p-2\alpha p}\right)^{(2+p)/(2p)} \quad \text{a.s.}$$

for $\alpha < (2+p)/2p$. It follows from the estimates given in (2.1) and a rescaling argument along with an application of the Borel–Cantelli lemma. All of these are fairly standard and well understood once one has the necessary probability estimate (2.1).

Example 2. Let $U(t)$ be the stationary Gaussian Markov process or the Ornstein–Uhlenbeck process with $\mathbb{E}\, U(s)U(t) = \sigma^2 \mathrm{e}^{-\theta|t-s|}$ for $\theta > 0$ and any $s,t \in [a,b]$, $-\infty < a < b < \infty$. Then we have

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_a^b |U(t)|^p\,\mathrm{d}t \leqslant \varepsilon^p\right) = -2\sigma^2\theta(b-a)^{(2+p)/p}\kappa_p.$$

In the case $p=2$, the above result and its refinement are given in Li (1992a) by using the Karhunen–Loéve expansion and a comparison theorem.

Example 3. For $0 \leqslant a < b < \infty$ and $1 \leqslant p \leqslant \infty$,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_a^b |W(t)|^p\,\mathrm{d}t \leqslant \varepsilon^p\right) = -(b-a)^{(2+p)/p}\kappa_p.$$

In the case $p=2$, the above result and its refinement are given in Li (1992b) with applications to Chung's type LIL over interval away from zero. Here we have by suing the standard arguments mentioned at the end of Example 1,

$$\liminf_{T\to\infty} \frac{(\log\log T)^{1/2}}{T^{(2+p)/(2p)}} \left(\int_{aT}^{bT} |W(t)|^p\,\mathrm{d}t\right)^{1/p} = \kappa_p^{1/2}(b-a)^{(2+p)/(2p)} \quad \text{a.s.}$$

Example 4. Let f be a locally bounded Borel function on $[0,\infty)$ such that $f \in L_2(R_+)$. Then from Revuz and Yor (1994, p. 135), the process

$$Z(t) = \int_0^t f(s)\,\mathrm{d}W(s), \quad t \geqslant 0$$

is a Gaussian Markov process with

$$\sigma_Z(s,t) = \mathrm{Cov}(Z(s)Z(t)) = \int_0^{\min(s,t)} f^2(u)\,\mathrm{d}u, \quad s,t \geqslant 0.$$

Thus we have for $1 \leqslant p \leqslant \infty$

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^\infty |Z(t)|^p\,\mathrm{d}t \leqslant \varepsilon^p\right) = -\kappa_p \left(\int_0^\infty |f(t)|^{2p/(2+p)}\,\mathrm{d}t\right)^{(2+p)/p}.$$

Example 5. Let

$$X(t) = \frac{|B(t)|}{\sqrt{t(1-t)}}$$

for $1 \leqslant p < \infty$. Then for $0 < s \leqslant t < 1$

$$\mathrm{Cov}(X(t)X(s)) = \left(\frac{s(1-t)}{(1-s)t}\right)^{1/2}.$$

Thus we have for $1 \leqslant p < \infty$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^1 |X(t)|^p \, \mathrm{d}t \leqslant \varepsilon^p\right) = -\kappa_p \left(B\left(\frac{2}{2+p}, \frac{2}{2+p}\right)\right)^{(2+p)/p}$$

where $B(x,y)$ is the beta function. Note that when $p = 2$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(|X(t)|_2 \leqslant \varepsilon\right) = -\pi^2/8$$

which can also be found from Anderson and Darling (1952) in their applications for asymptotic distribution of weighted von Mises criterion. The interesting weight function used here makes the process $X(t)$ constant variance.

Example 6. By using the well-known fact that $\{B(t), 0 \leqslant t \leqslant 1\} = \{(1-t)W(t/(1-t)), 0 \leqslant t \leqslant 1\}$ in law, we have in distribution

$$\sup_{0 \leqslant t \leqslant 1} \frac{|B(t)|}{t^\beta (1-t)^\beta} = \sup_{0 \leqslant t \leqslant 1} \frac{(1-t)W(t/(1-t))}{t^\beta (1-t)^\beta} = \sup_{0 \leqslant t < \infty} \frac{W(t)}{t^\beta (1+t)^{1-2\beta}}$$

for $0 \leqslant \beta < 1/2$. Thus

$$\begin{aligned}\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0 \leqslant t \leqslant 1} \frac{|B(t)|}{t^\beta (1-t)^\beta} \leqslant \varepsilon\right) &= \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0 \leqslant t < \infty} \frac{W(t)}{t^\beta (1+t)^{1-2\beta}} \leqslant \varepsilon\right) \\ &= -\frac{\pi^2}{8} B(1-2\beta, 1-2\beta).\end{aligned}$$

Interesting applications to empirical processes can be obtained similarly to those in Csáki (1994) by using the above estimate.

Next we state some lemmas we need in our proof. The first lemma is a special case of the so called de Bruijn's exponential Tauberian theorem in Bingham et al. (1987), Theorem 4.12.9. It connects the asymptotic behavior of Laplace transform of a positive random variable with the small ball probability of the random variable.

Lemma 2.1. *Let X be a positive random variable. Then for $\alpha > 0$*

$$\log \mathbb{P}(X \leqslant \varepsilon) \sim -C_X \varepsilon^{-\alpha} \quad \textit{as } \varepsilon \to 0^+$$

if and only if

$$\log \mathbb{E} \exp(-\lambda X) \sim -(1+\alpha)\alpha^{-\alpha/(1+\alpha)} C_X^{1/(1+\alpha)} \lambda^{\alpha/(1+\alpha)} \quad \textit{as } \lambda \to \infty.$$

As an easy consequence of the above exponential Tauberian theorem, we have the following lemma for sums of independent nonnegative random variables.

Lemma 2.2. *If X_i, $1 \leqslant i \leqslant n$, are independent nonnegative random variables such that*

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma} \log \mathbb{P}(X_i \leqslant \varepsilon) = -d_i, \quad 1 \leqslant i \leqslant n,$$

for $0 < \gamma < \infty$, *then*

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma} \log \mathbb{P}\left(\sum_{i=1}^{n} X_i \leqslant \varepsilon \right) = -\left(\sum_{i=1}^{n} d_i^{1/(1+\gamma)} \right)^{1+\gamma}.$$

Lemma 2.3. *For any* $1 \leqslant p \leqslant \infty$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\|W(t)\|_p \leqslant \varepsilon) = \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\|B(t)\|_p \leqslant \varepsilon) = -\kappa_p \tag{2.2}$$

where κ_p *is given in* (1.3).

Before we present detailed proof of this lemma a few remarks on related facts and history are needed. The case $p=2$ and $p=\infty$ with $\kappa_2 = 1/8$ and $\kappa_\infty = \pi^2/8$ are well known and the exact distributions in terms of infinite series are known, see Smirnov (1937) and Doob (1949). The only other case, for which the exact distribution is given in terms of Laplace transform, is in Kac (1946) for $p=1$. Namely, for $\lambda \geqslant 0$

$$\mathbb{E} \exp\left\{ -\lambda \int_0^1 |W(s)| \, \mathrm{d}s \right\} = \sum_{j=1}^{\infty} \theta_j \exp\{-\delta_j \lambda^{2/3}\} \tag{2.3}$$

where $\theta_j = (3\delta_j)^{-1}(1 + 3\int_0^{\delta_j} P(y)\,\mathrm{d}y)$ and $\delta_1, \delta_2, \ldots$ are the positive roots of the derivative of

$$P(y) = 3^{-1}(2y)^{1/2}(J_{-1/3}(3^{-1}(2y)^{3/2}) + J_{1/3}(3^{-1}(2y)^{3/2})) = 2^{1/3}\mathrm{Ai}(-2^{1/3}y),$$

$J_\alpha(x)$ are the Bessel functions of parameter α and $\mathrm{Ai}(x)$ is the Airy function. The extension of (2.3) to values of $\lambda < 0$ remains to be open as far as we know. By using the exponential Tauberian theorem, Lemma 2.1, we have from (2.3) $\kappa_1 = (4/27)\delta_1^3$ where δ_1 is the smallest positive root of the derivative of $P(y)$.

Now from asymptotic point of view for the Laplace transform, it was shown in Kac (1951) by Feynman–Kac formula and the eigenfunction expansion that

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ -\int_0^t |W(s)|^p \, \mathrm{d}s \right\} = -\lambda_1(p) \tag{2.4}$$

and $\lambda_1(p)$ is the smallest eigenvalue of the operator

$$Af = -\tfrac{1}{2} f''(x) + |x|^p f(x) \tag{2.5}$$

on $L_2(-\infty, \infty)$. Thus from (2.5) and the classical variation expression for eigenvalues, we obtain (1.4). A different and extremely powerful approach was given in Donsker and Varadhan (1975) so that the direct relation between (2.4) and (1.4)

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ -\int_0^t |W(s)|^p \, \mathrm{d}s \right\}$$
$$= -\inf\left\{ \int_{-\infty}^{\infty} |x|^p \phi^2(x) \, \mathrm{d}x + \frac{1}{2} \int_{-\infty}^{\infty} (\phi'(x))^2 \, \mathrm{d}x \right\} \tag{2.6}$$

holds as a very special case of their general theory on occupation measures for Markov processes. Both approaches work for more general function $V(x)$ than the ones we used here with $V(x)=|x|^p$, $1\leqslant p<\infty$, and thus the statement for W in Lemma 2.3 also holds for $0<p<1$.

On the other hand, from small ball probability or small deviation point of view, Borovkov and Mogulskii (1991) obtained

$$\mathbb{P}(\|W\|_p\leqslant\varepsilon)\sim c_1(p)\varepsilon\exp\{-\lambda_1(p)\varepsilon^{-2}\}$$

by using similar method as Kac (1951), but more detailed analysis for the polynomial term. Unfortunately, they did not realize the variation expression (1.4) for $\lambda_1(p)$ and the polynomial factor ε is missing in their original statement due to an algebraic error.

So our Lemma 2.3 is formulated explicit here for the first time. And it follows from (2.4) or (2.6), which is by Brownian scaling

$$\lim_{\lambda\to\infty}\lambda^{-2/(2+p)}\log\mathbb{E}\exp\left\{-\lambda\int_0^1|W(s)|^p\,\mathrm{d}s\right\}=-\lambda_1(p),$$

and the exponential Tauberian theorem, Lemma 2.1 with $\alpha=2/p$.

Now we turn to the proof of remaining parts of Lemma 2.3, i.e. the proof for Brownian Bridge. An abstract argument given in Li (1999b, Theorem 1.2), implies the conclusion easily. But here we give a traditional argument which provides slightly more information. By Anderson's inequality and the fact that $W(t)-tW(1)$ is independent of $W(1)$, we have

$$\begin{aligned}\mathbb{P}(\|W\|_p\leqslant\varepsilon)&=\mathbb{P}(\|W(t)-tW(1)+tW(1)\|_p\leqslant\varepsilon)\\&\leqslant\mathbb{P}(\|W(t)-tW(1)\|_p\leqslant\varepsilon)=\mathbb{P}(\|B\|_p\leqslant\varepsilon).\end{aligned}$$

For the upper bound, we have for any $\delta>0$

$$\begin{aligned}\mathbb{P}(\|W\|_p\leqslant(1+\delta)\varepsilon)&\geqslant\mathbb{P}(\|W(t)\|_p\leqslant(1+\delta)\varepsilon,|W(1)|\leqslant(1+p)^{1/p}\delta\varepsilon)\\&\geqslant\mathbb{P}(\|W(t)-tW(1)\|_p\leqslant\varepsilon,|W(1)|\leqslant(1+p)^{1/p}\delta\varepsilon)\\&=\mathbb{P}(\|B\|_p\leqslant\varepsilon)\cdot\mathbb{P}(|W(1)|\leqslant(1+p)^{1/p}\delta\varepsilon)\end{aligned}$$

where the second inequality follows from triangle inequality of the L_p-norm. Thus

$$\limsup_{\varepsilon\to\infty}\varepsilon^2\log\mathbb{P}(\|B\|_p\leqslant\varepsilon)\leqslant\limsup_{\varepsilon\to\infty}\varepsilon^2\log\mathbb{P}(\|W\|_p\leqslant(1+\delta)\varepsilon)=-(1+\delta)^{-2}\kappa_p$$

and the result follows by taking $\delta\to0$.

One last lemma is a Gaussian correlation inequality given in Li (1999b). The ways we use it in the next section are typical in various applications of the inequality to small ball estimates.

Lemma 2.4. *Let μ be a centered Gaussian measure on a separable Banach space E. Then for any $0<\lambda<1$, any symmetric, convex sets A and B in E.*

$$\mu(A\cap B)\geqslant\mu(\lambda A)\mu((1-\lambda^2)^{1/2}B).$$

A very short proof of this useful inequality is given in Li and Shao (2000) based on Anderson's inequality.

3. Proof of Theorem 1.2

Our proof is given in three steps. First we assume that $\rho(t)$ is bounded and $\rho(t)^{2p/(2+p)}$ Riemann integrable on $[0,T]$ and $\rho(t)=0$ for $t\geqslant T$. In the second step, we assume $\rho(t)$ is nonincreasing on $[0,a]$ for some $a>0$ small and $\rho(t)=0$ for $t\geqslant T$. Our Gaussian correlation Lemma 2.4 is used but it is not critical here since we can form independent increment by introducing the value at $t=a$. In the last step, we show the theorem over the whole positive half line under the condition (i) or (ii). In this step, the Gaussian correlation Lemma 2.4 helps to simplify the arguments and it seems hard to do the argument without Lemma 2.4. Note also that almost all the proofs we give below are for $1\leqslant p<\infty$, but the case for $p=\infty$ can be obtained easily with the nature modification.

Our first step is the following proposition for nice weight function over finite interval.

Proposition 3.1. *Let* $\rho:\ [0,T]\to[0,\infty)$ *be a bounded function on* $[0,T]$, $0<T<\infty$. *Then*

$$\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\left(\int_0^T|\rho(t)W(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)$$

$$\geqslant-\kappa_p\inf\left(\sum_{i=1}^n M_i^{2p/(2+p)}(t_i-t_{i-1})\right)^{(2+p)/p} \tag{3.1}$$

and

$$\limsup_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\left(\int_0^T|\rho(t)W(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)$$

$$\leqslant-\kappa_p\sup\left(\sum_{i=1}^n m_i^{2p/(2+p)}(t_i-t_{i-1})\right)^{(2+p)/p} \tag{3.2}$$

where the infimum and supremum being taken over all partitions $P=\{t_i\}_0^n$ *and*

$$m_i=\inf_{t_{i-1}\leqslant t\leqslant t_i}\rho(t)\quad\text{and}\quad M_i=\sup_{t_{i-1}\leqslant t\leqslant t_i}\rho(t).$$

In particular, if $\rho(t)^{2p/(2+p)}$ *is Riemann integrable, then*

$$\lim_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\left(\int_0^T|\rho(t)W(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)=-\kappa_p\left(\int_0^T\rho(t)^{2p/(2+p)}\,\mathrm{d}t\right)^{(2+p)/p}. \tag{3.3}$$

Proof. Fix a finite partition $P=\{t_i\}_0^n$ of $[0,T]$ such that

$$0=t_0<t_1<\cdots<t_n=T.$$

Let $B_1(t),B_2(t),\ldots,B_n(t)$, $0\leqslant t\leqslant 1$, be independent standard Brownian bridges that are also independent of $W(t)$. Define for $t_{i-1}\leqslant t\leqslant t_i$

$$\widehat{W}(t)=W(t_{i-1})+(W(t_i)-W(t_{i-1}))\frac{t-t_{i-1}}{t_i-t_{i-1}}+\sqrt{t_i-t_{i-1}}B_i\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right). \tag{3.4}$$

Then it is well known and easy to check that $\{\widehat{W}(t),\ 0\leqslant t\leqslant T\}$ is a standard Brownian motion. Thus we have

$$\begin{aligned}
&\mathbb{P}\left(\left(\int_0^T |\rho(t)W(t)|^p\,\mathrm{d}t\right)^{1/p}\leqslant\varepsilon\right)\\
&\quad=\mathbb{P}\left(\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\rho^p(t)|\widehat{W}(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad\leqslant\mathbb{P}\left(\sum_{i=1}^n m_i^p\int_{t_{i-1}}^{t_i}|\widehat{W}(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad=\mathbb{P}\left(\sum_{i=1}^n m_i^p(t_i-t_{i-1})\int_0^1|(1-t)W(t_{i-1})+tW(t_i)\right.\\
&\qquad\quad\left.+\ \sqrt{t_i-t_{i-1}}B_i(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad\leqslant\mathbb{P}\left(\sum_{i=1}^n m_i^p(t_i-t_{i-1})\int_0^1|\sqrt{t_i-t_{i-1}}B_i(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad=\mathbb{P}\left(\sum_{i=1}^n m_i^p(t_i-t_{i-1})^{1+p/2}\int_0^1|B_i(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)
\end{aligned}\tag{3.5}$$

where the second inequality follows from Anderson's inequality and the fact that $B_i(t)$, $1\leqslant i\leqslant n$, are independent of $W(t)$. Thus by Lemma 2.2 and Lemma 2.3,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\ \varepsilon^2\log\mathbb{P}\left(\left(\int_0^T|\rho(t)W(t)|^p\,\mathrm{d}t\right)^{1/p}\leqslant\varepsilon\right)\\
&\quad\leqslant\limsup_{\varepsilon\to 0}\ \varepsilon^2\log\mathbb{P}\left(\sum_{i=1}^n m_i^p(t_i-t_{i-1})^{1+p/2}\int_0^1|B_i(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad=-\kappa_p\left(\sum_{i=1}^n m_i^{2p/(2+p)}(t_i-t_{i-1})\right)^{(2+p)/p}
\end{aligned}$$

which proves the upper estimate (3.2). Next we turn to the lower estimate. Let

$$Q_i=M_i^{-p/(2+p)}\left(\sum_{i=1}^n M_i^{2p/(2+p)}(t_i-t_{i-1})\right)^{-1/p}>0,\quad 1\leqslant i\leqslant n\tag{3.6}$$

and pick any $0<\delta<\min_{1\leqslant i\leqslant n}Q_i$. Then

$$\begin{aligned}
&\mathbb{P}\left(\int_0^T|\rho(t)W(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad\geqslant\mathbb{P}\left(\sum_{i=1}^n M_i^p(t_i-t_{i-1})\int_0^1|(1-t)W(t_{i-1})+tW(t_i)+\sqrt{t_i-t_{i-1}}B_i(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \mathbb{P}\left(\int_0^1 |(1-t)W(t_{i-1})+tW(t_i)+\sqrt{t_i-t_{i-1}}B_i(t)|^p\,\mathrm{d}t \leqslant Q_i^p\varepsilon^p,\ 1\leqslant i\leqslant n\right)\\
&\geqslant \mathbb{P}\left(\int_0^1 |(1-t)W(t_{i-1})+tW(t_i)|^p\,\mathrm{d}t \leqslant \delta^p\varepsilon^p,\right.\\
&\qquad\left.\int_0^1 |\sqrt{t_i-t_{i-1}}B_i(t)|^p\,\mathrm{d}t \leqslant (Q_i-\delta)^p\varepsilon^p,\ 1\leqslant i\leqslant n\right)\\
&\geqslant \mathbb{P}\left(|W(t_i)|\leqslant\delta\varepsilon, \int_0^1 |\sqrt{t_i-t_{i-1}}B_i(t)|^p\,\mathrm{d}t \leqslant (Q_i-\delta)^p\varepsilon^p,\ 1\leqslant i\leqslant n\right)\\
&= \mathbb{P}\left(\max_{1\leqslant i\leqslant n}|W(t_i)|\leqslant\delta\varepsilon\right)\cdot\prod_{i=1}^n \mathbb{P}\left(\int_0^1 |B_i(t)|^p\,\mathrm{d}t\leqslant (t_i-t_{i-1})^{-p/2}(Q_i-\delta)^p\varepsilon^p\right)
\end{aligned}
$$

where the equality follows from the independence of $B_i(t)$, $1\leqslant i\leqslant n$, and $W(t)$. Now by Khatri–Sidak's lemma (Khatri, 1967; Sidak, 1967),

$$
\mathbb{P}\left(\max_{1\leqslant i\leqslant n}|W(t_i)|\leqslant\delta\varepsilon\right)\geqslant\prod_{i=1}^n\mathbb{P}(|W(t_i)|\leqslant\delta\varepsilon).
$$

Thus by combining the above estimates together and using Lemma 2.3

$$
\begin{aligned}
&\liminf_{\varepsilon\to0}\varepsilon^2\log\mathbb{P}\left(\int_0^T|\rho(t)W(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad\geqslant\sum_{i=1}^n\lim_{\varepsilon\to0}\varepsilon^2\log\mathbb{P}\left(\int_0^1|B_i(t)|^p\,\mathrm{d}t\leqslant(t_i-t_{i-1})^{-p/2}(Q_i-\delta)^p\varepsilon^p\right)\\
&\qquad+\sum_{i=1}^n\liminf_{\varepsilon\to0}\varepsilon^2\log\mathbb{P}(|W(t_i)|\leqslant\delta\varepsilon)\\
&\quad=-\kappa_p\sum_{i=1}^n(Q_i-\delta)^{-2}(t_i-t_{i-1}).
\end{aligned}
$$

Taking $\delta\to0$ and substituting in Q_i from (3.6), we finish the proof of the lower estimates.

Next we observe that the upper estimate of our Theorem 1.2 follows easily from Proposition 3.1. Namely, we have for any $0<a<T<\infty$

$$
\begin{aligned}
&\limsup_{\varepsilon\to0}\varepsilon^2\log\mathbb{P}\left(\int_0^\infty|\rho(t)W(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad\leqslant\limsup_{\varepsilon\to0}\varepsilon^2\log\mathbb{P}\left(\int_0^T|1_{(a,T)}(t)\rho(t)W(t)|^p\,\mathrm{d}t\leqslant\varepsilon^p\right)\\
&\quad=-\kappa_p\left(\int_a^T\rho(t)^{2p/(2+p)}\,\mathrm{d}t\right)^{(2+p)/p}.
\end{aligned}
$$

Taking $a\to0$ and $T\to\infty$ gives the desired result. So the rest of this section is on the lower bound over the positive half line and allows the weight function to be unbounded near zero.

Proposition 3.2. *Assume the conditions of Theorem* 1.2 *with* $\rho(t)$ *nonincreasing on* $[0,a]$ *for a small. Then*

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant \varepsilon^p\right) \geqslant -\kappa_p \left(\int_0^T \rho(t)^{2p/(2+p)}\,\mathrm{d}t\right)^{(2+p)/p}.$$

Proof. For any $0<\delta<1/2$ and $0<\lambda<1$, we have

$$\begin{aligned}
&\mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant \varepsilon^p\right)\\
&\quad\geqslant \mathbb{P}\left(\int_0^a |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant \delta\varepsilon^p, \int_a^T |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant (1-\delta)\varepsilon^p\right)\\
&\quad\geqslant \mathbb{P}\left(\int_0^a |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant \delta(1-\lambda^2)^{p/2}\varepsilon^p\right)\\
&\qquad\cdot \mathbb{P}\left(\int_a^T |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant (1-\delta)\lambda^p\varepsilon^p\right)
\end{aligned} \tag{3.7}$$

by the Gaussian correlation Lemma 2.4. For the second term above, we have by Proposition 3.1

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_a^T |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant (1-\delta)\lambda^p\varepsilon^p\right)\\
&\quad = -(1-\delta)^{-2/p}\lambda^{-2}\kappa_p \left(\int_a^T \rho(t)^{2p/(2+p)}\,\mathrm{d}t\right)^{(2+p)/p}.
\end{aligned} \tag{3.8}$$

For the first term in (3.7) above, we have to estimate it similar to what we did in the bounded case. Note that we do not need to capture the exact constant in this case, but just up to a constant which depends on a and goes to 0 as $a\to 0$. Let

$$s_0 = a, \quad s_j = 2^{-j}a, \quad j = 0,1,\ldots$$

and

$$\tilde{M}_j = \sup_{s_{j+1}\leqslant t\leqslant s_j} \rho(t), \quad j\geqslant 0.$$

Take $a>0$ small enough such that

$$\tilde{Q}_j = \tilde{M}_j^{-p/(2+p)} \left(\sum_{j=0}^{\infty} \tilde{M}_j^{2p/(2+p)}(s_j - s_{j+1})\right)^{-1/p} < \infty, \quad j\geqslant 0. \tag{3.9}$$

Then, following the estimates below (3.6) with $\delta_j = \tilde{Q}_j/2$, we have for the first term in (3.7),

$$\begin{aligned}
&\mathbb{P}\left(\int_0^a |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant \delta(1-\lambda^2)^{p/2}\varepsilon^p\right)\\
&\quad\geqslant \prod_{j\geqslant 0} \mathbb{P}(|W(s_j)| \leqslant 2^{-1}\tilde{Q}_j\delta^{1/p}(1-\lambda^2)^{1/2}\varepsilon)\\
&\qquad\cdot \prod_{j=0}^{\infty} \mathbb{P}\left(\int_0^1 |B_j(t)|^p \,\mathrm{d}t \leqslant \frac{2^{-p}\tilde{Q}_j^p\delta(1-\lambda^2)^{p/2}\varepsilon^p}{(s_j - s_{j+1})^{p/2}}\right).
\end{aligned} \tag{3.10}$$

Now by Lemma 2.3, we obtain for the first products above that

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \prod_{j=0}^{\infty} \mathbb{P}(|W(s_j)| \leqslant 2^{-1}\tilde{Q}_j \delta^{1/p}(1-\lambda^2)^{1/2}\varepsilon)$$

$$\geqslant \liminf_{\varepsilon\to 0} \varepsilon^2 \sum_{j=0}^{\infty} \log \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |W(t)| \leqslant 2^{-1} s_j^{-1/2} \tilde{Q}_j \delta^{1/p}(1-\lambda^2)^{1/2}\varepsilon\right)$$

$$\geqslant -4\kappa_\infty \sum_{j=0}^{\infty} s_j \tilde{Q}_j^{-2} \delta^{-2/p}(1-\lambda^2)^{-1}$$

$$= -8\kappa_\infty \delta^{-2/p}(1-\lambda^2)^{-1} \sum_{j=0}^{\infty} \tilde{Q}_j^{-2}(s_j - s_{j+1})$$

$$= -8\kappa_\infty \delta^{-2/p}(1-\lambda^2)^{-1} \left(\sum_{j=0}^{\infty} \tilde{M}_j^{2p/(2+p)}\right)^{(2+p)/p}. \tag{3.11}$$

Next by Lemma 2.3 again, we have for the second product term in (3.10)

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \sum_{j=0}^{\infty} \log \mathbb{P}\left(\int_0^1 |\tilde{B}_j(t)|^p \leqslant \frac{2^{-p}\tilde{Q}_j^p \delta(1-\lambda^2)^{p/2}\varepsilon^p}{(s_j - s_{j+1})^{p/2}}\right)$$

$$\geqslant -4\kappa_\infty \sum_{j=0}^{\infty} \tilde{Q}_j^{-2} \delta^{-2/p}(1-\lambda^2)^{-1}(s_j - s_{j+1})$$

$$= -4\kappa_\infty \delta^{-2/p}(1-\lambda^2)^{-1} \left(\sum_{j=0}^{\infty} \tilde{M}_j^{2p/(2+p)}\right)^{(2+p)/p}. \tag{3.12}$$

Now putting (3.10)–(3.12) together, we have

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^a |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant \delta(1-\lambda^2)^{p/2}\varepsilon^p\right)$$

$$\geqslant -12\kappa_\infty \delta^{-2/p}(1-\lambda^2)^{-1} \left(\sum_{j=0}^{\infty} \tilde{M}_j^{2p/(2+p)}\right)^{(2+p)/p}. \tag{3.13}$$

Combining (3.13) with (3.7) and (3.8), we thus obtain

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p \,\mathrm{d}t \leqslant \varepsilon^p\right)$$

$$\geqslant -(1-\delta)^{-2/p}\lambda^{-2}\kappa_p \left(\int_a^T \rho(t)^{2p/(2+p)}\,\mathrm{d}t\right)^{(2+p)/p}$$

$$-12\kappa_\infty \delta^{-2/p}(1-\lambda^2)^{-1} \left(\sum_{j=0}^{\infty} \tilde{M}_j^{2p/(2+p)}\right)^{(2+p)/p}. \tag{3.14}$$

Since $\lim_{a\to 0}\sum_{j=0}^{\infty} \tilde{M}_j^{2p/(2+p)} = 0$, we finish the proof by taking $a \to 0$ first and then $\delta \to 0$, $\lambda \to 1$.

Note that there is a simple but less precise way of estimating (3.10). Namely, by using

$$\int_0^a |\rho(t)W(t)|^p \, dt \leqslant \int_0^a \rho(t)^p \, dt \cdot \left(\sup_{0 \leqslant t \leqslant a} |W(t)| \right)^p$$

under the stronger condition $\int_0^a \rho(t)^p \, dt < \infty$, we have

$$\begin{aligned} &\mathbb{P}\left(\int_0^a |\rho(t)W(t)|^p \, dt \leqslant \delta(1-\lambda^2)^{p/2}\varepsilon^p \right) \\ &\quad \geqslant \mathbb{P}\left(\sup_{0 \leqslant t \leqslant a} |W(t)| \leqslant \delta^{1/p}(1-\lambda^2)^{1/2} \left(\int_0^a \rho(t)^p \, dt \right)^{-1/p} \varepsilon \right) \end{aligned}$$

and thus by Lemma 2.3

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p \, dt \leqslant \varepsilon^p \right) \geqslant -\kappa_\infty \delta^{-2/p}(1-\lambda^2)^{-1} \left(\int_0^a \rho(t)^p \, dt \right)^{2/p}.$$

In fact this is more or less the argument used in Shi (1999) in the case of Brownian motion for $p = 2$.

Finally, we can handle the lower bound over the whole half line $(0, \infty)$ under the assumptions given in the Theorem 1.2. Applying the Gaussian correlation Lemma 2.4, we have for any $0 < \lambda < 1$ and $0 < \delta < 1$,

$$\begin{aligned} &\mathbb{P}\left(\int_0^\infty |\rho(t)W(t)|^p \, dt \leqslant \varepsilon^p \right) \\ &\quad \geqslant \mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p \, dt \leqslant (1-\delta)\varepsilon^p, \int_T^\infty |\rho(t)W(t)|^p \, dt \leqslant \delta\varepsilon^p \right) \\ &\quad \geqslant \mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p \, dt \leqslant \lambda^p(1-\delta)\varepsilon^p \right) \\ &\qquad \cdot \mathbb{P}\left(\int_T^\infty |\rho(t)W(t)|^p \, dt \leqslant (1-\lambda^2)^{p/2}\delta\varepsilon^p \right). \end{aligned} \tag{3.15}$$

For the second term from the equation above, we have by using the time inversion representation $\{W(t),\ t > 0\} = \{tW(1/t),\ t > 0\}$ in the distribution for Brownian motion

$$\begin{aligned} &\mathbb{P}\left(\int_T^\infty |\rho(t)W(t)|^p \, dt \leqslant (1-\lambda^2)^{p/2}\delta\varepsilon^p \right) \\ &\quad = \mathbb{P}\left(\int_0^{1/T} t^{-2-p}\rho^p(t^{-1})|tW(t^{-1})|^p \, dt \leqslant (1-\lambda^2)^{p/2}\delta\varepsilon^p \right) \\ &\quad = \mathbb{P}\left(\int_0^{1/T} |t^{-1-2/p}\rho(t^{-1})W(t)|^p \, dt \leqslant (1-\lambda^2)^{p/2}\delta\varepsilon^p \right). \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16), we obtain by the lower bound estimate we already have,

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^\infty |\rho(t)W(t)|^p \, dt \leqslant \varepsilon^p \right)$$

$$
\begin{aligned}
&\geqslant \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p \, dt \leqslant \lambda^p(1-\delta)\varepsilon^p\right) \\
&\quad + \liminf_{\varepsilon\to 0} \varepsilon^2 \mathbb{P}\left(\int_0^{1/T} |t^{-1-2/p}\rho(t^{-1})W(t)|^p \, dt \leqslant (1-\lambda^2)^{p/2}\delta\varepsilon^p\right) \\
&= -\lambda^{-2}(1-\delta)^{-2/p}\kappa_p \left(\int_0^T \rho(t)^{2p/(2+p)} \, dt\right)^{(2+p)/p} \\
&\quad -(1-\lambda^2)^{-1}\delta^{-2/p}\kappa_p \left(\int_0^{1/T} (t^{-1-2/p}\rho(t^{-1}))^{2p/(2+p)} \, dt\right)^{(2+p)/p} \\
&= -\lambda^{-2}(1-\delta)^{-2/p}\kappa_p \left(\int_0^T \rho(t)^{2p/(2+p)} \, dt\right)^{(2+p)/p} \\
&\quad -(1-\lambda^2)^{-1}\delta^{-2/p}\kappa_p \left(\int_T^{\infty} \rho(t)^{2p/(2+p)} \, dt\right)^{(2+p)/p}.
\end{aligned}
$$

Taking $T \to \infty$ first and then $\delta \to 0$ and $\lambda \to 1$ last, we obtain the desired lower estimate and thus finish the whole proof.

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