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Small ball probabilities for integrals of weighted Brownian motion

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Abstract

Let $X_\psi(t) := \int_0^t \psi(s)W(s)\,\mathrm{d}s$, $t\geqslant 0$, where $W(t)$, $t\geqslant 0$, is a standard Brownian motion and ψ is a weight function. We determine the rate of $-\log \mathbb{P}(\sup_{t\in[0,1]}|X_\psi(t)| < \varepsilon)$, as $\varepsilon\to 0$, for a large class of weight functions. The methods of our proofs are general and can be applied to many other problems. As an application, a Chung-type law of the iterated logarithm is given for $X_\psi(t)$ with $\psi(t)=t^{-\alpha}$, $\alpha<\frac{3}{2}$.

Keywords: Small ball probabilities; Integrated Brownian motion

1. Introduction

Let $W(t)$, $t\geqslant 0$, be a standard Brownian motion. Consider the Gaussian process

$$X_\psi(t) := \int_0^t \psi(s)W(s)\,\mathrm{d}s, \quad t\geqslant 0, \tag{1}$$

where ψ is a nonnegative function. In the monograph of Revuz and Yor (1991) one can find considerations on the a.s. existence of integral (1). Assuming that $\psi\in L_1^{\mathrm{loc}}((0,\infty))$ a criterion for the a.s. finiteness of the integral $\int_0^t \psi(s)|W(s)|\,\mathrm{d}s$, $t>0$, is that

$$\int_0^t \psi(s)s^{1/2}\,\mathrm{d}s < \infty. \tag{2}$$

When $\psi(t)\equiv 1$, the small ball probability

$$\lim_{\varepsilon\to 0} -\varepsilon^{2/3}\mathbb{P}\left(\sup_{t\in[0,1]}\left|\int_0^t W(s)\,\mathrm{d}s\right| < \varepsilon\right) = c \tag{3}$$

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for some constant $c \in (0, \infty)$ and the related Chung's law was studied in a work by Khoshnevisan and Shi (1998). The main idea in their paper is an indirect approach to (3) by using local time techniques and related limit laws since other techniques available at that time did not yield a satisfying answer. In particular, the general lower bound established in Talagrand (1993) and reformulated in Ledoux (1996, p. 257) fails to produce a sharp estimate for this example.

The main purpose of this note is to show the power and usefulness of the close connection between the small ball probabilities of Gaussian processes and the entropy numbers of certain corresponding operators. This relation was discovered by Kuelbs and Li (1993) and improved to its full extend in a recent work by Li and Linde (1998). The method of using the connection to give sharp lower bound is general and can be applied to many other problems.

In order to find the small ball probabilities for more general weight function ψ, we need some regularity conditions. Namely, we require that

(ψ1) ψ is nonincreasing in a neighborhood of zero when ψ is unbounded, and

(ψ2) we have $\int_0^1 \psi(s)^{2/3}\,\mathrm{d}s < \infty$.

Keep in mind that the example $\psi(t) = t^{-\alpha}$ satisfies (ψ1), (ψ2) and (2) when $\alpha < \frac{3}{2}$, and violates (ψ2) and (2) when $\alpha = \frac{3}{2}$.

Now we can state our results with the notation $f(\varepsilon) \lesssim g(\varepsilon)$ as $\varepsilon \to 0$ if $\limsup_{\varepsilon \to 0} f(\varepsilon)/g(\varepsilon)$ is bounded, and $f(\varepsilon) \approx g(\varepsilon)$ as $\varepsilon \to 0$ if $f(\varepsilon) \lesssim g(\varepsilon)$ and $g(\varepsilon) \lesssim f(\varepsilon)$.

Theorem 1. *Suppose that $\psi : (0,1] \to [0,\infty)$ fulfills (ψ1) and (ψ2) or that ψ is bounded then we have*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X_\psi(t)| < \varepsilon\right) \lesssim \varepsilon^{-2/3}$$

for ε tending to zero.

The above estimate is sharp for all "nice" weight function ψ. Namely, it holds:

Theorem 2. *Assume that there is an interval $[a,b] \subset [0,1]$ such that it holds $\inf_{t\in[a,b]} |\psi(t)| > 0$. Additionally, we require that $\psi'(t)$ exists for a.e. $t \in [a,b]$ and $\psi' \mathbf{1}_{[a,b]} \in L_2([0,1])$. Then we have*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X_\psi(t)| < \varepsilon\right) \gtrsim \varepsilon^{-2/3}$$

for ε tending to zero.

The above Theorems are a bit surprising since the asymptotic behaviors are the same (up to constants) for weights like $\psi(t) = t^{-\alpha}$, $\alpha < \frac{3}{2}$. As an application, we have the following Chung-type law of the iterated logarithm. It follows from the estimates given above and a rescaling argument along with an application of the Borel–Cantelli lemma. We will forgo the proof since it is fairly standard once one has the necessary probability estimates.

Theorem 3.

$$\liminf_{T\to\infty} T^{\alpha-3/2} (\log\log T)^{3/2} \sup_{0\leqslant t\leqslant T} \left| \int_0^t W(s)/s^\alpha \,\mathrm{d}s \right| = C_\alpha \quad \textit{a.s.},$$

where $0 < C_\alpha < \infty$ and $\alpha < \frac{3}{2}$. The exact value of C_α is unknown.

Finally, we would like to stress that the methods of proofs of Theorems 1 and 2 given in the next section are the most powerful for Gaussian processes as we understand it at this time. To prove Theorem 1, we use the entropy connections back and forth. We are unaware of any pure probabilistic proof of Theorem 1. To prove Theorem 2, we construct a large enough number of orthonormal functions and then apply Anderson's inequality.

2. Proofs of the theorems

Before we formulate the theorems from Li and Linde (1998) which we will apply in our proofs, let us recall the definition of the (dyadic) entropy numbers. Let $T:E\to F$ be an operator acting between two Banach spaces E and F. We denote by B_E and B_F the unit balls of E and F, respectively. Then the nth entropy number of T is defined as

$$e_n(T):=\inf\left\{\varepsilon>0:\exists x_1,\dots,x_{2^{n-1}}\in F \text{ such that } TB_E\subset\bigcup_{i=1}^{2^{n-1}}(x_i+\varepsilon B_F)\right\}.$$

In the sequel H denotes a Hilbert space and $f_1,f_2,\dots$ shall be a complete orthonormal system in H. The variables $g_1,g_2\dots$ shall always stand for a sequence of independent $\mathcal{N}(0,1)$-distributed random variables.

Theorem 4 (Li and Linde, 1998). *Let $T:H\to E$ be an operator mapping a Hilbert spaces into a Banach space and suppose that $\sum_{i=1}^{\infty}g_iTf_i$ converges a.s. in E. Then for $\alpha\in(0,2)$ the following are equivalent*:

(i) $e_n(T)\lesssim n^{-1/\alpha}$ (*respectively* $\approx n^{-1/\alpha}$) *and*

(ii) $-\log\mathbb{P}(\|\sum_{i=1}^{\infty}g_iTf_i\|<\varepsilon))\lesssim\varepsilon^{-2\alpha/(2-\alpha)}$ (*respectively* $\approx\varepsilon^{-2\alpha/(2-\alpha)}$).

Theorem 5 (Li and Linde, 1998). *Let $T:H\to E$ be as in Theorem 4 and assume that*

$$-\log\mathbb{P}\left(\left\|\sum_{i=1}^{\infty}g_iTf_i\right\|<\varepsilon\right)\lesssim\varepsilon^{-\beta}$$

for some $\beta>0$. If S is another operator mapping E into another Banach space F with $e_n(S)\lesssim n^{-1/\gamma}$, for some $\gamma>0$, then we have

$$-\log\mathbb{P}\left(\left\|\sum_{i=1}^{\infty}g_iS(Tf_i)\right\|<\varepsilon\right)\lesssim\varepsilon^{-\beta\gamma/(\beta+\gamma)}.$$

In the formulation of Theorems 4 and 5 we restricted ourselves to the situation needed in the proofs below. For more general statements and other applications, we refer the reader to Li and Linde (1998). Next we need the following lemma whose proof is also instructive.

Lemma 6. *For $\varphi\in L_2([0,1])$ we define an integral operator $I_\varphi:L_2([0,1])\to C([0,1])$ by*

$$(I_\varphi f)(t):=\int_0^t\varphi(s)f(s)\,\mathrm{d}s.$$

Then we have for the entropy numbers the estimate $e_n(I_\varphi)\lesssim n^{-1}$.

Proof. Without loss of generality, we may assume that $\varphi\not\equiv0$ since for $\varphi\equiv0$ the statement of the lemma is trivial. Now, let $(Y_\varphi(t))_{t\in[0,1]}$ be the process generated by the operator I_φ, i.e. $Y_\varphi=\sum_{i=1}^{\infty}g_iI_\varphi f_i$, and set

$F(t) := \int_0^t \varphi(s)^2\,\mathrm{d}s$. Then we observe

$$\begin{aligned}\mathbb{E}Y_\varphi(s)Y_\varphi(t) &= \int_0^1 \mathbf{1}_{[0,s]}(x)\varphi(x)\mathbf{1}_{[0,t]}(x)\varphi(x)\,\mathrm{d}x \\ &= F(s\wedge t) = F(s)\wedge F(t) \\ &= \mathbb{E}W(F(s))W(F(t)),\end{aligned}$$

i.e. $(Y_\varphi(t))_{t\in[0,1]}$ and $(W(F(t)))_{t\in[0,1]}$ have the same covariance function. Consequently, we can deduce

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |Y_\varphi(t)| < \varepsilon\right) = -\log \mathbb{P}\left(\sup_{t\in[0,F(1)]} |W(t)| < \varepsilon\right) \approx \varepsilon^{-2}$$

using the well-known small ball behavior of the Brownian motion. Finally, we conclude by Theorem 4 that $e_n(I_\varphi) \lesssim n^{-1}$. $\square$

Proof of Theorem 1. For $\varphi \in L_2([0,1])$ we introduce a multiplication operator M_φ mapping f to φf. By Hölder's inequality, M_φ is a bounded operator from $C([0,1])$ into $L_2([0,1])$. Note that

$$(I_{\psi^{1/3}} \circ M_{\psi^{1/3}})(\psi^{1/3}W)(t) = \int_0^t \psi(s)W(s)\,\mathrm{d}s = X_\psi(t).$$

By Lemma 6 it follows from ($\psi 2$) that

$$e_n(I_{\psi^{1/3}} \circ M_{\psi^{1/3}}) \leqslant \|M_{\psi^{1/3}}\| e_n(I_{\psi^{1/3}}) \lesssim n^{-1}.$$

It was shown in Berthet and Shi (1998) that ($\psi 1$) and ($\psi 2$) imply

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |\psi(t)^{1/3}W(t)| < \varepsilon\right) \lesssim \varepsilon^{-2}.$$

The same statement was shown for bounded weight functions in Mogulskii (1974), and the critical case (condition ($\psi 2$) is violated) was treated in Li (1998). Combining these two results Theorem 5 yields our assertion. $\square$

Proof of Theorem 2. First of all, it is clear that

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X_\psi(t)| < \varepsilon\right) \geqslant -\log \mathbb{P}\left(\sup_{t\in[0,b]} |X_\psi(t)| < \varepsilon\right).$$

Then, we decompose the process $(X_\psi(t))_{t\in[0,b]}$ in the following way:

$$X_\psi(t) = X_\psi(t\wedge a) + \mathbf{1}_{(a,b]}(t)W(a)\int_a^t \psi(s)\,\mathrm{d}s + \mathbf{1}_{(a,b]}(t)\int_a^t \psi(s)(W(s)-W(a))\,\mathrm{d}s.$$

Since the first two summands are independent of the last one, it follows from Anderson's inequality (see Anderson, 1955) that

$$-\log \mathbb{P}\left(\sup_{t\in[0,b]} |X_\psi(t)| < \varepsilon\right) \geqslant -\log \mathbb{P}\left(\sup_{t\in[a,b]} \left|\int_a^t \psi(s)(W(s)-W(a))\,\mathrm{d}s\right| < \varepsilon\right).$$

Now, rescaling of the Brownian motion shows that there is no loss of generality if we assume that the interval $[a,b]$ is the whole unit interval.

Recall that the Brownian motion is generated (in the sense discussed above) by the operator $f \mapsto \int_0^\cdot f(s)\,\mathrm{d}s$. Thus, we can define an operator $T_\psi : L_2([0,1]) \to C([0,1])$ by

$$(T_\psi f)(t) := \int_0^t \psi(s) \int_0^s f(x)\,\mathrm{d}x\,\mathrm{d}s$$

and obtain the following representation $X_\psi = \sum_{i=1}^\infty g_i T_\psi f_i$. Next, we construct for each $n \in \mathbb{N}$ an orthonormal system. For this purpose, we define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) := \int_0^t \int_0^s \mathbf{1}_{(0,1/4)\cup(3/4,1)}(x) - \mathbf{1}_{(1/4,3/4)}(x)\,\mathrm{d}x\,\mathrm{d}s$$

which is compactly supported in [0, 1], and we set

$$\tilde{f}_{n,k} := \mathbf{1}_{[0,1]} D\left(\frac{1}{\psi(\cdot)} D\varphi\left(n\left(\cdot - \frac{k}{n}\right)\right)\right)$$

for $k = 0, \ldots, n-1$, where D is the operator of differentiation. Finally, we normalize these functions by setting $f_{n,k} = \tilde{f}_{n,k}/\|\tilde{f}_{n,k}\|_2$. Observe that for fixed n the functions $f_{n,k}$, $k = 0, \ldots, n-1$, have essentially disjoint supports. The norm $\|\tilde{f}_{n,k}\|_2$ can be estimated from above as follows. Let $c := \inf_{t\in[0,1]} |\psi(t)| > 0$ then we have

$$\begin{aligned}\|\tilde{f}_{n,k}\|_2 &\leqslant \frac{n}{c^2}\left\|\psi'\varphi'\left(n\left(\cdot - \frac{k}{n}\right)\right)\right\|_2 + \frac{n^2}{c}\left\|\varphi''\left(n\left(\cdot - \frac{k}{n}\right)\right)\right\|_2 \\ &\leqslant \frac{n}{c^2}\|\psi'\|_2\|\varphi'\|_\infty + \frac{n^2}{c}n^{-1/2}\|\varphi''\|_2 \leqslant Cn^{3/2}.\end{aligned}$$

Using again Anderson's inequality, we conclude for $\varepsilon = C^{-1}\|\varphi\|_\infty n^{3/2}$

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,1]} |X_\psi(t)| < \varepsilon\right) &\leqslant \mathbb{P}\left(\sup_{t\in[0,1]}\left|\sum_{k=0}^{n-1} g_k (T_\psi f_{n,k})(t)\right| < \varepsilon\right) \\ &\leqslant \mathbb{P}\left(\sup_{0\leqslant k<n}\sup_{t\in[0,1]}\left|g_k \frac{\varphi(t)}{\|\tilde{f}_{n,k}\|_2}\right| < \varepsilon\right) \\ &\leqslant \prod_{k=0}^{n-1} \mathbb{P}(|g_k| < Cn^{3/2}\|\varphi\|_\infty^{-1}\varepsilon) \\ &\leqslant \exp(-C'n) = \exp(-C''\varepsilon^{-2/3})\end{aligned}$$

which completes the proof. □

For further reading see Berthet and Shi (1998).

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